Octonions, complex structures and Standard Model fermions

Kirill Krasnov (Nottingham)

Motivations: Physics

Based on 2209.05088

Elementary particles (from which we are all made) are described by fields (Spin(1,3) Lorentz group spinors) representations of

$$G_{\mathrm{SM}} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)/\mathbb{Z}_6$$

Particles	SU(3)	SU(2)	U(1)
$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	$\overline{\mathbb{C}^3}$	$\mathbb{C}^2$	$1/6$
$\bar{u}$ $\bar{d}$	$\mathbb{C}^3$	trivial	$-2/3$ $1/3$
$L = \begin{pmatrix} \nu \\ e \end{pmatrix}$	trivial	$\mathbb{C}^2$	$-1/2$
$\bar{e}$	trivial	trivial	1

Known since '73

Times 3 for three generations

This looks complicated, and naturally people searched for some principle explaining this pattern

SU(5) Unification

1) All particles fit into two irreducible representations of SU(5) — Georgi-Glashow '74

$$\mathbb{C}^5 \oplus \Lambda^3 \mathbb{C}^5$$

Split $\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2 = (\bar{d}, L)$

$$\mathrm{SU}(5) \ni \begin{pmatrix} e^{i\phi/3} g_3 & 0 \\ 0 & e^{-i\phi/2} g_2 \end{pmatrix}$$

$$\Lambda^3 \mathbb{C}^5 = \Lambda^3 \mathbb{C}^3 \oplus \Lambda^2 \mathbb{C}^3 \otimes \mathbb{C}^2 \oplus \mathbb{C}^3 \otimes \Lambda^2 \mathbb{C}^2$$

Charge calculation:

$$3 * \frac{1}{3} = 1 \qquad 2 * \frac{1}{3} - \frac{1}{2} = \frac{1}{6} \qquad \frac{1}{3} - 2 * \frac{1}{2} = -\frac{2}{3}$$

$$\bar{e} \qquad Q \qquad \bar{u}$$

There is clearly some truth here

Spin(10) Unification

2) All particles fit into a single irreducible representations of Spin(10) Fritzsch-Minkowski '75

$\mathrm{SU}(5) \subset \mathrm{Spin}(10)$ $\mathbb{C} \oplus \Lambda^3\mathbb{C}^5 \oplus \Lambda^5\mathbb{C}^5 = S_+$ Weyl spinor representation of Spin(10)

This is particularly compelling, because all particles fit into a single irreducible representation

Also predicts that there is a new particle, completely neutral with respect to all interactions - right-handed neutrino

One needs to select some mechanism by which GUT gauge group is broken to the SM gauge group, and write down a concrete physics model for this, which is then to be phenomenologically tested.

This is where things go murky. Too many possibilities, too many models. SU(5) models have been ruled out experimentally. Too many parameters in the Spin(10) model. Some additional principle is needed.

This talk: New characterisation of $G_{\mathrm{SM}} \subset \mathrm{Spin}(10)$

Complex structures

1) Let $J : \mathbb{R}^{10} \to \mathbb{R}^{10}, \quad J^2 = -1$ be an orthogonal complex structure on $\mathbb{R}^{10}$

$G_J := \{g \in \mathrm{SO}(10) : \quad gJ = Jg\}$ We have $G_J = \mathrm{U}(5)$

The space splits into two eigenspaces of J $\mathbb{R}^{10} = E_J \oplus \overline{E_J}$ $E_J = \{v \in \mathbb{R}^{10}_{\mathbb{C}} : Jv = +iv\}$ $E_J = \mathbb{C}^5$

2) Let J_1, J_2 be two commuting complex structures $J_1 J_2 = J_2 J_1$

Then $K = J_1 J_2$ is an orthogonal product structure $K^2 = 1$

We obtain a further split $E_{J_1} = E^+_{J_1} \oplus E^-_{J_1}$ where $E^-_{J_1} \subset E_{J_2}$

± 1 eigenspaces of K

2) For $\mathbb{R}^{10}$ there are just two options at this stage

$$\mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2 \qquad (*)$$

$$\mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C}^1 \qquad (**)$$

<u>(Simple) proposition:</u> Subgroup of SO(10) that commutes with J_1, J_2 in case (*) is $\mathrm{U}(3) \times \mathrm{U}(2)$
in case (**) is $\mathrm{U}(4) \times \mathrm{U}(1)$

To proceed further we need pure spinors

Pure spinors

These are in the Spin(10) spinor orbits of smallest dimension (largest stabiliser)

Proposition: A Weyl spinor of Spin(10) is pure if the stabiliser is $\mathrm{SU}(5)$

Proposition:

Pure spinors of Spin(10) of given parity (modulo rescaling)	$\Leftrightarrow$	Orthogonal complex structures on $\mathbb{R}^{10}$ Inducing a given orientation

$S_+ \sim \mathbb{C}^{16}$ Pure spinor satisfies 5 quadratic constraints $\dim(\mathrm{Spin}(10)/\mathrm{SU}(5)) = 45 - 24 = 21$

Theorem: Let $\psi_1, \psi_2 \in S_+$ Be two pure spinors that are orthogonal $\langle \hat{\psi}_1, \psi_2 \rangle = 0$

$\hat{}: S_+ \to S_-$ anti-linear map

And such that $\psi_1 + \psi_2$ is also a pure spinor

Then the subgroup of Spin(10) that stabilises ψ_1 and projectively stabilises ψ_2 is G_{SM}

Octonionic model of Spin(10)

The Weyl spinor irreducible representation of Spin(10) is $\mathbb{C}^{16}$

There is an octonionic model in which this is identified with $\mathbb{O}_{\mathbb{C}}^2$

Concretely, we have the following Clifford generators, viewed as matrices acting on $\mathbb{O}_{\mathbb{C}}^4$

$$\Gamma_x = \begin{pmatrix} 0 & 0 & 0 & L_{\bar{x}} \\ 0 & 0 & L_x & 0 \\ 0 & L_{\bar{x}} & 0 & 0 \\ L_x & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma_9 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \Gamma_{10} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$x \in \mathbb{O}$ $\quad L_x$ - the operator of left multiplication by an octonion x

A simpler description is available at the level of the Lie algebra $\mathfrak{spin}(10)$

$$\mathfrak{spin}(10) \ni \begin{pmatrix} A + ia & -L_{\bar{x}} + iL_{\bar{y}} \\ L_x + iL_y & A' - ia \end{pmatrix}$$

$$A, A' \in Spin^{\pm}(8), \quad x, y \in \mathbb{O}, \quad a \in \mathbb{R}$$

This is the restriction to the space of Weyl spinors of one chirality

$$S_+ \sim \mathbb{O}_{\mathbb{C}}^2$$

Pure spinors of Spin(10)

The action of Spin(10) on spinors of one chirality is that on $\mathbb{O}_{\mathbb{C}}^2$

two-component column with complexified octonion entries $$\psi = \begin{pmatrix} \alpha_1 + i\alpha_2 \\ \beta_1 + i\beta_2 \end{pmatrix}$$

There is an invariant Hermitian pairing on $S_+ \sim \mathbb{O}_{\mathbb{C}}^2$ $$\langle \hat{\psi}, \psi \rangle = |\alpha_1|^2 + |\alpha_2|^2 + |\beta_1|^2 + |\beta_2|^2$$

<u>Proposition:</u>

octonionic multiplication ↙

A Weyl spinor $\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is pure iff $(\alpha, \alpha) = (\beta, \beta) = 0$ $\alpha \cdot \bar{\beta} = 0$

A pair of null complexified octonions whose product is zero

the bar denotes octonionic conjugation not touching i $\bar{\beta} = \bar{\beta}_1 + i\bar{\beta}_2$

Description of $\mathrm{Spin}(10) \to G_{\mathrm{SM}}$

Let us take $\psi_1 = \begin{pmatrix} 1 + i\mathbf{u} \\ 0 \end{pmatrix}$ $\psi_2 = \begin{pmatrix} 0 \\ 1 + i\mathbf{u} \end{pmatrix}$ $\mathbf{u} \in S^6$ - unit imaginary octonion

$\psi_1 + \psi_2 = \begin{pmatrix} 1 + i\mathbf{u} \\ 1 + i\mathbf{u} \end{pmatrix}$ $(1 + i\mathbf{u}) \cdot (1 - i\mathbf{u}) = 0$ So, still a pure spinor

The stabiliser of ψ_1 and projective stabiliser of ψ_2 is G_{SM}

Conclusion: in the octonionic model of Spin(10), the choice of G_{SM} is parametrised by a choice of a unit imaginary octonion

Thank you!