3D/4D Gravity as the Dimensional Reduction of a theory of 3-forms in 6D/7D

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LQG community is well aware of the importance of BF theory

- 3D gravity is BF theory

this is why we know how to quantise it - Ponzano-Regge, Turaev-Viro models

- 4D gravity is constrained BF theory

Chiral Plebanski formulation → Ashtekar Hamiltonian formulation

Non-chiral full Lorentz group based Plebanski → Spin Foam Models

This talk is about an unusual perspective on 3D/4D BF theory (and gravity)

They are seen to emerge via dimensional reduction from a theory of a very different nature

Dimensional reduction on S^3 of a theory 3-forms in 6D/7D

These ideas are new

related to ideas in the subject of topological strings ~2002

But rely on beautiful geometry > 100 years old

I don't yet know if they are useful for physics

But they are natural and beautiful

So may be telling us something important about gravity

Plan

- Review of (chiral) Plebanski formulation

  and its generalisations

- Differential forms in Riemannian geometry
- G2 structures, dimensional reduction 7D to 4D
- From 3-forms in 6D to 3D gravity
- Summary

Chiral Plebanski formulation of 4D gravity

Plebanski '77, CDJM '91

$$S[A, B, \Psi] = M_p^2 \int_M B^i \wedge F^i - \frac{1}{2}\left(\frac{\Lambda}{3}\delta^{ij} + \Psi^{ij}\right) B^i \wedge B^j$$

$i = 1, 2, 3$ SO(3) indices

Euler-Lagrange equations

$$d_A B^i = 0$$

$$F^i = \left(\frac{\Lambda}{3}\delta^{ij} + \Psi^{ij}\right) B^i$$

$$B^i \wedge B^j \sim \delta^{ij}$$

On-shell becomes the self-dual part of Weyl curvature

Urbantke metric is Einstein, with cosmological constant Λ

A is the self-dual part of the spin connection compatible with the Urbantke metric

Urbantke metric

$$g_B(\xi, \eta)\mathrm{vol}_B = -\frac{1}{6}\epsilon^{ijk} i_\xi B^i \wedge i_\eta B^j \wedge B^k$$

Formulation of complex GR, need reality conditions depending on the desired signature

metric that makes B's self-dual

Remarks:

- Hamiltonian formulation of Plebanski gives Ashtekar's new Hamiltonian formulation of GR

- Convenient to make the action dimensionless

Dimensionless metric measures dimensionful distances

$$M_p^2 B^i_{old} = B^i_{new}$$

Dimensionful metric measures dimensionless distances

(in units of Planck length)

Also

$$\Lambda_{old}/M_p^2 = \Lambda_{new} \sim 10^{-120}$$

$$\Psi_{old}/M_p^2 = \Psi_{new} \ll 1$$

dimensionless cosmological constant and Weyl curvature

very small in classical situations where can trust GR

$$S[A, B, \Psi] = \int_M B^i \wedge F^i - \frac{1}{2}\left(\frac{\Lambda}{3}\delta^{ij} + \Psi^{ij}\right) B^i \wedge B^j$$

there is no scale in classical GR
not coupled to any matter

$$[B] = 2, \quad [A] = 1, \quad [\Lambda] = [\Psi] = 0$$

Generalisations of Plebanski

CDJ '91 in the context of "pure connection" formulation of GR

realised that there are <u>"neighbours of GR"</u>

$$S_{\rm CDJ}[A,\eta] = \int \eta \left({\rm Tr} M^2 - \frac{1}{2} ({\rm Tr} M)^2 \right)$$

change this coefficient

later studied by Capovilla, and extensively by Bengtsson

come in an infinite -parameter family

"cosmological constants" of Bengtsson

were mostly studied in the Hamiltonian formulation

geometrical interpretation was obscure

In 2006 I rediscovered these theories and studied since then

they are all 4D gravity theories describing just two propagating polarisations of the graviton

BF formulation of "deformations of GR"

$$S[A,B,M,\mu] = \int B^i F^i - \frac{1}{2} M^{ij} B^i B^j - \mu(f(M) - \lambda)$$

integrating out M gives V(B)
gravity as "BF plus potential"

different choices of f(M) give different theories

$$f_{\mathrm{GR}}(M) = \mathrm{Tr}(M)$$

← General Relativity in Plebanski formulation

$$f_{\mathrm{SDGR}}(M) = \mathrm{Tr}(M^{-1})$$

← Self-Dual Gravity

other interesting choice

$$f_{\det} = \det(M)$$

can "integrate out" B,M to get the "pure connection formulation"

useful alternative is to just "integrate out" B

These theories no longer have a "preferred" metric - only the conformal class is fixed

GR as the low energy limit of deformations of GR

Deformations of GR are UV modifications of GR

Any of these theories will at sufficiently low energies be indistinguishable from GR

Parametrise

$$M^{ij} = \frac{\Lambda}{3}\delta^{ij} + \Psi^{ij}$$

Solve $f(M) = \lambda$ for $\Lambda = \Lambda(\Psi)$

For $\Psi \ll 1$ the solution will be of the form

$$\Lambda(\Psi) = \Lambda_0 + \alpha \mathrm{Tr}(\Psi^2) + \ldots$$

can neglect $\mathrm{Tr}(\Psi^2)$ compared to Ψ

E.g.

$$\det(M) = (\lambda/3)^3 \qquad \Lambda(\Psi) = \lambda + \frac{3}{2\lambda}\mathrm{Tr}(\Psi^2) + \ldots$$

indistinguishable from GR for $\Psi \ll \lambda$

but λ is also the cosmological constant!

in a generic theory with only one scale corrections to GR appear at the same scale as the effective cosmological constant scale

Remarks:

- It was always strange to have an infinite-parameter family of 4D gravity theories with similar properties
- The observation that GR is the low energy limit of any one of them explains specialness of GR
- But then there remains the question of what is the "right" theory from this class, if any

The new development:

Theories of this type arise by dimensional reduction from a certain theory of 3-forms in 7 dimensions

What comes out is a theory with specific f(M)

Moreover, the size of the extra dimension provides another scale

Frame field

Eli Cartan method of moving frames

introduced differential forms into geometry

$$e^I, \qquad I = 1, \ldots, n$$

$$n = \dim(M)$$

collection of 1-forms that are declared orthonormal and this defines the metric

$$ds^2 = \eta_{IJ} e^I \otimes e^J$$

frame (generalised tetrad, vielbein) as the square root of the metric

$$\mathrm{GL}(D)/\mathrm{SO}(D)$$

metric as coset

Cartan's structure equations

$$de^I + w^I{}_J \wedge e^J = 0$$

torsion-free condition determines the spin connection

$$F^{IJ} := dw^{IJ} + w^I{}_K \wedge w^{KJ}$$

curvature of the spin connection is Riemann curvature

Cartan connection

Both frame and the spin connection are one-forms

Cartan himself realised that there is a useful construction that puts the two together - Cartan connection

Cartan geometry is a generalisation of Riemannian and Kleinian geometry (based on homogeneous group spaces G/H)

book by Sharpe

CS description of 3D gravity

$$\Lambda^2\mathbb{R}^3 \sim \mathbb{R}^3$$

$$f^i = dw^i + \frac{1}{2}\epsilon^{ijk} w^j \wedge w^k$$

$$i = 1, 2, 3$$

$$w^{ij} = \epsilon^{ikj} w_k$$

$$f^{ij} = \epsilon^{ikj} f_k$$

$$F^i = 0 \quad \Leftrightarrow$$

$$\Lambda < 0$$

$$A^i := w^i + \sqrt{-1}\, e^i$$

$$de^i + \epsilon^{ijk} w^j \wedge e^k = 0$$

$$f^i = \frac{1}{2}\epsilon^{ijk} e^j \wedge e^k$$

real and imaginary parts of the zero curvature condition

$\mathrm{SL}(2,\mathbb{C})$ connection

Einstein equations in 3D

Flat connections are extrema of Chern-Simons functional

$$S_{\mathrm{CS}}[A] = \int A^i \wedge dA^i + \frac{1}{3}\epsilon^{ijk} A^i \wedge A^j \wedge A^k$$

We understand quantum gravity in 3D mostly due to formulation in terms of differential forms!

Similar trick is possible in 4D - Mac Dowell-Mansouri formulation

but does not seem to be so useful as in 3D

3-Forms

New idea - there is a different way of putting the spin connection and the frame together

Start by forming Lorentz group Lie algebra valued 2-forms

$$B^{IJ} := e^I \wedge e^J$$

of course not every Lie algebra valued 2-form is of this type, will need to deal with this

Introduce Maurer-Cartan 1-forms on the Lorentz group

$$m^{IJ} := (g^{-1}dg)^{IJ} \qquad g \in \mathrm{SO}(n)$$

Then w^{IJ}, B^{IJ} are different components of a 3-form in $M \times \mathrm{SO}(n)$

$$[m, m]^{IJ} \wedge w_{IJ}$$

$$m^{IJ} \wedge B_{IJ}$$

Kaluza-Klein spirit - gauge group arising from geometry of “internal” space

but there are other components as well

I am going to apply this idea to 4D gravity

There is no interesting theory of 3-forms in 4+6=10D

But can use self-dual 2-forms instead

$$\mathrm{SO}(4) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)$$

$$\Sigma^i = \left(e^I \wedge e^J\right)_{sd}$$
$$A^i = \left(w^{IJ}\right)_{sd}$$

fields of chiral Plebanski formulation of GR

3-forms are most interesting in 4+3=7D!

Geometry of 3-forms in 7D

$$\mathrm{GL}(7)/\mathrm{G}_2$$

stable forms as coset

Stable forms define a metric

$$g_C(\xi,\eta)\mathrm{vol}_C = -\frac{1}{6} i_\xi C \wedge i_\eta C \wedge C$$

G_2 first exceptional Lie algebra, rank 2

C as the "cube root" of the metric

3-form in 7D = G2 structure

Can also write an explicit formula for the volume

$$\mathrm{vol}_C \sim (\tilde\epsilon^{\mu_1\ldots\mu_7}\tilde\epsilon^{\nu_1\ldots\nu_7}\tilde\epsilon^{\rho_1\ldots\rho_7} C_{\mu_1\nu_1\rho_1}\ldots C_{\mu_7\nu_7\rho_7})^{1/3}$$

When over reals, metric is either Riemannian or signature (4,3)

Canonical expression

$$C = e^{123} + e^1\Sigma^1 + e^2\Sigma^2 + e^3\Sigma^3$$

$$\Sigma^1 = e^{45} - e^{67}$$
$$\Sigma^2 = e^{46} - e^{75}$$
$$\Sigma^3 = e^{47} - e^{56}$$

ASD 2-forms

Intimate relation with 7D spinors

3-form = metric + unit spinor

$$C_{\mu\nu\rho} = \psi^T \gamma_\mu\gamma_\nu\gamma_\rho\psi \qquad \psi^T\psi = 1$$

Theory of 3-forms in 7D

$$S[C] = \int_M C \wedge dC + 6\lambda \mathrm{vol}_C$$

can always set $\lambda = 1$
by rescaling 3-form

Euler-Lagrange equations $dC = \lambda {}^*C$

cone over such metrics can be shown to have holonomy Spin(7)

Can be shown to imply that the metric g_C is Einstein
(with non-zero scalar curvature)

But they are more restrictive

- also imply vanishing of some Weyl components

This is a theory with 3 propagating DOF

Unlike 7D GR which has $7*8/2 - 2*7 = 14$ DOF

Claim: the dimensional reduction of this theory on S^3 is 4D gravity theory coupled to a scalar field

canonical example:
Hopf fibration
$S^7 \to S^4$

Dimensional reduction to 4D

Assume 7D manifold to have SU(2) act on it without fixed pts

Has the structure of the principal SU(2) bundle over 4D base

Assume C is invariant 3-form

$$C = -2\mathrm{Tr}\left(\frac{\phi^3}{3}W^3 + \phi W B\right) + c$$

scalar field on the base

3-form on the base

connection 1-form in the total space

Lie algebra valued 2-form on the base

Easiest to start with

$$\phi = const, \quad c = 0$$

Then

$$g_C\Big|_{base} = \text{Urbantke metric}$$

Urbantke metric has 7-dimensional origin!

Topological theory in 7D

$$S[C] = \int C \wedge dC$$ for $\lambda = 0$ we get a topological theory

$$dC = 0, \qquad C \bmod dB$$

analog of Abelian CS in 3D

Dimensional reduction on S^3

$$C = -2\mathrm{Tr}\left(\frac{\phi^3}{3}W^3 + \phi W B\right) + c$$

$$dC = -2\mathrm{Tr}\Big(\phi^2 d\phi W^3 + (\phi^3 F + \phi B)W^2 + d_A(\phi B)W + \phi F B\Big) + dc$$

$$\frac{1}{2}\int CdC = \int_{\mathrm{SU}(2)} -\frac{2}{3}\mathrm{Tr}(\mathbf{m}^3) \times \int -2\mathrm{Tr}(\phi^4 \mathbf{BF} + (\phi^2/2)\mathbf{BB}) + \phi^3 dc$$

topological BF theory (with non-zero cosmological constant)

coupled to topological 3-form field

Dimensional reduction of the full theory

The dimensional reduction of $\lambda \neq 0$ 7D theory gives

$$f(M) = \frac{1}{\phi^3}\det(\mathbb{I} + \phi^2 M)$$

for small $M\big|_{tf}$ gives GR!

with constraint f(M)=1

Then expanding $$\frac{\Lambda(\Psi)}{3} = \frac{\phi - 1}{\phi^2} + \frac{\phi}{2}\mathrm{Tr}(\Psi^2) + \ldots$$

The S^3 dim reduction of theory of 3-forms in 7D is 4D gravity theory (in general coupled to a scalar field) indistinguishable from GR for small Weyl curvatures $\Psi \ll 1/\phi$

For $\phi \approx 1$ get small cosmological constant and Planck scale modifications!

size of the internal manifold gives an extra scale!

Geometry of 3-forms in 6D (after Hitchin)

A stable (=generic) 3-form $\Omega \in \Lambda^3(M)$ in 6D of the right sign defines an almost complex structure

$$\mathrm{GL}(6)/\mathrm{SL}(3)\times\mathrm{SL}(3)$$

stable forms as coset

Defines an endomorphism $K_\Omega : TM \to TM$

$$i_\xi(K_\Omega(\alpha)) := \alpha\wedge i_\xi\Omega\wedge\Omega/v$$

Here defined using its action on one-forms rather than vector fields

Here v is an arbitrary volume form on M

A computation shows that its square is multiple of identity

$$K_\Omega(\alpha)^2 = \lambda(\Omega)\,\mathbb{I}$$

$\lambda(\Omega)\neq 0 \Leftrightarrow$ form is stable

For $\lambda(\Omega)<0$ get an almost complex structure

$$J_\Omega := \frac{1}{\sqrt{-\lambda(\Omega)}}K_\Omega \qquad J_\Omega^2 = -\mathbb{I}$$

Does not depend on v chosen in definition

For the negative sign the canonical expression is

$$\Omega = 2\mathrm{Re}\left(\alpha^1 \wedge \alpha^2 \wedge \alpha^3\right) \qquad J_\Omega(\alpha^i) = \frac{1}{\mathrm{i}}\alpha^i$$

where the complex 1-forms are unique modulo SL(3,C)

Can apply J to all 3 slots of Ω

$$\hat{\Omega} := -J_\Omega(\Omega) \qquad \hat{\Omega} = 2\mathrm{Im}\left(\alpha^1 \wedge \alpha^2 \wedge \alpha^3\right)$$

Theorem (Hitchin): the almost complex structure J_Ω is integrable iff

$$d\Omega = 0$$
$$d\hat{\Omega} = 0$$

Theorem (Hitchin): integrable almost complex structures are critical points of

$$S[\Omega] = \frac{1}{2}\int_M \Omega \wedge \hat{\Omega}$$

when variation is taken within a cohomology class

Relation to 3D gravity with $\Lambda < 0$

starting with $A^i := w^i + \sqrt{-1}\, e^i$

form a lift to the total space of the principal SU(2) bundle

$$\mathbf{m} := g^{-1} dg, \qquad g \in \mathrm{SU}(2)$$

$$\mathbf{w} := w^i \tau^i, \qquad \mathbf{e} = e^i \tau^i \qquad \tau^i := (-i/2)\sigma^i$$

connection 1-form in the total space of the bundle

$$W = \mathbf{m} + g^{-1}\mathbf{w} g, \qquad E = g^{-1}\mathbf{e} g$$

Chern-Simons (Cartan) connection $A := W + \sqrt{-1}\, E$

Define

$$\Omega = \mathrm{Re}\left(-\frac{1}{3}\mathrm{Tr}(A^3)\right) = \mathrm{Tr}\left(-\frac{1}{3}W^3 + W E^2\right)$$

Theorem: J_Ω is integrable iff

$$d_{\mathbf{w}}\mathbf{e} = 0 \quad \text{and} \quad \mathbf{f}(\mathbf{w}) = \mathbf{e} \wedge \mathbf{e}$$

i.e. the 3D metric has constant negative curvature

canonical example

$\mathrm{SL}(2, \mathbb{C}) \to H^3$

Theory of forms in 6D

Consider a theory of 2- and 3-forms in 6D

$$S[B,C] = \int_M B \wedge dC + \mathrm{vol}_C$$

topological theory

$$C \in \Lambda^3(M) \qquad B \in \Lambda^2(M)$$

$$\mathrm{vol}_C := \frac{1}{2} C \wedge \hat{C}$$

Euler-Lagrange equations

$$dC = 0$$

$\Rightarrow J_C$ is integrable

$$dB = \hat{C} \qquad \Rightarrow d\hat{C} = 0$$

Dimensional reduction to 3D

$$C = -2\mathrm{Tr}\left(\frac{\phi^3}{3}W^3 + \phi W B\right) + c$$

For simplicity we set

$$\phi = 1, \qquad c = 0$$

In 3D $\quad B = \pm E \wedge E$

So, consider

$$C = -2\mathrm{Tr}\left(\frac{1}{3}W^3 - W E^2\right) = \mathrm{Re}\left(-\frac{2}{3}\mathrm{Tr}\left(A^3\right)\right)$$

$$A = W + iE$$

So, 6D field equations imply 3D Einstein equations

in general get 3D gravity coupled to topological 2-form field

Summary

- Can put Lie algebra valued 1- and 2-forms together into a 3-form in a bigger space - variant of KK idea

  one necessarily gets more than gravity - scalar in 4D

- The origin of 4D SU(2) BF theory - topological theory of 3-forms in 7D

- 4D gravity (coupled to a scalar field) as the dimensional reduction of a theory of 3-forms in 7D

  Urbantke metric has 7D origin!

- 3D gravity arises as the dimensional reduction of a topological theory of 2- and 3-forms in 6D

  would be interesting to quantise this theory

Problems of this approach

(if to apply it to real world gravity)

- No known mechanism to drive $\phi \to 1$ dynamically

  There are 2 solutions with constant ϕ

  $\phi = 2$ Round sphere S^7

  $\phi = 6/5$ Squashed sphere S^7

  $S^7 \to S^4$ Hopf bundle

  Both give very large cosmological constant

  (order one in Planck units)

  Exactly the same problem exists in Kaluza-Klein supergravity

  $AdS_4 \times S^7$ Freund-Rubin solution

- Also (less so) coupling to matter and "reality conditions"

  but matter arises by generalising form content and/or number of dimensions

  reality conditions are either irrelevant or clear in many situations

Thank You!