Formulations of General Relativity

Gravity, Spinors and Differential Forms

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To the memory of two Peter Ivanovichs in my life:
Peter Ivanovich Fomin, who got me interested in gravity,
and Peter Ivanovich Holod, who formed my taste for mathematics.

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Preface

Give thanks to God, who made necessary things simple, and complicated things unnecessary.

Gregory Skovoroda, *Ukrainian Thinker, 1722-1794*

There is always another way to say the same thing that doesn't look at all like the way you said it before. I don't know what the reason for this is. I think it is somehow a representation of the simplicity of nature? Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing.

Richard Feynman, *Nobel Lecture, 1965*

Theories of the known, which are described by different physical ideas may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in ones attempt to understand what is not yet understood. I, therefore, think that a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory available to him.

Richard Feynman, *Nobel Lecture, 1965*

Formulations of General Relativity. Facing this title the prospective reader should be thinking: What is there to formulate General Relativity? GR can be formulated in one sentence: GR action functional is the integral of the scalar curvature over the manifold. Everything else that is there to say about GR is the consequence of the

Euler-Lagrange equations one obtains by extremising this action, together with the action for matter fields. How can there be a book about ”formulations”? And why plural? Is not there just the usual Einstein-Hilbert formulation as stated above?

A more sophisticated reader will know that there are several equivalent formulations of General Relativity. There is the usual metric formulation, and then there is an equivalent formulation in terms of tetrads. But this is all well-known. General Relativity is about physical consequences of the dynamical postulate that fixes the theory. There may be several equivalent ways to define the dynamics. But this does not change the physics. So, one formulation is sufficient to unravel all the physics predicted by the theory. The usual metric formulation is by far the most studied and best understood. Why bother about developing any other equivalent formulation? And then why write a book about such unnecessary alternatives?

This is when the above two quotes from the Richard Feynman Nobel lecture become relevant. The first is about an empirical observation that theories that are relevant for describing the world around us tend to admit many different equivalent, but not obviously so, reformulations. The example Feynman has in mind is classical electrodynamics, not gravity. Feynman also notices that there is a deep link between the ”simplicity” of a theory, and the availability of many different not manifestly equivalent descriptions. He goes further and proposes this as the criterion of simplicity. This suggests that one can never fully appreciate the simplicity and beauty of General Relativity without absorbing all the different available formulations of this theory.

The second quote is a different, but not unrelated thought. There may be equivalent formulations of a theory, all leading to the same physical predictions. But such reformulations may be inequivalent if one decides to generalise. The example of most relevance for Feynman is the Hamiltonian and Lagrangian description of classical mechanics. The quantum generalisation of the Hamiltonian description leads to the usual operator formalism for quantum theory. The generalisation of the Lagrangian description leads to path integrals, which is arguably one of Feynman’s main contributions to physics. These two equivalent formulations of classical mechanics are certainly not equivalent in terms of the new structures that can be generated from them. The same may well apply to gravity. We do not yet know which of the many available formulations of gravity will lead to the next big step in the quest for understanding the world around us.

So, the purpose of this book is to describe all the ”equivalent” formulations of General Relativity that are known to the author, and that also put the geometry of differential forms and fibre bundles at the forefront of the description of gravity. What is meant by a ”formulation” here is a Lagrangian description, in which the dynamical equations are obtained by extremising the corresponding action. This gives the most economic way of defining the theory.

Some of these equivalent formulations will likely be known to many readers. In particular, this is the already mentioned formulation in terms of tetrads. If this was the complete list, there would be no good reason to write this book.

What is known much less, and what really motivated this author to embark on the present project, is that there are some special features of General Relativity in four spacetime dimensions. These special features are related to coincidences that occur precisely in four dimensions. Thus, in any dimension the Riemann curvature can be viewed as a matrix mapping anti-symmetric rank two tensors again into such tensors. And in four dimensions one also has the Hodge star operator that maps anti-symmetric rank two tensors into such tensors. One can ask how these two operations are related or compatible. It is then a simple to check but deep fact that a metric is Einstein if and only if these two operations commute. This fact leads to a whole series of *chiral* formulations of four dimensional General Relativity that have no analogs in higher spacetime dimensions. It is the development of these formulations, and contrasting them with the more known ones, that will occupy us for the large part of this book. There is no coherent account of these developments in the literature, certainly not in any book on General Relativity. It is our desire to make such a coherent account available that was one of the main motivations for writing this monograph.

Another motivation for writing this exposition was our desire to promote the formalism(s) for GR that place the differential forms rather than metrics at the forefront. Differential forms are arguably the simplest and most natural geometric objects that can be placed on a smooth manifold, and are certainly simpler objects than a metric. It turns out to be possible to describe GR using the powerful calculus of differential forms and fibre bundles, which is largely due to Élie Cartan, see the next Chapter for more on this. This book is in particular aimed at giving an expositions of possible formalisms that achieve this.

A related theme is that of spinors and spinorial description. As is well-known, and as we will also emphasise in the book, spinors and differential forms are essentially the same thing, with the Dirac operator being intimately related to the exterior derivative operator . This means that as soon as differential forms are being used as variables to describe the theory, the description has an interesting spinor translation. Viewed in this way, the kinetic operators arising in the field equations of formulations that use differential forms are various versions of the Dirac operator. This becomes especially pronounced in the so-called first-order formulations where field equations are first order in derivatives. These spinor aspects of gravity (and, as we shall see, Yang-Mills theory too), absent in the usual metric description, is another unifying theme of this book. In addition, the spinor description of gravity simplifies link to some recent developments in the field of scattering amplitudes, as we will touch on.

The more familiar of formalisms that use differential forms rather than metrics is that of tetrad (or vielbein, or moving frame, or soldering form) introduced by Cartan. Historically, this formalism was first discovered in the context of two dimensions by the French mathematician Jean-Gaston Darboux (Cartan's PhD supervisor) in late 19th century. It is particularly powerful in this context, as the two one-forms that encode the metric information can be combined into a single complex-valued one-form on the manifold. This is related to the fact that any 2-manifold is a complex manifold. There is no direct analog of such a complexification trick in four dimensions because there is no longer a unique choice of an almost complex structure. But one gets a computationally powerful formalism in four dimensions via chiral formulations referred to above. These formulations, in the case of Lorentzian signature, bring into play complex-valued objects and in a certain sense provide the analog of the complexification trick that works so well in two dimensions. They also make a link to the twistor description of gravity, as we shall explain.

Our final introductory remark is about the Einstein's cornerstone idea that gravity is geometry. At the time when Einstein invented his theory, the only available to him geometry was Riemannian geometry of metrics, described via the tensor calculus of Ricci and Levi-Civita. Einstein learned this mathematics guided by his friend and classmate Marcel Grossmann. It is thus no surprise that General Relativity was formulated in the language of Riemannian geometry and tensor calculus. It is still being developed and also taught to graduate students in that way. However, already at the time of Einstein's invention of GR Élie Cartan was developing a very different type of geometry, the geometry in which the key role was played by differential forms and connections. His works, and works of those around him, strongly influenced the subject of differential geometry, and it is now far more rich and sophisticated than it was a hundred years ago. The Riemannian geometry is now only its relatively small corner. This discussion is related to the theme of the present book because various different formulations of GR that we develop place various different geometric constructions at the forefront. In particular, the geometry of fibre bundles plays much more important role than it does in the usual description of GR. It is thus certainly true that gravity continues to be geometry in the developments on this book, it is only that the word geometry is being understood more broadly than in the metric GR context. We do not yet know which of these "geometries" is more fundamental than others, but a good researcher will certainly want to keep his/her mind open and learn all the available options.

The target audience for this book are postgraduate students interested in gravity, as well as already established researchers. To give encouraging words to the first audience, the we would like to recall our own experience as a student. This author remembers very distinctly that it was easiest to study, understand and prepare for exams on classical mechanics by reading Vladimir Arnold's book on the subject.

And Paul Dirac's book played similar role for quantum mechanics. Both books present their respective subjects in a beautiful and logical way, and both are inspired by mathematics. The moral here is that there are some students that learn best by understanding the overall logic of the formalism first, and only then embark on applications and problem solving. This is certainly not a universal way to learn, and most likely not the way to approach the subject for the first time. But it was important to the present author in his time as a student to have accounts of the usual subjects that concentrate more on the overall logic and the mathematical formalism, rather then on concrete problems that can be solved. The author hopes that there are similar minds out there, and that the present exposition will help such students to understand what General Relativity is about.

In terms of the specialised knowledge that is required to understand this book, we do not assume any more than is usually assumed for graduate level courses. Familiarity with concepts of differential geometry is desirable, but the aspects of this subject that are required to understand the present text are reviewed in the first chapter. So, a good graduate student should be able to follow this exposition without too much difficulty.

Thus, this book is mainly about different possible formalisms for doing calculations with GR, rather than about different possible physical consequences of this theory. So, this book certainly does not compete with the standard textbook expositions of GR, and the student must also study these more standard sources to understand the physics as predicted by General Relativity. Excellent books on the subject that became the standard sources are "General Relativity" by R. Wald and "Spacetime and Geometry: An Introduction to General Relativity" by S. Carroll for GR in general, and "Physical Foundations of Cosmology" by V. Mukhanov for applications to cosmology.

For the experienced researchers, the author suggests this book as a source on aspects of General Relativity that are important about this theory, especially in four spacetime dimensions, but are not covered in any standard book on the subject. Thus, the book can be used as a compendium on different available formalisms for GR, as well as on some less standard aspects of geometry that are required to develop these formulations. Additional motivations for why this or that different formulation of GR may lead to new developments and/or new generalisations are given in the concluding chapter.

We end by explaining why it is the quote from Gregory Skovoroda that we chose to be an epigraph for this whole exposition. First, the present author is a Ukrainian, and it gives him a distinct pleasure to be able to quote Skovoroda, who was a deep thinker years ahead of his times, and who is still relevant today. He is almost unknown in the West, and maybe one of the readers will remember the name, and read his texts.

Second, we aim here to explain only simple, but in our view important things about General Relativity in four spacetime dimensions. There is much more that can be said, and there is a great wealth of physical phenomena that the theory predicts and describes, and that we omit. Not because they are unimportant - on the contrary, they are the reason why physicists learn the subject. But rather because they are unnecessary to understand the overall logic of the theory. It is this overall logic and the facts likely needed to ”move to the unknown” that will concern us in the present book. And so we concentrate here only on things necessary to understand the overall logic of gravity, and hence only on things simple. We hope the reader will take this as a word of encouragement to follow the development of different formalisms described here.

Finally, I would like to thank my collaborators, from whom I learned a lot and without whose insight this book would not exist. Particular thanks are to Joel Fine, Yannick Herfray, Yuri Shtanov and Carlos Scarinci. Thanks also go to my family for their support of ”papa” working on his ”kniga”.

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Introduction

[The tensor calculus] is the debauch of indices.

Élie Cartan, *from Introduction to "Lecons sur la Geometrie des Espaces de Riemann", 1928*

In 1907, while still working as a clerk in a patent office in Bern, Albert Einstein had what he later referred to as "the happiest thought" of his life. He realised that a freely falling observer does not experience gravity, and thus effects of gravity are indistinguishable from those arising in an accelerating frame. These ideas were developed in two papers he published in 1908 and 1911. In these papers Einstein argued that the rules of special relativity must continue to be applicable in an accelerated reference frame. This in particular led him to analyse experiences of an observer performing experiments on a rotating turntable. Einstein concluded that the ratio of the circumference of a circle to its diameter that this observer would measure would be different from π. What this meant for Einstein was that if effects of gravity are those of a non-inertial coordinate system, and the geometry in the later is different from the Euclidean one, then gravity is geometry.

Einstein then searched for a mathematical description of this idea. On return in 1912 to his *alma mater* ETH Zurich he turned for help to his friend and classmate, now a professor of mathematics Marcel Grossmann. Grossmann directed Einstein's attention to Riemannian geometry, the only developed at that time type of geometry that had its origin in Gauss' work on the intrinsic geometry of 2-surfaces in 3-dimensional space. Bernhard Riemann lay the foundation of the subject in his famous 1854 Göttingen habilitation lecture "On the hypotheses that underlie geometry". In this lecture he described the way to extend the Gauss' notion of curvature to an "n-ply extended magnitude". Thus, by the time Einstein studied this subject, it was far from being new. Einstein learned it in the form described in 1900 exposition by Gregorio Ricci and Tulio Levi-Civita "Methods of the absolute differential calculus and their applications". In a joint 1913 paper with Grossmann,

Figure 0.1 Bernhard Riemann

Einstein described "an outline" of a new gravity theory using precisely this language. The final version of the new theory of gravity was developed by late 1915, still using the language of tensor calculus. By this time Einstein was already in Berlin, and this work appeared single-authored. It is this 1915 theory that is now known as Einstein's General Relativity. To a considerable extent, even this day it is taught and applied using the 19th century language of tensor calculus.

Bernhard Riemann was born on September 17, 1826 in Breselenz, a village in the Kingdom of Hanover. His father was a poor Lutheran pastor. Riemann was the second of six children, shy and of not very strong health. His mother died when he was 20, and his brother and 3 of his sisters all died young, as eventually did he. Riemann exhibited exceptional mathematical skills, such as calculation abilities, from an early age but suffered from a fear of speaking in public.

Even though Riemann was very gifted in mathematics, he planned to study theology and become a pastor, like his father. In 1846 his father gathered enough money to send him to Göttingen to study theology. However, once there Riemann started attending mathematics lectures by Gauss. The latter recommended that Riemann gives up his theological work and goes into mathematics. After gaining father's approval Riemann transferred to Berlin in 1847, and returned to Göttingen in 1849. He defended his doctoral dissertation in 1851, on what we now call Riemann surfaces. He held his first lectures in 1854. His habilitation lecture has founded the field of Riemannian geometry. In 1859, following Dirichlet's death, who occupied Gauss' chair since 1855, Riemann became the head of mathematics at Göttingen.

In 1862 Riemann married Elise Koch and they had a daughter. He fled Göttingen in 1866 when the armies of Prussia and Hanover clashed there. He died in Italy the same year from tuberculosis. Riemann was a dedicated Christian, and saw his life as a mathematician as another way to serve God. During his life, he held closely to his Christian faith and considered it to be the most important aspect of his life. At the time of his death, he was reciting the Lord's Prayer with his wife and died before they finished saying the prayer.

But roughly around the same time a French mathematician Élie Cartan was developing a very different type of geometry. In Cartan's work on differential geometry, the notions of differential forms and fibre bundles, both of which he to a large extent established, played central role. Both of these will play a crucial role in this book too. Also, in 1913, constructing linear representations of Lie groups, Cartan discovered spinors. This will be important in our exposition as well. It was realised much later, in 1954 book "The algebraic theory of spinors" by another French mathematician (and one of the founding members of Bourbaki group) Claude Chevalley, that spinors and differential forms are very closely related. We will explain this fact in due course.

Cartan was led to the notion of differential forms in his 1901 work developing a geometric approach to partial differential equations. What Cartan was after was a formalism that is invariant under arbitrary changes of variables. Cartan's main tool for this was the calculus of differential forms. Cartan then worked on problems of group theory, and in particular, as we already mentioned, discovered the spinor representations of the orthogonal groups in 1913.

Theory of Lie groups is intimately related to geometry. It is thus no surprise that Cartan turned to the later. He was also motivated by Einstein's theory of gravity that came to prominence in 1919. It is in Cartan's works of 1920's that his most important contributions to differential geometry were developed. Cartan's main realisation was that it is fruitful and necessary to consider other "bundles" apart from the tangent bundle, and other "connections" apart from the Levi-Civita connection. We took the words bundles and connections in quotes because these notions were only beginning to be understood in Cartan's works. In particular, Cartan himself, while working with different bundles extensively, never explicitly defined what is now known as a (principal) fibre bundle. Cartan was also responsible for a notion of what is now known as the (principal) connection, and in particular realised that such a connection is best described as a (Lie algebra valued) one-form. Cartan was thus able to disassociate the notion of the connection and parallel transport from the very restricted form these take in the context of affine connections in the tangent bundle. This led him to discovery of many new types of geometry, thus finding probably the most fruitful generalisation of Riemannian geometry. This was searched by

Figure 0.2 Élie Cartan

many around the same time, in particular by Hermann Weyl, but it was Cartan who achieved this goal. As a bonus of his general programme on connections, Cartan was also able to give a very powerful and simple description of Riemannian geometry, in his 1925 paper "La géométrie des espaces de Riemann". In the preface to his 1928 book "Leçons sur la géométrie des espaces de Riemann" he stated his aim as that of bringing out the simple geometrical facts which have often been hidden under a debauch of indices. It is this Cartan's description of Riemannian geometry that we will present under the name of "tetrad" formalism for GR.

Élie Cartan was born on 9 April 1869 in Dolomieu (near Chambéry), region Rhône-Alpes, France. His father was a blacksmith. The family was very poor, and it would be impossible for Élie to get good education if not for his talent for mathematics that was early spotted. Already at primary school Élie impressed his teachers. One of them later said: "Élie Cartan was a shy boy, but his eyes shone with an unusual light of great intelligence". Still, Cartan could have never become a great mathematician if not of a young school inspector, later important politician Antonin Dubost. Dubost was visiting the school where the young Élie was taught and was impressed with young boy's talent. He encouraged Élie to participate in a competition for state funds that could enable him to study in a Lycée. Élie's school teacher M Dupuis prepared him for the competitive examinations that were held in Grenoble. Excellent performance allowed Élie to study in good schools, and then later at the École Normale Supériere in Paris.

In ENS Cartan became a student of Gaston Darboux, the inventor of the moving frame method, which Cartan later greatly developed. Cartan's friend Arthur Tresse was studying under Sophus Lie in Leipzig, and told Cartan about remarkable work by Wilhelm Killing on classification of finite groups of continuous transformations. Cartan then set to complete Killing's work, and corrected some important mistakes and omissions in it. This became Cartan's doctoral dissertation. In one way or another, Cartan's whole scientific career revolved around the questions related to Lie groups and their geometry.

Cartan was a lecturer at the University at Montpellier from 1894 to 1896, and a lecturer at the University of Lyons, where he taught from 1896 to 1903. In 1903 he married Marie-Louise Bianconi (1880-1950), the daughter of a professor of chemistry there. The family moved to Paris in 1909, where Cartan was appointed first at Sorbonne and later at ENS. Cartans had four children. The eldest son Henri has become a renowned mathematician of his own. The second son Jean, a composer of fine music, died of tuberculosis in 1932 at the age of 25. Their third son Louis became a physicist. He was a member of the Resistance fighting in France against the occupying German forces, and was arrested and executed by the nazis in 1943. Cartan was 75 at the time when he learned of his third son's fate, and this was a devastating blow for him. The fourth child of the family was a daughter Hélène who became a teacher of mathematics.

Cartan died in Paris in 1951, at the age of 82. Cartan's obituary by Chern and Chevalley opens with the words: "Undoubtedly one of the greatest mathematicians of this century, his career was characterized by a rare harmony of genius and modesty".

Cartan's more general connections were rediscovered by physicists only much later, in 1954 work by Yang and Mills. Every known interaction in Nature is now described by a gauge field or connection, of precisely the type that was first introduced by Cartan in his differential geometry work of 1920's. Of course Cartan did not write the Yang-Mills field equations, as his motivations were entirely different from those of particle physicists of the 1950's. It was thus Cartan who developed mathematics that is necessary to formulate gauge theories, and that can also be used to describe gravity. It is rather unfortunate that the theory of gravity is usually taught in the 19th century language of tensor calculus and not in the 20th century language of principal connections in fibre bundles. Not only this second language is more clear – the debauch of indices is no longer there – but it is also more computationally efficient due to its usage of differential forms, and brings gravity closer in form to all the other interactions. We hope this book will serve to promote Cartan's language of differential forms and connections as the most appropriate one not just for Yang-Mills theory but also for gravity.

It must be admitted that for someone who is raised on notions of indices and tensor calculus, absorbing Cartan's geometric ideas is a rather difficult task. This is in particular manifested by the fact that Cartan's work on differential geometry was recognised to be of importance only late in his life. Quoting Cartan's obituary by Shiing-Shen Chern and Claude Chevalley, written in 1951, Cartan's "death came at a time when his reputation and the influence of his ideas were in full ascent". However, even in 1938 Hermann Weyl, in reviewing one of Cartan's books, wrote: "Cartan is undoubtedly the greatest living master in differential geometry. . . . I must admit that I found the book, like most of Cartan's papers, hard reading. . . ." This sentiment was shared by many geometers at the time. The situation has changed however. Differential geometry is now taught, at least to mathematicians, in a way that incorporates Cartan's geometric ideas from the start. It is time that this powerful language is also taken on board by the (gravitational) physicists.

Having given the praise to Cartan's ideas, it should be said that the tetrad formalism *is* described in most standard textbooks on GR, often under the name of "non-coordinate bases", see e.g. Sean Carroll's book and/or "Geometry, topology and physics" by Mikio Nakahara. This formalism, however, is described only as secondary to the usual metric one. In particular, the spin connection, which is the central object that the tetrad formalism introduces, is considered to be only an object derived from the usual Christoffel connection. Also, the conceptual change that the tetrad formalism brings with itself, namely the fact that it works with a vector bundle different from the tangent bundle, is rarely emphasised, while this is the central point. Moreover, the presentation of the tetrad formalism in GR literature in fact avoids introducing any other bundle. The presentation of the tetrad formalism to be given in this book is different from the standard treatment in GR texts and is closer to the ones appearing in the mathematical literature.

Moreover, while a description of the tetrad formalism can often be found in the GR literature, it is rarely given any significance. Indeed, the usual attitude is that it is only a reformulation of GR, and moreover one that increases the number of field components that one has to work with, from 10 metric components in four dimensions in metric GR, to 16 tetrad components. This is clearly in the direction of loss of economy, and is this appears to be a clear reason against using the tetrads. Furthermore, the tetrad formalism uses two different types of indices, the spacetime indices for vectors and forms on a manifold, and "internal" indices for objects valued in the vector bundle on which the tetrad formalism is based. The usual attitude to this is that this leads to a notational nightmare. Why then use a formalism with two types of indices if in the metric GR it is possible to work with only spacetime indices? Thus, the usual attitude to tetrads in the GR community is that this is a cumbersome formalism, which brings with it nothing new, and is therefore not worth the effort. It is nevertheless admitted that spinors can only be coupled to gravity by using the tetrads. But one is rarely interested in gravity effects caused by spinor matter, usually an effective description of matter using perfect fluids is completely sufficient to extract interesting physics. So, even though spinors do require tetrads, one rarely needs spinors in GR.

Yet another seemingly compelling reason to ignore tetrads is the description of the linearised excitations of the gravitational field. These carry spin two. As such, it appears to be natural to describe them by rank two tensors. The linearised dynamics is then readily available by either linearising the Einstein equations, or by looking for a second order differential operator that is invariant under the linearised diffeomorphisms. Both procedures uniquely lead to the same linearised dynamics. The attitude of the particle physics community is then that Einstein's theory gives a non-linear completion of this linearised description, which is moreover to a very large extent unique. This point of view has been advocated in Weinberg's book "Gravitation and Cosmology". From this point of view it appears to be unnatural to use any other object to describe gravity other than the metric.

Both arguments agains the usage of tetrads actually underestimate the power of the formalism of differential forms. Yes, the tetrad carries more components, but the amount of gauge has also increased. And it is often the case in mathematics that a formalism that uses more independent functions allows for a simpler description. That this is the case with the tetrad formalism is manifested by the fact that the gravitational action in the tetrad formalism is just quartic in the basic fields, while the Einstein-Hilbert metric action is non-polynomial in the metric. Thus, the tetrad formalism gives an algebraically simpler description of the gravitational field. And working with objects with different types of indices is not a problem once an appropriate formalism is developed. Indeed, having fields with two different types of indices does not cause any problems in the treatment of Yang-Mills theory. Finally, for the description of the linearised dynamics, it turns out that not only the tetrad formalism does not make things more complicated, on the contrary, the usage of differential forms brings with it simpler differential operators as compared to those that arise in the metric formalism. In fact, using differential forms one achieves a description of the spin two linearised fields that is analogous to the description in the case of the Maxwell theory, as we shall see in Chapter 8. There is no such analogy when one works with the metric variables. So, all in all, the formalism of differential forms does introduce simplifications in GR ranging from the full non-linear dynamics to the linearised treatment. So, it is brushed aside in the usual GR texts for the wrong reasons, as we hope will become clear from the treatment in this book.

As we have already said in the preface, this book is more than just about the tetrad formalism. Its unifying theme is the formalisms for GR (in particular GR in four spacetime dimensions) that are based on vector valued differential forms. Towards the end of the book we will develop an even more exotic alternative, in which gravity in four dimensions will be seen to arise as the dimensional reduction of a theory of ”pure” differential forms, i.e. differential forms valued in $\mathbb{R}$, in seven dimensions. The development of all these different formulations would be impossible without Cartan’s ideas and the example of the tetrad formalism, historically the first description of GR in terms of differential forms. This explains the considerable attention given to Cartan’s type differential geometry in this book. To put it provocatively, this book attempts to develop the theory of gravity using the 20th century differential geometry of Cartan, forgetting Einstein’s theory of General Relativity formulated using the 19th century language of tensor calculus as much as possible.

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Concluding Remarks

Our journey took us from the usual formalism that views GR as a dynamical theory of Riemannian geometry of metrics through a sequence of formalisms based on connections and differential forms to more exotic 6D and 7D constructions. It is now time to attempt to summarise what has been learned.

In all formalisms related to Cartan's tetrads gravity becomes very similar to Yang-Mills gauge theory. The geometric structures that make this possible are essentially invisible in the usual metric formulation. But gravity is not Yang-Mills. From the geometric point of view the main difference is presence in gravity of an object that solders the geometry of the manifold to the geometry of whatever abstract bundle that is used. This geometric object is different in different formalisms. It is useful to summarise this as the following table:

Table 10.1 *Table of formalisms with objects that implement soldering*

Formalism	Soldering object
Cartan formalism	Frame field or tetrad
BF formalism	2-Form field valued in the Lie algebra of Lorentz group
MacDowell-Mansouri formalism	De Sitter / Anti-De Sitter connection
Pure spin connection formalism	Curvature of the spin connection
Plebanski formalism	Triple of self-dual 2-forms
Chiral pure connection formalism	Curvature of the chiral part of the spin connection

Thus, in all these descriptions there is a geometric object that ties the geometry of an abstract fibre bundle over a manifold to the geometry of the tangent bundle. The metric is then constructed from this object. There is no such soldering in Yang-Mills theory. We can therefore say that

Gravity is Gauge Theory with Soldering

We have also seen that formalisms based on differential forms allow the equations of gravity to be rewritten in index-free notations. In 2D this is achieved by introducing a complex linear combination of the two frame 1-forms, see (3.40). In 3D this is achieved by constructing 1-forms with values in the Lie algebra of the appropriate "Lorentz" group, concretely 1-forms with values in 2×2 tracefree matrices, both for the frame field as well as for the connection, see (4.11), (4.13). Finally, in 4D the closest one gets to an index-free formalism is via the chiral Plebanski setup. For instance, the index-free relation (9.138) is the Einstein equation describing the 4-sphere. In general, however, when there is also Weyl curvature present, 4D Einstein equations can't be naturally written in a completely index-free notation due to the presence of the matrix Ψ^{ij} representing the chiral part of the Weyl curvature on the right-hand-side of Plebanski equations (5.162). Thus, field equations of 4D gravity are like those of Yang-Mills theory in the sense that they can't be written solely in terms of wedge products of Lie algebra valued differential forms. In the case of Yang-Mills theory one needs the operation of the Hodge dual to write $d^*F = 0$. In the case of gravity the analogous operation is the one required to form the right-hand-side of Plebanski second equation in (5.162) from Σ^i. Schematically, the Plebanksi equations are $d_A\Sigma = 0$, which is written solely in terms of wedge product of forms, as well as $F =$ "$^*\Sigma$", where the "Hodge star" in quotes is the operation that produces the Lie algebra valued 2-form $(\Psi^{ij} + (\Lambda/3)\delta^{ij})\Sigma^j$ from the Lie algebra valued 2-form Σ^i.

The analogy with Yang-Mills indexYang-Mills becomes even more pronounced in the pure connection formalism, where the field equations take the form d_A"$^*F$" $= 0$. Now the "Hodge star" is the operation (6.14) that is necessary to produce the Lie algebra valued 2-form Σ^i_F from the curvature 2-forms. In both YM and GR it is the presence of these "Hodge stars" that prevents the equations to be writable solely in terms of wedge products of differential forms.

In terms of the computational efficiency, we have seen that 4D *chiral* formalisms are clearly superior in terms of their economy. In these formalisms, the connection components necessary for the computation of the curvature are stored very compactly and computations required to write Einstein equations proceed with minimal effort. This is true both in the case of the original Plebanski description that works with 2-forms Σ^i and connection A^i, as well as for the pure connection formalisms that work with either solely A^i or A^i and the auxiliary matrix M^{ij}.

We have also seen that the description of the linearised gravity and the gravitational perturbation theory simplifies greatly by the use of the chiral formalisms. First, the usage of chiral objects brings with it completely new types of differential operators, see Figure 8.1. This allows to write the familiar spin one and spin two kinetic terms in a completely new way, see e.g. (8.158) for how the usual linearised

Lagrangian for the spin two perturbation $h_{\mu\nu}$ gets compactly re-written by the use of the chiral 2-form fields $\Sigma^i_{\mu\nu}$.

The propagators and interaction vertices also get simplified by the chiral formalism. The gravitational action becomes polynomial in the fields in any first order formalism. However, all such formalisms apart from the chiral ones introduce "too many" auxiliary fields. This is manifested by the fact that the two-point function of the auxiliary field with itself is non-zero in all but the chiral formalisms. This is the case in the chiral description of YM, see (8.98), as compared to the non-chiral version, see (8.158), as well as in the chiral description of GR as compared to standard GR, as we have verified in the Section on Plebanski perturbation theory. The chiral perturbation theory for GR that we have developed in this book may well hold a lot of potential. It would be interesting to try to use it to simplify computations ranging from quantum loops to the perturbative calculations that are necessary to extract the gravitational wave signals.

In the last Chapter we have developed an even more exotic viewpoint on 4D gravity, one that puts at the forefront the total space of the bundle of 2-component spinors over the 4-dimensional manifold M in question. The projective version of this bundle is known as the twistor space of M. The usual twistor story emphasises the complex analytic aspects of the twistor space. This, however, only works when the geometry of M is chiral in the sense that only one of the two chiral halves of the Weyl curvature is non-zero.

We have seen that there exists a version of the twistor story that works in the circle bundle over twistor space instead. This is a 7D manifold, and the geometric data on M define a certain natural 3-form C on it. There is then a natural first order differential equation that can be imposed on M, namely $dC = \lambda {}^*C$, where λ is a constant. Such 3-forms are called nearly parallel and define a 7D metric via (9.146). Moreover, this metric is automatically Einstein with non-zero scalar curvature. Requiring that this equation is satisfied for the 3-form that is defined by the 4D data imposes Einstein-like equations on these data. We have then seen that the usual twistor story with its integrable almost complex structures lifts naturally to this 7D description. In particular, the first order Cauchy-Riemann guaranteeing integrability of the almost complex structure on the twistor space follow from the first order nearly parallel condition $dC = \lambda {}^*C$ satisfied by the 3-form.

Importantly, the described 6D and 7D viewpoint on 4D gravity is crucially based on precisely its chiral version to which we devoted so much attention in this book. This is manifested particularly strongly by the example of the quaternionic Hopf fibration in Section 9.3. This example shows the chiral 4D description of the 4-sphere with its chiral 2-forms Σ and the chiral connection $\mathbf{A}$ arising from the geometry of the total space of the Hopf 3-sphere bundle over S^4. A related point is the fact that the Urbantke formula (5.37) that appears somewhat mysteriously in the chiral 4D descriptions gets explained by the observation that it is the dimensionally reduced to 4D version of the formula (9.146) for the metric defined by a generic 3-form in 7D, see (9.148).

At the same time, the higher dimensional descriptions that we developed suffer from a very serious defficiency - they only work for the Euclidean version of the 4D gravity. This is the case for both the usual twistor description, which is only capable of describing the half-flat Euclidean gravitational instantons, as well as for the 7D description in terms of 3-forms that we developed. It is clear that if there is any truth in the higher dimensional perspective of the type described, it should be possible to find also the version appropriate for the Lorentzian signature.

Let us end this discussion by listing questions that, in the opinion of this author, hold greatest potential to lead to a breakthrough in our understanding of gravity. The first question was already mentioned in Section 3 introducing formalisms based on differential forms. It is "Why non-zero metric?" To expand on this, we now know that if there is a non-zero metric filling the Universe, then its low-energy dynamics can only be described by General Relativity, at least in 4D. At the same time, GR is unable to answer the question as to why such a non-zero metric exists. The same is true about any of its reformulations described in this book, even though reformulations based on differential forms seem to come closer to an eventual answer because in these formulations one can at least talk about zero field configurations. So, it is clear that answering the "Why non-zero metric?" question will require radically new ideas. It is possible that the puzzle of gravity can only be solved by answering this question.

The second question that we believe is also of fundamental importance is more well-posed, and so can probably be answered in the near future. This is the question of interpretation of the Lorentzian signature Urbantke formula (5.47). In our discussion following (9.148) we have seen that the Euclidean signature Urbantke formula can be understood as being a consequence of (9.146) defining a 7D metric from a stable 3-form. Thus, we have seen that assuming that the 7D manifold is fibered by 3-dimensional submanifolds on which the 3-form is non-zero exhibits the 4D Urbantke metric as the one induced on the 4D slices transverse to the fibres. The same interpretation exists for the Spit signature metrics in 4D. This also follows from the 7D formula (9.146) but for C lying in the orbit of different sign, the one for which the metric defined by C is of signature $(3, 4)$. However, there is no such interpretation to the Lorentzian signature Urbantke that works with complex-valued 2-forms but still produces a real-valued metric. It is clear that if there is an interpretation that is related to 3-forms in seven dimensions, it must involve complex-valued forms in some way. We believe that finding such an interpretation, if it exists, holds potential for a breakthrough in understanding of 4D Lorentzian signature gravity, as providing a deeper geometric structure behind it.

We end this book by a provocative remark. General Relativity is the unique low energy theory of interacting massless spin two particles. This statement holds independently of any Lagrangian formulation that may be used to describe it. The usual metric formalism is by far the most explored one. But in this book we have see that, surprisingly, GR admits many not obviously equivalent formulations. In fact, GR appears to be the theory that admits by far many more reformulations than any other known theory. This is one ”experimental fact” about GR that is rarely emphasised, and that we believe becomes strikingly apparent from the developments we have followed. We don’t know what is the significance of this fact, if any, but it may be that gravity is trying to tell us something by this fact. It is possible that the message is: ”I am more than just an effective low energy theory of massless spin two particles, I hold the key to the puzzle of why the Universe can be so successfully described in geometric terms”.

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