

Octonions, complex structures and Standard Model fermions

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Motivations: Physics

Based on 2209.05088

Elementary particles (from which we are all made) are described by fields (Spin(1,3) Lorentz group spinors) representations of

$$G_{\text{SM}} = \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)/\mathbb{Z}_6$$

Particles	SU(3)	SU(2)	U(1)
$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	$\overline{\mathbb{C}^3}$	\mathbb{C}^2	1/6
$\begin{matrix} \bar{u} \\ \bar{d} \end{matrix}$	\mathbb{C}^3	trivial	$\begin{matrix} -2/3 \\ 1/3 \end{matrix}$
$L = \begin{pmatrix} \nu \\ e \end{pmatrix}$	trivial	\mathbb{C}^2	-1/2
\bar{e}	trivial	trivial	1

Known since ‘73

Times 3 for three generations

This looks complicated, and naturally people searched for some principle explaining this pattern

SU(5) Unification

1) All particles fit into two irreducible representations of SU(5)

Georgi-Glashow '74

$$\mathbb{C}^5 \oplus \Lambda^3 \mathbb{C}^5$$

Split $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2 = (\bar{d}, L)$

$$\text{SU}(5) \ni \begin{pmatrix} e^{i\phi/3} g_3 & 0 \\ 0 & e^{-i\phi/2} g_2 \end{pmatrix}$$

Charge calculation:

$$\Lambda^3 \mathbb{C}^5 = \Lambda^3 \mathbb{C}^3 \oplus \Lambda^2 \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus \mathbb{C}^3 \otimes \Lambda^2 \mathbb{C}^2$$

$$3 * \frac{1}{3} = 1$$

\bar{e}

$$2 * \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$$

Q

$$\frac{1}{3} - 2 * \frac{1}{2} = -\frac{2}{3}$$

\bar{u}

There is clearly some truth here

Spin(10) Unification

2) All particles fit into a single irreducible representations of Spin(10)

Fritzsch-Minkowski '75

$$SU(5) \subset Spin(10)$$

$$\mathbb{C} \oplus \Lambda^3 \mathbb{C}^5 \oplus \Lambda^5 \mathbb{C}^5 = S_+$$

Weyl spinor representation of Spin(10)

This is particularly compelling, because all particles fit into a single irreducible representation

Also predicts that there is a new particle, completely neutral with respect to all interactions - right-handed neutrino

One needs to select some mechanism by which GUT gauge group is broken to the SM gauge group, and write down a concrete physics model for this, which is then to be phenomenologically tested.

This is where things go murky. Too many possibilities, too many models. SU(5) models have been ruled out experimentally. Too many parameters in the Spin(10) model. Some additional principle is needed.

This talk: New characterisation of $G_{\text{SM}} \subset Spin(10)$

Complex structures

1) Let $J : \mathbb{R}^{10} \rightarrow \mathbb{R}^{10}$, $J^2 = -1$ be an orthogonal complex structure on \mathbb{R}^{10}

$$G_J := \{g \in \text{SO}(10) : gJ = Jg\} \quad \text{We have} \quad G_J = \text{U}(5)$$

The space splits into two eigenspaces of J $\mathbb{R}^{10} = E_J \oplus \overline{E_J}$ $E_J = \{v \in \mathbb{R}_{\mathbb{C}}^{10} : Jv = +iv\}$ $E_J = \mathbb{C}^5$

2) Let J_1, J_2 be two commuting complex structures $J_1 J_2 = J_2 J_1$

Then $K = J_1 J_2$ is an orthogonal product structure $K^2 = 1$

We obtain a further split $E_{J_1} = E_{J_1}^+ \oplus E_{J_1}^-$ where $E_{J_1}^- \subset E_{J_2}$

± 1 eigenspaces of K

2) For \mathbb{R}^{10} there are just two options at this stage

$$\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2 \quad (*)$$

$$\mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C}^1 \quad (**)$$

(Simple) proposition: Subgroup of $\text{SO}(10)$ that commutes with J_1, J_2 in case (*) is $\text{U}(3) \times \text{U}(2)$
in case (**) is $\text{U}(4) \times \text{U}(1)$

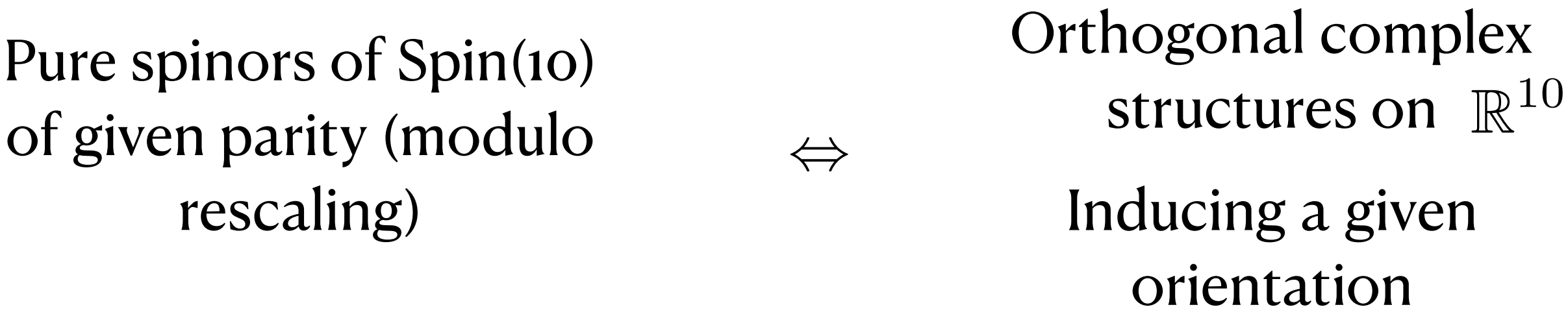
To proceed further we need pure spinors

Pure spinors

These are in the Spin(10) spinor orbits of smallest dimension (largest stabiliser)

Proposition: A Weyl spinor of Spin(10) is pure if the stabiliser is $SU(5)$

Proposition:



$S_+ \sim \mathbb{C}^{16}$	Pure spinor satisfies 5 quadratic constraints	$\dim(\text{Spin}(10)/SU(5)) = 45 - 24 = 21$
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Theorem: Let $\psi_1, \psi_2 \in S_+$ Be two pure spinors that are orthogonal $\langle \hat{\psi}_1, \psi_2 \rangle = 0$ $\hat{\cdot}: S_+ \rightarrow S_-$
And such that $\psi_1 + \psi_2$ is also a pure spinor anti-linear map

Then the subgroup of Spin(10) that stabilises ψ_1 and projectively stabilises ψ_2 is G_{SM}

Octonionic model of Spin(10)

The Weyl spinor irreducible representation of Spin(10) is \mathbb{C}^{16}

There is an octonionic model in which this is identified with $\mathbb{O}_{\mathbb{C}}^2$

Concretely, we have the following Clifford generators, viewed as matrices acting on $\mathbb{O}_{\mathbb{C}}^4$

$$\Gamma_x = \begin{pmatrix} 0 & 0 & 0 & L_{\bar{x}} \\ 0 & 0 & L_x & 0 \\ 0 & L_{\bar{x}} & 0 & 0 \\ L_x & 0 & 0 & 0 \end{pmatrix} \quad \Gamma_9 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \Gamma_{10} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$x \in \mathbb{O}$ L_x - the operator of left multiplication by an octonion x

A simpler description is available at the level of the Lie algebra $\mathfrak{spin}(10)$

$$\mathfrak{spin}(10) \ni \begin{pmatrix} A + ia & -L_{\bar{x}} + iL_{\bar{y}} \\ L_x + iL_y & A' - ia \end{pmatrix}$$

$$A, A' \in Spin^{\pm}(8), \quad x, y \in \mathbb{O}, \quad a \in \mathbb{R}$$

This is the restriction to the space
of Weyl spinors of one chirality

$$S_+ \sim \mathbb{O}_{\mathbb{C}}^2$$

Pure spinors of Spin(10)

The action of Spin(10) on spinors of one chirality is that on $\mathbb{O}_{\mathbb{C}}^2$

two-component column with complexified octonion entries $\psi = \begin{pmatrix} \alpha_1 + i\alpha_2 \\ \beta_1 + i\beta_2 \end{pmatrix}$

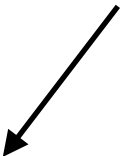
There is an invariant Hermitian pairing on $S_+ \sim \mathbb{O}_{\mathbb{C}}^2$

$$\langle \hat{\psi}, \psi \rangle = |\alpha_1|^2 + |\alpha_2|^2 + |\beta_1|^2 + |\beta_2|^2$$

Proposition:

A Weyl spinor $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is pure iff $(\alpha, \alpha) = (\beta, \beta) = 0$ $\alpha \cdot \bar{\beta} = 0$

octonionic multiplication



A pair of null complexified octonions whose product is zero

the bar denotes octonionic conjugation not touching i $\bar{\beta} = \bar{\beta}_1 + i\bar{\beta}_2$

Description of $\text{Spin}(10) \rightarrow G_{\text{SM}}$

Let us take $\psi_1 = \begin{pmatrix} 1 + i\mathbf{u} \\ 0 \end{pmatrix}$ $\psi_2 = \begin{pmatrix} 0 \\ 1 + i\mathbf{u} \end{pmatrix}$ $\mathbf{u} \in S^6$ - unit imaginary octonion

$$\psi_1 + \psi_2 = \begin{pmatrix} 1 + i\mathbf{u} \\ 1 + i\mathbf{u} \end{pmatrix} \quad (1 + i\mathbf{u}) \cdot (1 - i\mathbf{u}) = 0 \quad \text{So, still a pure spinor}$$

The stabiliser of ψ_1 and projective stabiliser of ψ_2 is G_{SM}

Conclusion: in the octonionic model of $\text{Spin}(10)$, the choice of G_{SM}
is parametrised by a choice of a unit imaginary octonion

Thank you!