The Spin Foam Lectures
2: Skein algebra

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v2

$U_q\mathrm{sl}2$

Algebra with generators K_+, K_-, K_0

$$[K_0, K_+] = 2K_+$$

$$[K_0, K_-] = -2K_-$$

$$[K_+, K_-] = \frac{e^{hK_0} - e^{-hK_0}}{e^h - e^{-h}} \qquad h \in \mathbb{C}$$

Two cases

1. $h = \frac{i\pi}{r}, \quad r \in \mathbb{Z}$, so $q = e^h = e^{i\pi/r}$
2. $h = 0 \quad (r = \infty) \quad$ Lie algebra $\mathrm{sl}2$

$U_q\mathrm{sl}2$ Coproduct

$$\Delta K_0 = K_0 \otimes 1 + 1 \otimes K_0$$

$$\Delta K_+ = K_+ \otimes e^{hK_0} + 1 \otimes K_+$$

$$\Delta K_- = K_- \otimes 1 + e^{-hK_0} \otimes K_-$$

The coproduct determines the tensor product of reps

$$\xi \otimes \eta \mapsto \Delta(K)\xi \otimes \eta$$

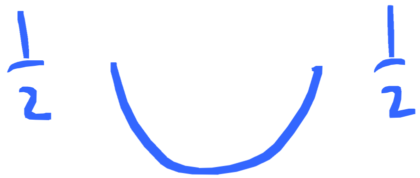
Two-dimensional representation "$\frac{1}{2}$"

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad K_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The tensor

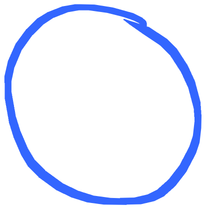
$$\epsilon = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - A^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

is invariant, if $A^2 = e^h$. Define this to be



$\frac{1}{2}$ is self-dual.

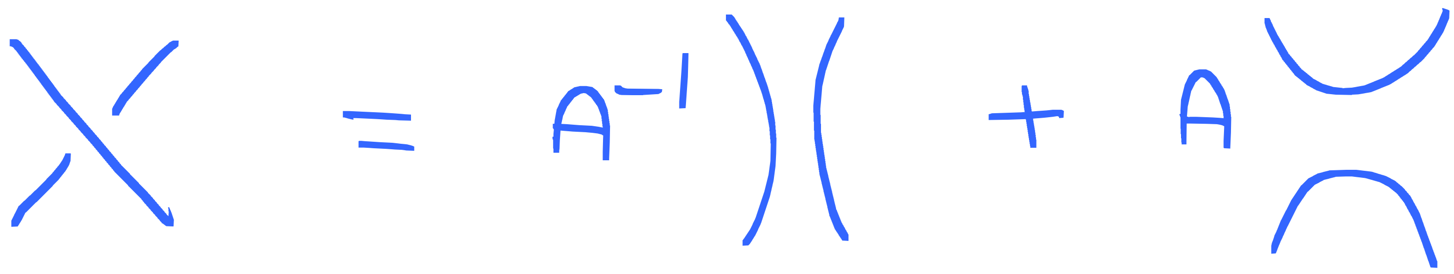
Unknot

$\frac{1}{2}$  $= -A^2 - A^{-2} = -q - q^{-1}$

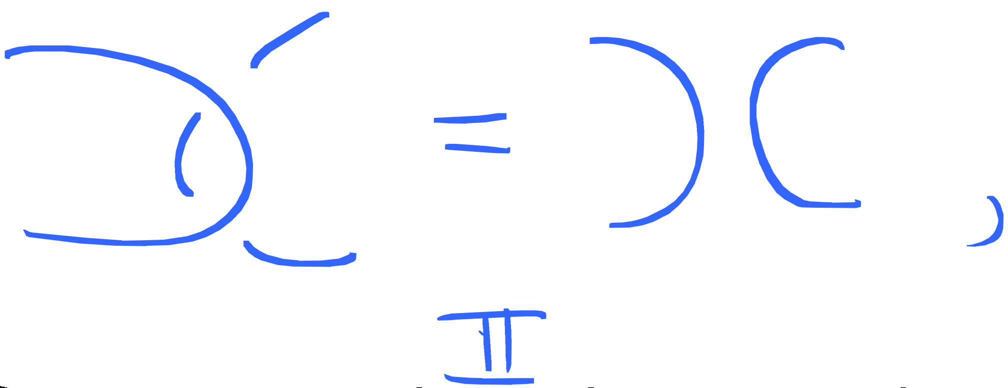
From now on, a line without a number is spin $\frac{1}{2}$.

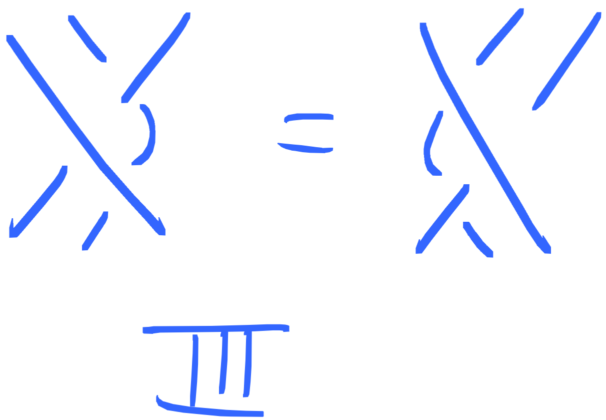
Crossing

Define the crossing intertwiner



This satisfies Reidemeister moves II and III



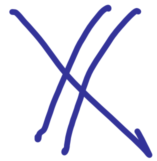


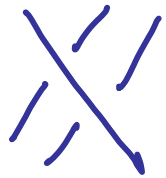
Crossings can always be removed.

Other irreducibles

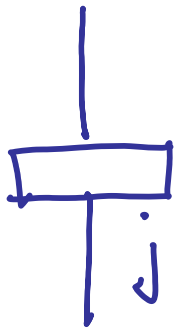
$$j \subset \frac{1}{2} \otimes \frac{1}{2} \otimes \ldots \otimes \frac{1}{2}, \quad 2j \text{ copies}$$

If $\sigma \in S_{2j}$, symmetric group, then $\widehat{\sigma}$ is a $+$ve braid:

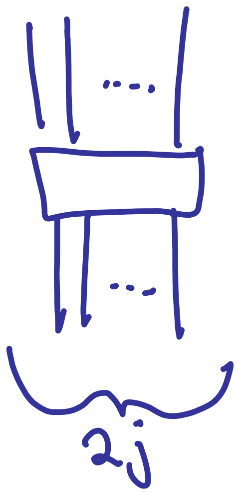
$\sigma =$ 

$\hat{\sigma}$ 

The projector onto the spin j irreducible

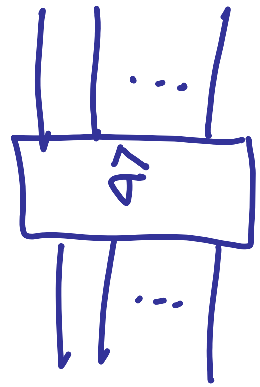


$=$



$$= c \sum_{\sigma \in S_{2j}} \left(A^{-3}\right)^{\#cr}$$

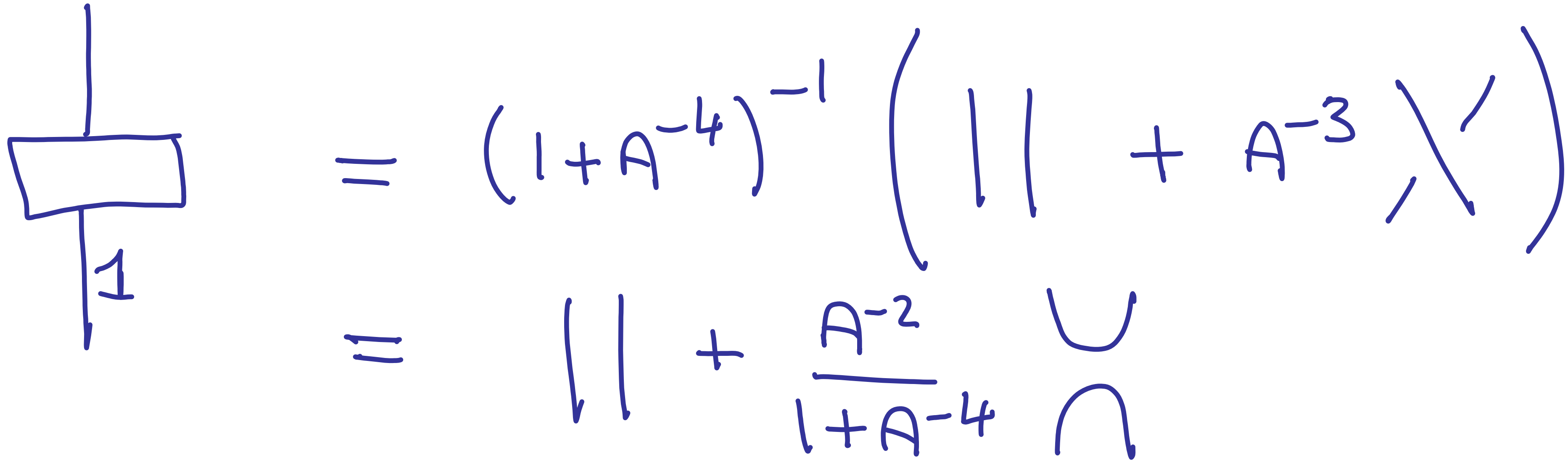
\# crossings



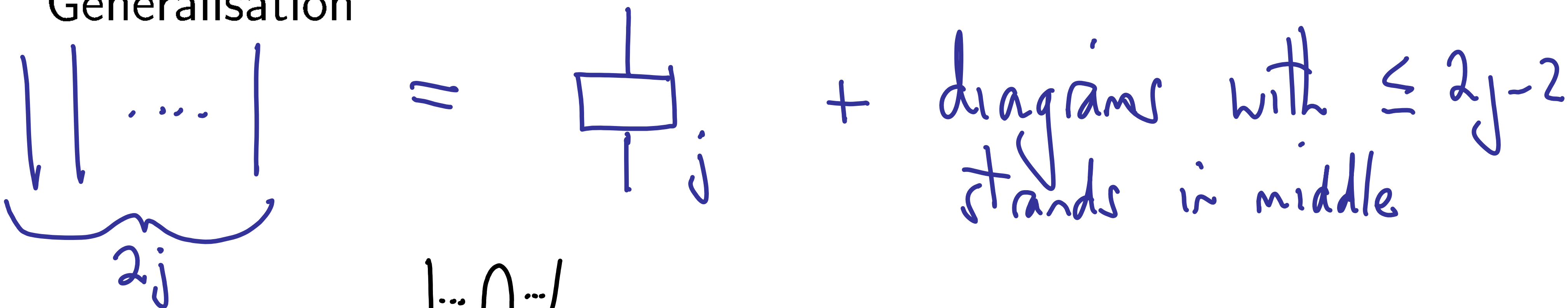
with

$$c^{-1} = \sum_{\sigma \in S_{2j}} \left(A^{-4}\right)^{\#cr}.$$

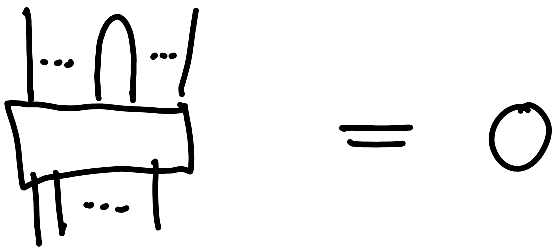
Example: spin 1



Generalisation

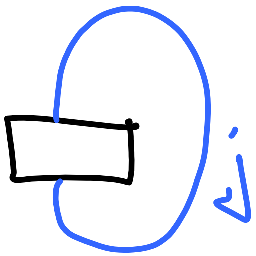


Lemma



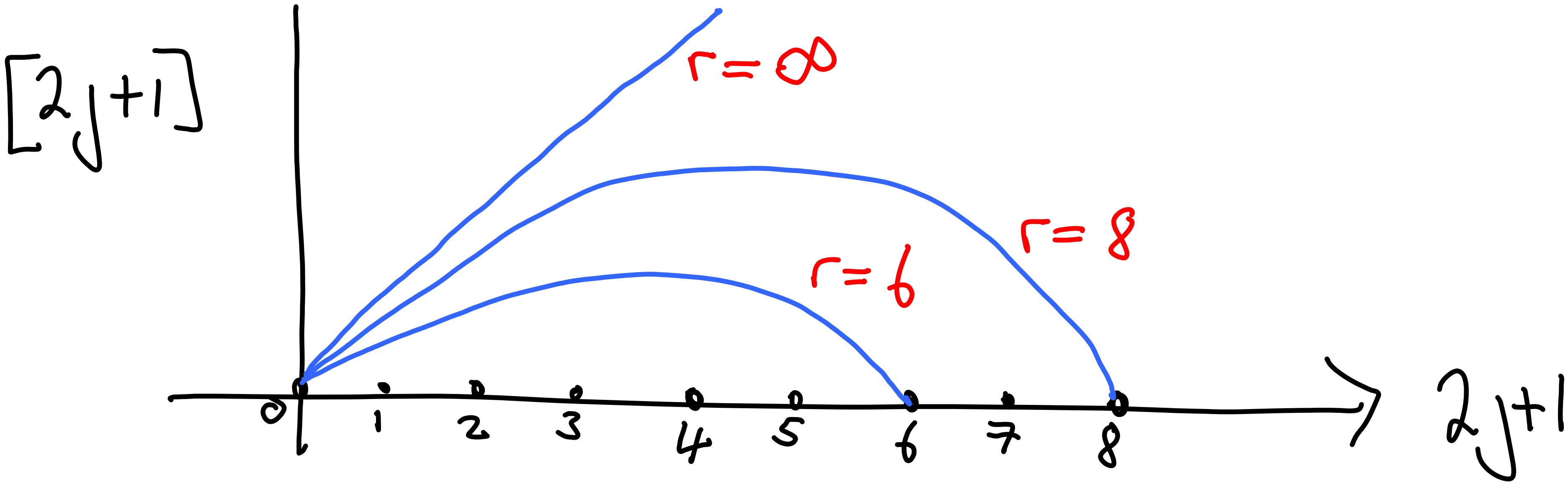
Quantum dimension

$$A = e^{i\pi/2r}$$

$$\Delta_j = $$ 

quantum integer

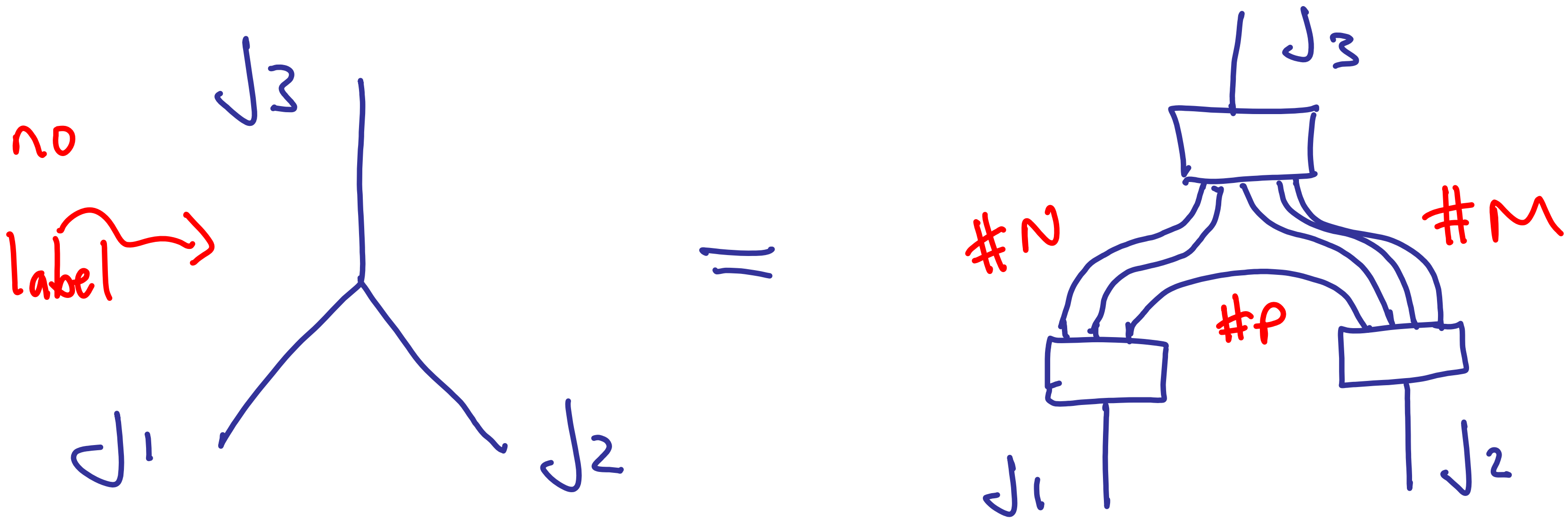
$$= (-1)^{2j}\frac{A^{4j+2}-A^{-4j-2}}{A^2-A^{-2}} = (-1)^{2j}\frac{\sin\frac{\pi}{r}(2j+1)}{\sin\frac{\pi}{r}} = (-1)^{2j}[2j+1]$$



Intertwiners

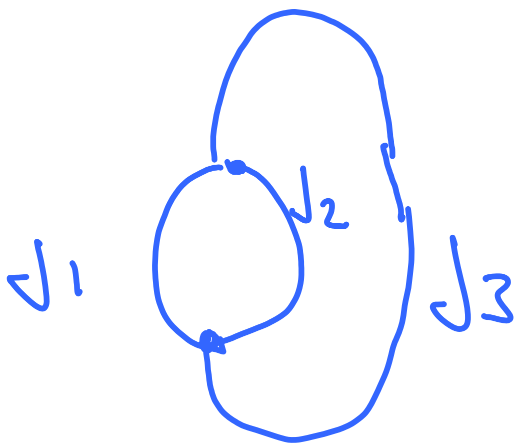
These are constructed from

$\mathrm{Hom}(j_1 \otimes j_2, j_3)$ has dimension 0 or 1. Canonical intertwiner



$$j_1 = \frac{1}{2}(N + P), \quad j_2 = \frac{1}{2}(M + P), \quad j_3 = \frac{1}{2}(N + M).$$

Theta



(j_1, j_2, j_3) admissible if $\Theta \neq 0$.

$$= \theta_{j_1 j_2 j_3} = (-1)^{j_1+j_2+j_3} \times \frac{[j_1+j_2+j_3+1]![j_1+j_2-j_3]![j_1+j_3-j_2]![j_2+j_3-j_1]!}{[2j_1]![2j_2]![2j_3]!}$$

using $[n]! = [n][n-1]\ldots[1]$.

Admissibility conditions

Conditions for $\theta_{j_1 j_2 j_3} \neq 0$, i.e., $\dim \mathrm{Hom} = 1$.

$$\#\text{strings} = j_1 + j_2 + j_3 \in \mathbb{Z} \tag{1}$$

$$M = j_3 + j_2 - j_1 \geq 0 \tag{2}$$

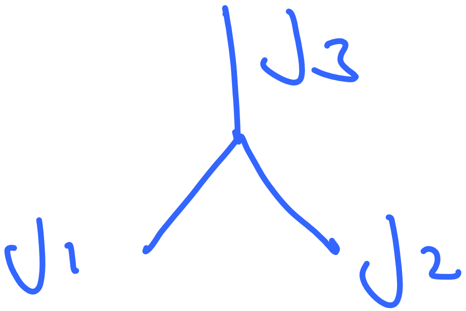
$$N = j_1 + j_3 - j_2 \geq 0 \tag{3}$$

$$P = j_1 + j_2 - j_3 \geq 0 \tag{4}$$

$$j_1 + j_2 + j_3 \leq r - 2 \tag{5}$$

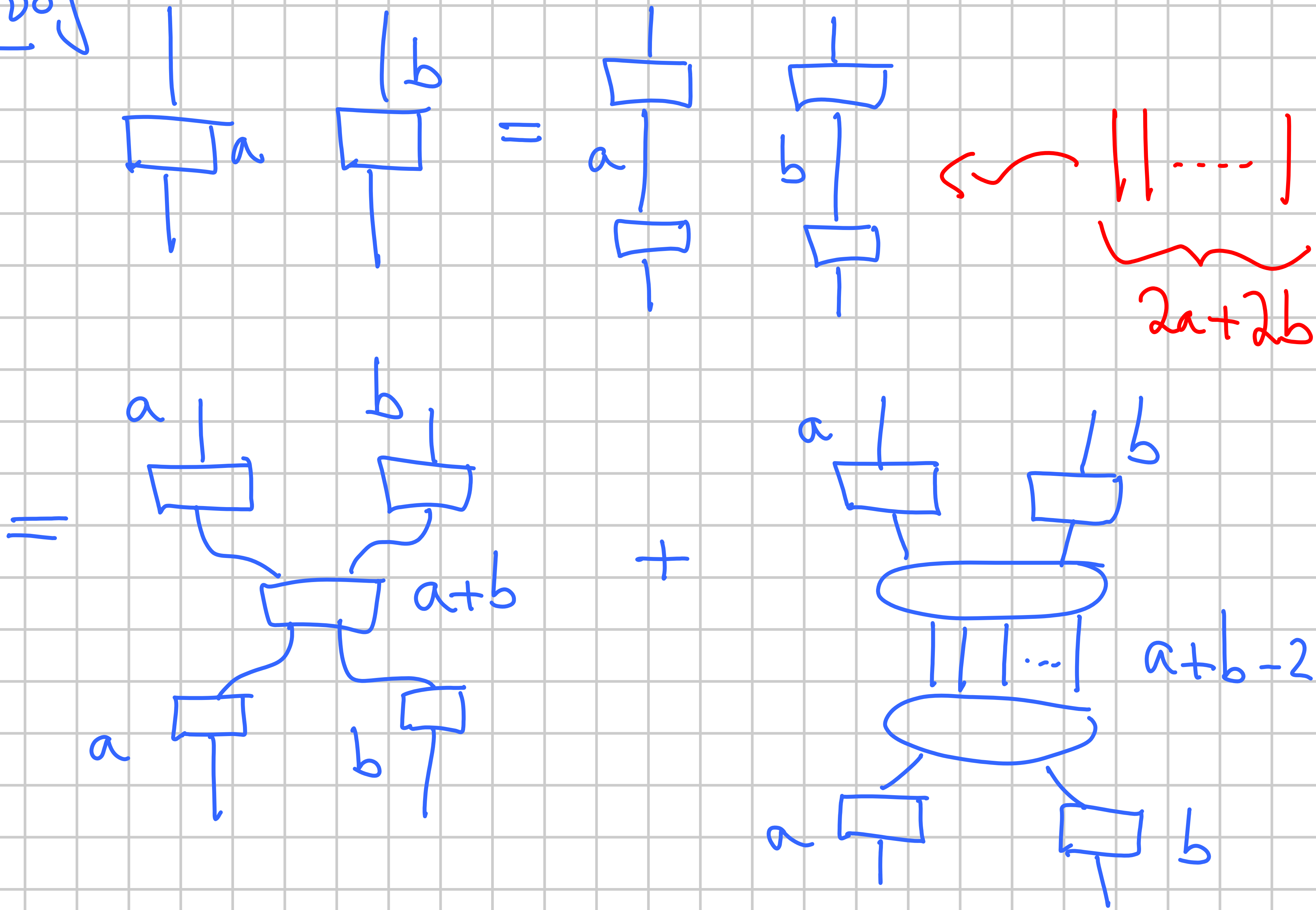
$$\Rightarrow \quad 2j_1, 2j_2, 2j_3 \leq j_1 + j_2 + j_3 \leq r - 2 \tag{6}$$

If not admissible,

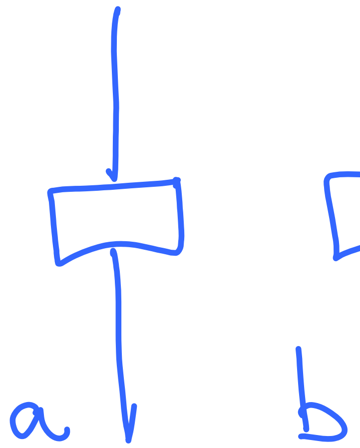


$= 0$ (equivalence)

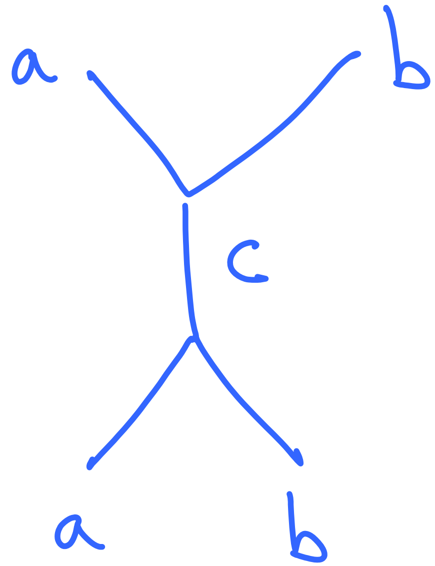
Proof



Semisimplicity



$$= \sum_{\substack{c \\ \text{admissible}}} \Gamma_c$$



Proof: use "generalisation".

Coefficients: $\Gamma_c = \Delta_c \Big/ \Theta_{abc}$.

Exercises - lecture 2

1. Show that the representation of $\mathrm{U}_q\mathrm{sl2}$ on $\mathbb{C}^2$ given in the lecture satisfies the relations for the algebra.
2. Show that if u and d are basis vectors in $\mathbb{C}^2$, then

$$\epsilon = Au \otimes d - Bd \otimes u$$

   is an invariant tensor, deriving the relation between A, B and $q = e^h$.
3. Prove the Reidemeister II and III moves for $\mathrm{U}_q\mathrm{sl2}$ spin 1 2. Calculate the crossing intertwiner in the classical cases $A = \pm 1$. In which case does it simply permute the two factors?

Exercises - lecture 2

4 Show that the admissibility conditions for $\mathrm{Hom}(j_1 \otimes j_2, j_3)$ are related to the inequalities on the edge-lengths $j_1 + 1/2$, $j_2 + 1/2$, $j_3 + 1/2$ of a non-degenerate triangle on a sphere. How is the radius of the sphere related to r?

Skein theory references

- Kauffman and Lins (book): Temperley-Lieb recoupling theory and invariants of 3-manifolds
- Roberts: Skein theory and Turaev-Viro invariants
- Lickorish: Skeins and handlebodies