Part III: BF formulations of GR

The idea is to replace the combination $\frac{1}{2}\epsilon_{IJKL}e^K e^L := B_{IJ}$ with a new field B_{IJ}. This is a Lie algebra valued 2-form field.

The action takes the form

$$S[e,\omega] = \frac{1}{16\pi G}\int B^{(e)}_{IJ} R^{IJ}(\omega) - \frac{\Lambda}{12} B^{(e)}_{IJ} B^{(e)}_{KL} \epsilon^{IJKL}$$

We now want to think of B_{IJ} as an independent field. However, there is a problem with this, as not any Lie-algebra valued 2-form field is of the form $\frac{1}{2}\epsilon_{IJKL}e^K e^L$ for some co-frame e^I. This is clear from e.g. the count of # of functions in $B^{IJ}_{\mu\nu}$ and e^I_μ

6×6 components   4×4 components

So, the idea is to impose constraints on B^{IJ} that guarantee that it is of the right form. We can impose these constraints with Lagrange multipliers. It is clear that we need $36-16=20$ constraints.

A possible set of constraints is

$$B^{IJ}\wedge B^{KL} \sim \epsilon^{IJKL}$$

This equation says that 20 out of 21 components of the matrix on the left are zero, while not constraining at all the remaining component

a 4-form with values in symmetric 6×6 matrices

$$\frac{6\cdot 7}{2} = 21 \text{ components}$$

There are 4 sets of solutions of these equations
Freidel de Pietri 1999

$$B^{IJ} = \pm \frac{1}{2} \epsilon_{IJKL} e^K e^L \quad \text{and} \quad B_{IJ} = \pm e_I e_J$$

for some coframe e^I

One of these sets is the desired one. The other leads to a "topological" theory where kinetic term, after substitution $\omega = \omega(e)$ reduces to a total derivative

With this in mind we write the following action

$$S_{FdP}[B,\omega,\Psi] = \frac{1}{16\pi G} \int B_{IJ} R^{IJ}(\omega) - \frac{1}{2}\left(\Psi_{IJKL} + \frac{\Lambda}{6}\epsilon_{IJKL}\right) B^{IJ} B^{KL}$$

The arising Euler-Lagrange equations $\Psi_{IJKL}\epsilon^{IJKL} = 0$

Lagrange multiplier field needed to impose the "metricity" constraints on B^{IJ}

$$R^{IJ} = \left(\Psi^{IJKL} + \frac{\Lambda}{6}\epsilon^{IJKL}\right) B_{KL}$$

$$d^\omega B^{IJ} = 0$$

$$B^{IJ} \wedge B^{KL} \sim \epsilon^{IJKL}$$

All solutions of Einstein-Cartan theory are also solutions of this theory.

The last 2 equations are also solved by $B_{IJ} = \pm e_I e_J$ and $\omega = \omega(e)$. But this is incompatible with the first equation, at least for $\Lambda \neq 0$. Indeed, in this case we get

$$R^{IJKL} = \Psi^{IJKL} + \frac{\Lambda}{6}\epsilon^{IJKL}$$

The full anti-symmetrization on the left-hand side vanishes (Bianchi identity), but is non-zero on the RHS. (for $\underline{\Lambda \neq 0}$)

So no more solutions in this theory than in Einstein-Cartan

Remarks:

- A cubic formulation of GR even with non-zero Λ. But one of the fields is a Lagrange multiplier without a kinetic term. Also kinetic term is of BF type and has more symmetry than the full action. Difficult (but not impossible) to quantize such theories perturbatively

- There is a more general formulation of the theory where the metricity constraints are replaced with

$$B^{IJ} B^{KL} \sim \alpha \epsilon^{IJKL} + \beta \delta^{I[K} \delta^{L]J}$$

two invariant bilinear forms on the Lorentz group Lie algebra

The solution is a linear combination of the two previously disjoint sets of solutions. The action on the solution is Einstein-Cartan plus the topological term $\int e^I e^J R_{IJ} \sim \int d^\omega e^I \wedge d^\omega e_I$

- Like Einstein-Cartan, there is no requirement that $B^{IJ} \neq 0$, but there is no good perturbative expansion around $B=0$ background.

- This formulation is the starting point for spin foam quantization of GR

- The nature of soldering in this formulation has changed completely: $B: TM \times TM \to so(1,3)$

Pure spin connection formulation revisited

It turns out to be possible to "integrate out" all the fields apart from the connection in this formulation.

Closed form of the pure spin connection formalism is available

More Lagrange multipliers

$$S[B,\omega,M,\mu] = \frac{1}{16\pi G}\int B_{IJ}R^{IJ}(\omega) - \frac{1}{2}M^{IJKL}B_{IJ}B_{KL}$$
$$+ \frac{\mu}{2}\left(M^{IJKL}\epsilon_{IJKL} - 4\Lambda\right)$$

For simplicity, to avoid carrying some signs, let us here consider the Euclidean signature

$$\epsilon^{IJKL}\epsilon_{IJKL} = 24$$

Integrating out B gives

$$S[\omega,M,\mu] = \frac{1}{32\pi G}\int (M^{-1})_{IJKL}R^{IJ}R^{KL} + \mu\left(M^{IJKL}\epsilon_{IJKL} - 4\Lambda\right)$$

The Euler-Lagrange equation for M is

$$M^{-1}XM^{-1} = \mu\epsilon \qquad \text{where } X^{IJKL} := R^{IJ}R^{KL}$$

4-form with values $so(1,3)\times so(1,$

Index-free notation. Every object is a symmetric Lie algebra $\times$ Lie algebra valued matrix

Because matrices here do not commute, for some time nobody knew how to solve this equation

The solution appeared only recently Ermis Mitsou 2019

$$M^{-1} = \pm \sqrt{\mu}\,\sqrt{\epsilon}\,(\sqrt{\epsilon}\,X\,\sqrt{\epsilon})^{-1/2}\sqrt{\epsilon}$$

assumed that all matrices are invertible.

Remark: in Euclidean signature $\sqrt{\epsilon}$ does not exist as a real matrix, but this will not be a problem - will disappear in final action

The final step is to find μ from the constraint

$$\sqrt{\mu} = \frac{1}{4\Lambda}\,\mathrm{Tr}\sqrt{\sqrt{\epsilon}\,X\sqrt{\epsilon}} = \frac{1}{4\Lambda}\,\mathrm{Tr}\sqrt{\epsilon X}$$

using cyclicity of the trace.

Substituting the solution get the pure connection action in closed form

Mitsou '19

$$S[\omega] = \frac{1}{128\pi G\Lambda}\int\left(\mathrm{Tr}\sqrt{\epsilon X}\right)^2$$

Remarks:

- Its linearization precisely reproduces the second order pure connection action obtained in the Einstein - Cartan formalism. Its value on the constant curvature background also matches the expected $\frac{\Lambda}{8\pi G}\int e$ value. Passes all checks!

- This action is explicitly non-negative. Moreover, the Hessian on the maximally symmetric solution is non-positive. The bell-shape. In striking contrast with the properties of the EH action

Proof of $\mathrm{Tr}\sqrt{\sqrt{E}\,X\sqrt{E}} = \mathrm{Tr}\sqrt{EX}$

We prove more general statement

$$\mathrm{Tr}\sqrt{MM'} = \mathrm{Tr}\, M^{-1}\sqrt{MM'}\,M = \mathrm{Tr}\sqrt{M^{-1}MM'M} = \mathrm{Tr}\sqrt{M'M}$$

Because

$$\sqrt{A^{-1}BA} = A^{-1}\sqrt{B}\,A$$

square root commutes with similarity transformations

Modifications of GR from BF formalism

Coming back to BF formulation, the action with Lagrange multipliers

$$S[B,\omega,M,\mu] = \frac{1}{16\pi G}\int B_{IJ} R^{IJ}(\omega) - \frac{1}{2} M^{IJKL} B_{IJ} B_{KL}$$

$$+ \frac{\mu}{2}\left(f_{GR}(M) - 4\Lambda\right)$$

where $f_{GR}(M) = M^{IJKL}\epsilon_{IJKL}$

As discussed, can replace this f_{GR} by more general trace

$$f_{GR}(M) = M^{IJKL}\left(\alpha\epsilon_{IJKL} + \beta\,\delta_{IK}\,\delta_{LJ}\right)$$

This still remains GR

However, can consider other gauge invariant functions of M

e.g. $\mathrm{Tr}\,M^2$ or $\det \underline{M}$

This modifies the theory drastically, introducing new DOF – generally becomes bigravity

analysed by Smolin, Speziale and collaborators

It is interesting that there is a 2-parameter family of choices of f that still gives GR but is very different from the f_{GR} above.

Field redefinitions

Coming back to actions of BF type, we can obtain a very surprising formulation of GR that does not have Lagrange multiplier fields. The first step is to consider some field redefinitions that become possible in BF formalism

Let us form a new 2-form field $\tilde{B}^{IJ}$ given by a linear combination of the "old" B^{IJ} and $R^{IJ}(\omega)$

B^{IJ} – vector with values in $so(1,3)$

$\tilde{B}^{IJ}, R^{IJ}$

$$B = G\tilde{B} + HR$$

index-free notation, G, H are matrices, or linear maps acting on $so(1,3)$

G^{IJKL}, H^{IJKL} – matrices 6×6

The action in terms of $\tilde{B}$ will contain R^2 terms. Can choose G, H so that the R^2 terms generated are just the 2 topological terms

$$\int R^t R \qquad \text{and} \qquad \int R^t \epsilon R$$

transpose of a column vector

$R^{IJ}R_{IJ}$ $\qquad$ $R^{IJ}\epsilon_{IJ}{}^{KL}R_{KL}$

Thus, we require the R^2 terms to be of the form

$$\int R^t \Upsilon R \qquad \Upsilon = t_1 \mathbb{1} + \frac{1}{2} t_2 \epsilon$$

$$\mathbb{1}^{IJ}{}_{KL} = \delta^I_{[K}\delta^J_{L]}$$

We will also require the $\tilde{B}R$ term to preserve its form. This gives 2 equations

$$H^t - \frac{1}{2} H^t M H = \Upsilon$$

$$G^t - G^t M H = \mathbb{1}$$

After some algebra, these equations can be solved

$$G^t = T^{-1/2}\left(\mathbb{1} - 2T^{1/2} M T^{1/2}\right)^{-1/2} T^{1/2}$$

$$H = T^{1/2}\left(\mathbb{1} + T^{1/2} M T^{1/2}\right)^{-1} T^{1/2}$$

We denote the matrix arising in the $\tilde{B}\tilde{B}$ term by $\tilde{M}$.

One then finds
$$M = \tilde{M}\left(\mathbb{1} + 2T\tilde{M}\right)^{-1}$$

Even though $T^{1/2}$ was present in intermediate expressions, it disappears on final formulas that matter

Omitting the tildas, the action in terms of new fields is

$$S[B,A,M,\mu] = \frac{1}{16\pi G}\int B^t R - \frac{1}{2} B^t M B + R^t T R$$
$$+ \frac{\mu}{2}\left(\mathrm{Tr}\left(\epsilon M \left(\mathbb{1} + 2TM\right)^{-1}\right) - 4\Lambda\right)$$

When $T = 0$ gives the original BF type action.
Now have a 2-parameter family of its "versions".
They all describe the same theory, it is just how changed what the 2-form field is. The new B does not satisfy the metricity constraints anymore!

$$f_{GR} = \mathrm{Tr}\left(\epsilon \underline{M} \left(\mathbb{1} + 2TM\right)^{-1}\right)$$

$$T = t_1 \mathbb{1} + \frac{t_2}{2}\epsilon$$

More general than could be expected family of theories still describing GR

BF type formulation with a potential for the 2-form field

As we will now show, one can eliminate all Lagrange multiplier fields from the above action. The result is a pure BF-type action, with no other fields apart from B^{IJ} and ω^{IJ}

The field equation obtained by varying with respect to M

$$(\mathbb{1} + 2TM)\, X_B\, (\mathbb{1} + 2MT) = \mu \epsilon$$

where $X_B = B \wedge B$

Its solution can be written as

$$T + 2TMT = \pm \sqrt{\mu}\, \sqrt{\epsilon}\, \left(\sqrt{\epsilon}\, T^{-1} X_B T^{-1} \sqrt{\epsilon}\right)^{-1/2} \sqrt{\epsilon}$$

One then determines M, then μ. The final arising action is

$$S[B,\omega] = \frac{1}{16\pi G} \int B^t R + \frac{1}{4} B^t T^{-1} B - \frac{\left(\mathrm{Tr}\sqrt{\epsilon T^{-1} X_B T^{-1}}\right)^2}{4\,\mathrm{Tr}(T^{-1}\epsilon) - 32\Lambda}$$

"potential" for the B field

The first line is a topological theory. Only the last term makes this into a theory with propagating DOF

The fact that such a formulation of GR should be possible is far from obvious! No Lagrange multipliers left, but the action is non-polynomial

Second-order Lagrangian for the 2-form field

Varying the BF Lagrangian wrt the connection gives

$$d^{\omega} B^{IJ} = 0$$

The number of equations here matches the number of unknown connection components, so one expects to be able to solve for $\omega = \omega(B)$. Substituting this back to the action would give a second order formalism with $B^{IJ}_{\mu\nu}$ as the only field.

This exercise has only been carried out in the chiral case considered next