Part I: Metric and related formulations

Historical remark:

Einstein's GR is formulated in the language of Riemannian geometry, the only type of geometry sufficiently developed in 1912 when Einstein returned to Zurich and was learning geometry with the help of his friend and ex-classmate M. Grossmann

Geometry is far reacher now as compared to what it was in 1912 thanks in particular to fundamental contributions by Cartan. Part of the motivation here is to learn to think about GR using the 20th century geometry of Cartan rather than 19th century tensor calculus of Ricci and Levi-Civita

Geometry of fibre bundles, differential forms and connections

Einstein's GR

Geometry $\rightleftarrows$ Matter

G - Newton's constant

$g = \det(g_{\mu\nu})$

Λ - Cosmological constant

$$S_{EH}[g] = \frac{1}{16\pi G}\int\sqrt{-g}\,(R-2\Lambda)$$

sign in front is signature dependent plus sign for the mostly plus signature

It is interesting to remark that GR is non-Machian - geometry exists even without matter

Related to "Why non-zero metric?

Conventions:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\alpha\mu} V^\alpha \qquad [\nabla_\mu,\nabla_\nu]V^\rho = R^\rho{}_{\alpha\mu\nu} V^\alpha$$

$$R^\sigma{}_{\rho\mu\nu} = \partial_\mu \Gamma^\sigma_{\rho\nu} - \partial_\nu \Gamma^\sigma_{\rho\mu} + \Gamma^\alpha_{\rho\nu}\Gamma^\sigma_{\alpha\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\alpha\nu}$$

Einstein equations (in vacuum) — Einstein metrics

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - 2\Lambda) = 0 \Rightarrow R_{\mu\nu} = \Lambda g_{\mu\nu}$$

(in 4 dimensions)

Expansion around flat space (Minkowski) - schematically

$$g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G} h_{\mu\nu}$$

$$\mathcal{L} = (\partial h)^2 + \sqrt{G} h (\partial h)^2 + \dots$$

negative mass dimension coupling, power counting non-renormalizable

derivative interactions

Propagating DOF

4 components of $g_{\mu\nu}$ are Lagrange multipliers for 4 constraints

$$10 - 4 - 4 = 2 \text{ DOF}$$

2 propagating polarizations of the graviton

Metric Affine formulation - first order Palatini

$$S_{Palatini}[g, \Gamma] = \frac{1}{16\pi G} \int \sqrt{-g} \left(g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda \right)$$

Variation wrt $\Gamma^{\rho}_{\mu\nu}$ gives $\nabla_\rho g^{\mu\nu} = 0 \Rightarrow \Gamma = \Gamma(g)$

- Only first derivatives in the action - like Hamiltonian formulation
- Note $R_{\mu\nu}(\Gamma)$ is not automatically symmetric even when $\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{(\mu\nu)}$ torsion free
- When $\Lambda = 0$ introducing $\sigma^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ gives a cubic Lagrangian for GR!
- Too many fields $10 + 40$ to be of practical use

Second order pure affine connection – Eddington formulation

When $\Lambda \neq 0$ can "integrate out" the metric instead of $\Gamma^{\rho}_{\mu\nu}$

Field equation for the metric $R_{\mu\nu}(\Gamma) = \Lambda g_{\mu\nu}$

and so $$g_{\mu\nu}(\Gamma) = \frac{1}{\Lambda} R_{\mu\nu}(\Gamma)$$

Metric algebraically constructed from the curvature of the connection

only possible when $\Lambda \neq 0$

Substituting back into the action gives

$$S_{\text{Eddington}}[\Gamma] = \frac{1}{8\pi G \Lambda} \int \sqrt{-\det R_{(\mu\nu)}(\Gamma)}$$

Field equation for Γ

$$\nabla_\mu R^{(\alpha\beta)}(\Gamma) = 0$$

Second order PDE for the affine connection

The metric constructed from solutions of this PDE is automatically Einstein

- Too many fields to be useful
- Can't expand around flat space ($\Lambda \neq 0$)
- Matter can be added without problems, the metric can be always "integrated out".
- The Lagrangian is no longer unique – can construct other invariants using the $R_{[\mu\nu]}$ part of the curvature. This is a rather general feature – other formulations typically have more ambiguity than Einstein-Hilbert.

Less familiar second order formulations - $\Gamma\Gamma$

$$\int \sqrt{-g} R = \int \sqrt{-g}\, \Gamma\Gamma + \int \partial_\mu (\sqrt{-g}\, \omega^\mu)$$

$(\partial R)^2$ type Lagrangian

Modulo surface term can rewrite the GR action in the form $(\partial g)^2$. There is no covariant way to do this, and this is why the only covariant action is of the form $g \partial^2 g$

$\Gamma^\rho_{\mu\nu}(g)$ here

Explicitly

$$S_{\Gamma\Gamma}[g] = \frac{1}{16\pi G} \int \sqrt{-g} \left(g^{\rho\sigma} \left(\Gamma^\mu{}_{\nu\rho} \Gamma^\nu{}_{\mu\sigma} - \Gamma^\mu{}_{\rho\sigma} \Gamma^\nu{}_{\nu\mu} \right) - 2\Lambda \right)$$

Action explicitly in terms of the metric

$$S[g] = \frac{1}{16\pi G} \int \sqrt{-g} \Big(\partial_\mu g^{\rho\alpha} \partial_\nu g_{\sigma\alpha} \left(\frac{1}{4} g^{\mu\nu} \delta^\sigma_\rho - \frac{1}{2} g^{\mu\sigma} \delta^\nu_\rho \right) - g^{\mu\nu} \partial_\mu \partial_\nu (\ln \sqrt{-g}) - 2\Lambda \Big)$$

Convenient starting point for linearization around Minkowski, but introducing $\sigma^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$ can also be used for getting a very compact perturbative expansion to any order. The number of terms at every order does not proliferate, unlike on the usual $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ expansion.

Cheung, Remmen 2017

Part II: "Tetrad and related formulations

Historical remarks: In 1920's Cartan, motivated partly by Einstein's GR that came to prominence in 1919, partly by his previous work on Lie algebras and differential forms, came up with a completely new way of thinking about Riemannian geometry. In fact what Cartan discovered was the most natural generalization of Riemannian geometry, sought by many during the same period, in particular by Weyl.

In Cartan's description fibre bundles and connections as Lie algebra valued 1-forms play key role. In fact, Cartan discovered connections (gauge fields) much before they appeared in physics. Mathematicians now learn geometry a la Cartan. GR can be formulated using this powerful language. Differential forms and exterior derivative simplify things greatly, even technically. It is time grav. physicists take this language on board. Things are much more deep than the non-coordinate bases typically mentioned in this context in GR books.

Soldering form

The natural object that lives in fibre bundles (principal bundles, vector bundles) is a connection

But connection does not tie (at least not in general) the geometry of the fibre to geometry of the base in any way.

Cartan's idea is to introduce precisely such an object. This is the general idea of soldering

As we shall see, soldering has many different incarnations. We start with the most familiar one

Let $V \hookrightarrow E \to M$ be a vector bundle over our manifold M, with fibres $V \sim \mathbb{R}^n$ $\dim M = n$

preparing for the fact that there will be a relation to geometry of M

We will also require that E is of the same topological type as TM

Can put many different connections on E

Definition: Co-frame or a soldering form at $x \in M$ is an isomorphism

$$e : T_x M \to V$$

Locally, it is a 1-form on M with values in V

$$e^I = e^I_\mu \, dx^\mu$$

Extending to all points of M get an isomorphism

$$e : TM \to E$$

This is an object that ties geometry of the fibre to that of the base

Note that $GL(n,\mathbb{R})$ acts (transitively) on the space of coframes at a point

$$GL(n,\mathbb{R}) \ni g : e \to g^{-1} e$$

coframe map followed by g^{-1} on V

This makes the space of coframes on M into the principal $GL(n,\mathbb{R})$ bundle over M

We did not yet put any geometric structure either on V or on M. If we do this we are led to the notion of G-structures.

Geometric structures

We now want to put some geometric structure on M.
This can be e.g. metric, but it can also be a non-degenerate 2-form, or an almost complex structure
$I : TM \to TM$, $I^2 = -\mathbb{1}$

It will be very convenient to fix some model object of this type in V. Thus, we take either

Q – non-degenerate symmetric tensor in $V^* \otimes V^*$ (metric)
ω – non-degenerate anti-symmetric tensor in $V^* \otimes V^*$
I – map $I : V \to V$ s.t. $I^2 = -\mathbb{1}$

In each case the group $GL(n, \mathbb{R})$ acting on V reduces to a subgroup of transformations that preserve the model object

$O(V, Q)$ orthogonal group
$Sp(V, \omega)$ symplectic group
$GL(m, \mathbb{C})$ general linear complex $2m = n$

Let us now put on M a geometric structure of one of the above types – metric, non-degenerate 2-form, ACS

Definition: A co-frame (soldering) is called adapted to a given geometric structure if this structure arises from the model structure on V by pull-back with the soldering map.

I.e. $e : TM \to E$ is adapted to a metric $g \in S^2 TM$ if g is the pull back via e from Q in E

$$g_{\mu\nu} = e^I_\mu e^J_\nu Q_{IJ}$$

metric on M the model metric

Similarly, $e: TM \to E$ is adapted to $B \in \Lambda^2 M$ if

$$B = e^*(\omega) \qquad B_{\mu\nu} = e^I_\mu e^J_\nu \omega_{IJ}$$

$e: TM \to E$ is adapted to $J \in TM \otimes TM^*$ if

$$J_\mu{}^\nu = e^I_\mu e^\nu_J I_I{}^J$$

inverse soldering (or frame)

The space of coframes adapted to a given geometric structure becomes a principal G bundle over M, where G is one of the $GL(n,\mathbb{R})$ subgroups, i.e. $O(V,G)$, $Sp(V,\omega)$ or $GL(m,\mathbb{C})$

We can now put a connection on this principal G bundle. This gives rise to a connection in E that preserves the model geometric structure on V. Concretely, a 1-form on M with values in the Lie algebra of $O(V,G)$, $Sp(V,\omega)$, $GL(m,\mathbb{C})$

The pull-back of this connection to TM via the soldering map gives an affine connection on TM

This is how soldering relates the vector bundle with some geometric structure and a connection that preserves this structure to a geometric structure on the manifold, and an affine connection.

Concretely $\quad u^I = e^I_\mu u^\mu$

$$e^I_\mu \nabla u^\mu := d^\omega u^I = du^I + \omega^I{}_J u^J$$

or

$$e^I{}_\sigma \Gamma^\sigma_{\nu\mu} = \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu$$

"total" cov. derivative is zero

Can rewrite as $\boxed{\nabla^\omega_\mu e^I_\nu = \partial_\mu e^I_\nu + \omega_\mu{}^I{}_J e^J_\nu - \Gamma^\sigma_{\nu\mu} e^I_\sigma = 0}$

The above story is completely general, and allows to treat metric, symplectic and complex geometry in parallel.

We now specialize to the metric case

Torsion: $T^I = d^\omega e^I$ 2-form on M with values in V

torsion only exists because of soldering.
There is no such thing in a general bundle

Fundamental lemma:

There exists a unique torsion free and metric connection in E. It is called the spin connection. Explicitly

$$\omega_\mu{}^I{}_J = e^{\rho I} e^\sigma_J \left(- C_{\mu\rho\sigma} + C_{\rho\sigma\mu} + C_{\sigma\mu\sigma} \right)$$

where $C_{\mu\rho\sigma} = e_{\mu I} \partial_{[\rho} e_{\sigma]}^I$

The objects $e_{\mu I} = e^J_\mu Q_{IJ}$ $\quad e^{\rho I} = e^\rho_J Q^{IJ}$

so only defined when there is a metric

There is no unique (torsion-free) connection in the symplectic and complex cases, and this is why metric geometry is somewhat exceptional

It is then easy to check that for the metric $g = e^*(Q)$ the metric and torsion free affine connection Γ is the pull-back of the metric and torsion free connection on E

Indeed $\nabla g_{\mu\nu} = \nabla^\omega g_{\mu\nu} = \nabla^\omega \left(e^I_\mu e^J_\nu Q_{IJ} \right) = 0$

$$e^I_\sigma \Gamma^\sigma_{[\mu\nu]} = \partial_{[\mu} e_{\nu]}^I + \omega_{[\mu}{}^I{}_J e^J_{\nu]} = T^I = 0$$

Riemann curvature is curvature of the spin connection

$$0 = 2 \nabla^\omega_{[\mu} \nabla^\omega_{\nu]} e^I_\rho = R^{(\omega)}_{\mu\nu}{}^I{}_J e^J_\rho - R^{\alpha(\Gamma)}{}_{\rho\mu\nu} e^I_\alpha$$

where $R^I{}_J = d\omega^I{}_J + \omega^I{}_K \omega^K{}_J$ curvature 2-form

This translates all operations in computing connection and curvature to working with differential forms e^I, $\omega^I{}_J$ and computing the curvature $R^I{}_J(\omega)$

This is much more efficient for explicit computations than the usual $\Gamma^\rho_{\mu\nu}$ way. 24 components of $\omega_\mu{}^I{}_J$ as compared to 40 components of $\Gamma^\rho_{\mu\nu}$

Torsion-free affine connection $\Gamma^\rho_{\mu\nu}$ is not a principal connection, and is not a differential form. Soldering maps it to a principal connection and a differential form. Working with differential forms is easy

If one wants to compute the Riemann curvature of some concretely given metric (e.g. spherically symmetric, to derive Schwarzschild solution), Cartan's approach is easier by far!

Einstein-Cartan first order formulation

Can write the Lagrangian in terms of the wedge product of differential forms

4D theory (other dimensions analogously)

$$S_{EC}[e,\omega] = \frac{1}{32\pi G}\int \epsilon_{IJKL}\, e^I e^J \left(R^{KL}(\omega) - \frac{\Lambda}{6} e^K e^L\right)$$

need to choose orientation of both M and V to make sense of this integral

The second index on $R^K{}_L$ is raised using the metric in V

Varying wrt spin connection gives the torsion-free condition

$$d^\omega e^I = 0$$

Varying wrt the coframe gives Einstein equations

$$\epsilon_{IJKL}\, e^J R^{KL} = \frac{\Lambda}{3}\epsilon_{IJKL}\, e^J e^K e^L$$

- First order formulation with 16+24 field components
- Polynomial (quartic) even when $\Lambda \neq 0$
- This formulation is unavoidable if wants to couple spinors to gravity, as spinors couple directly to the spin connection (hence the name)
- A drawback of this formulation is mismatch in #'s components of e^I_μ and $\omega_\mu{}^I{}_J$. There are too many "momenta" variables. Some of them are redundant and eliminated by second class constraints
- In contrast with metric GR, this formulation does not require $e^I_\mu \neq 0$. Indeed, all field being zero is still a solution of all the equations. This is a step closer to addressing "why non-zero metric" question. However, this question is of course not answered by this formalism, as it does not explain why e should be non-zero rather than zero. Also, there is no kinetic term around $e=0$ background

Teleparallel formulation

As in the metric case it was possible to rewrite the Lagrangian in the $\Gamma\Gamma$ form (plus boundary terms), it is possible to write the Einstein-Cartan Lagrangian with $\omega = \omega(e)$ in the schematic form $\omega\omega$

$$S_{EC}[e] = \frac{1}{8\pi G}\int e\left(-e^{[\mu}_K e^{\nu]}_L \omega_\mu{}^{K}{}_M \omega_\nu{}^{ML} - \Lambda\right)$$

torsion free connection

One can further rewrite everything in terms of derivatives of the coframe. Introduce

$$t^I_{\mu\nu} := 2\partial_{[\mu} e^I_{\nu]}$$

Coordinate derivative. To make this object transform covariantly under gauge, use any flat spin connection

Then $2e^{\mu}_{S}\,\omega_{\mu}{}^{I}{}_{J} = t_{SJ}{}^{I} + t^{I}{}_{JS} + t_{JS}{}^{I}$

where $t^{I}{}_{JK} := e^{\mu}{}_{J}\, e^{\nu}_{K}\, t^{I}{}_{\mu\nu}$

Then the Lagrangian can be written explicitly in terms of the torsion t

schematically $t = \partial e$

$$S_{EC}[e] = \frac{1}{16\pi G}\int e\left(-\frac{1}{4}t_{LMK}t^{LMK} + \frac{1}{2}t_{MLK}t^{LMK} + t^{I}{}_{KI}\,t^{JK}{}_{J} - 2\Lambda\right)$$

The spin connection that is used to compute torsion $t^{I} = d^{\omega} e^{I}$ is flat (zero)

Curvature has been traded for torsion in this way of thinking about GR

The Lagrangian is no longer unique - modifying the coefficients above get 2 parameter family of modifications of GR (with extra DOF)

Pure spin connection formulation

As it was possible to integrate out the metric when $\Lambda \neq 0$ to obtain Eddington formulation, it is similarly possible to integrate out the coframe.

However in 4D there is a technical difficulty. Need to solve

$$\epsilon_{IJKL}\, e^{J} R^{KL} = \frac{\Lambda}{3}\epsilon_{IJKL}\, e^{J} e^{K} e^{L}$$

This is a cubic equation for the co-frame e^{I}

Nobody knows how to solve it in closed form.

However, a perturbative solution around a maximally symmetric (constant curvature) background is possible

Zinoviev hep-th/0504210

Background $\quad R^{IJ} = \frac{\Lambda}{3} e^I e^J$

Linearization of Einstein equations

$$\epsilon_{IJKL}\, e^J\, d^\omega a^{KL} = \frac{2\Lambda}{3}\, \epsilon_{IJKL}\, \delta e^J e^K e^L$$

where we denoted $\delta\omega^{IJ} = a^{IJ}$

Solving this get

$$\delta e^I = \frac{3}{2\Lambda} \hat{f}^I{}_J\, e^J \qquad \text{where} \quad \hat{f}^I{}_J = f^I{}_J - \frac{1}{6}\delta^I_J f$$

$$f^I{}_J = f^{IK}{}_{JK}$$

Substituting into the action, and after some integration by parts manipulations get 1512.09060

$$f^{IJ}{}_{KL} := 2 e^\mu{}_K e^\nu{}_L\, d^\omega_{[\mu} a_{\nu]}^{IJ}$$

$$S^{(2)}[a] = -\frac{3}{64\pi G\Lambda} \int e\, C^{KL}{}_{IJ}(a)\, C^{IJ}{}_{KL}(a)$$

In Euclidean signature this action has sign – non-positive

where $C^{IJ}{}_{KL}$ is a Weyl-like tensor

$$C^{IJ}{}_{KL} = f^{IJ}{}_{KL} - \left(\delta^I_{[K} f^J_{L]} - \delta^J_{[K} f^I_{L]} \right) + \frac{f}{3}\, \delta^I_{[K} \delta^J_{L]}$$

In contrast with the Einstein Hilbert action, whose Hessian is indefinite

The pure connection formalism does not have the "conformal mode problem" that plagues the Euclidean path integral

Typical of all pure connection actions – even the Eddington action (being integral of a square root) has fixed sign

$$S_{\text{Eddington}}[\Gamma] \sim \int \sqrt{-\det R_{(\mu\nu)}(\Gamma)}$$

MacDowell-Mansouri formulation

Essentially the viewpoint advocated by Cartan half a century before. Cartan connection is a new type of relation between tangent and abstract vector bundles

Cartan connection

Let P be the total space of the principal H bundle over M. The principal connection in P would be a Lie algebra of H valued 1-form in P (with some invariance properties and reducing to standard Mauer-Cartan form on H on vertical vectors)

In contrast, Cartan connection is a $\mathfrak{g}$-valued 1-form in P, where $\mathfrak{g}$ is the Lie algebra of $G \supset H$. Moreover, Cartan connection is required to be an isomorphism

$$A : T_p P \to \mathfrak{g}$$

So, Cartan connection A solders the geometry of the principal H bundle over M to that of Lie group $G \supset H$.
In other words, it identifies the tangent space to P at every point with the Lie algebra, and thus tries to infinitesimally identify P with the group G

Canonical examples of Cartan's connections arise via coset constructions of manifolds

$$M = G/H$$

Then the total space of the principal H bundle over M is the group manifold G
Cartan's connection on M is $\quad A = g^{-1}dg \quad g \in G$

For the case relevant for us here H is the Lorentz group of appropriate signature, and G is de Sitter or AdS group of isometries, depending on the sign of Λ

Cartan connection is an object that wants to locally identify the total space of the principal Lorentz bundle over spacetime with G, or locally identify M with the group coset

$$M = \begin{cases} SO(1,4)/SO(1,3) & \Lambda > 0 \\ SO(2,3)/SO(1,3) & \Lambda < 0 \end{cases}$$

Concretely, Cartan connection mixes the spin connection with the coframe

$$A = \omega^{IJ} K_{IJ} + e^I P_I \sqrt{\frac{|\Lambda|}{3}}$$

generators of Lorentz

generators of "translations"

$$F(A) = \left(R^{IJ}(\omega) - \frac{\Lambda}{3} e^I e^J\right) K_{IJ} + d^\omega e^I \, P_I \sqrt{\frac{|\Lambda|}{3}}$$

Flatness of A is the constant curvature + torsion-free

Introduce $F^{IJ} = R^{IJ}(\omega) - \frac{\Lambda}{3} e^I e^J$

the Lorentz part of the curvature of Cartan connection

MacDowell-Mansouri action

$$S_{MM}[e,\omega] = -\frac{3}{64\pi G\Lambda} \int \epsilon_{IJKL} F^{IJ} F^{KL}$$

reduces to Einstein-Cartan + Euler characteristic

This action is of the form

$$S[A] = \int Q_{AB}\, F^A F^B$$

A, B – Lie algebra indices for G gauge group

Q_{AB} is an H invariant symmetric tensor

In our case $A = \begin{cases} [IJ] \\ I5 \end{cases}$ and $Q_{AB} = \epsilon_{IJKL}$

Lorentz, but not G-invariant

If we used a G-invariant Q_{AB} would obtain topological invariant – Pontryagin number of the bundle on which Cartan connection lives

Clear that this idea is more general than just the formulation of GR reviewed. Can also obtain supergravity this way $G = Osp(1|4)$

Remarks:

- The action is zero on the constant curvature background
- Total divergence term expressed as a boundary term is just the correct one for volume renormalization in asymptotically AdS or hyperbolic cases.
- Much easier to obtain the linearized pure connection action starting with MM – no need for integration by parts manipulations. Directly get the $(\mathrm{Weyl})^2$ form of the Lagrangian
- The MM action is likely the best (non-chiral) first order Lagrangian for GR

Stelle–West version of the MacDowell–Mansouri Lagrangian

The idea is to write everything in manifestly G-invariant form, but introduce a field that breaks the symmetry to H

Let $a, b, \dots$ be $\mathbb{R}^{1,4}$ or $\mathbb{R}^{2,3}$ indices

so that the Lie algebra index for G is a pair $[ab]$

$$S_{SW}[A,v] = -\frac{3}{64\pi G\Lambda}\int v^e \epsilon_{abcde} F^{ab} F^{cd} \quad - \frac{\mu}{2}(|v|^2 - 1)$$

Lagrange multiplier field that fixes v to lie on an appropriat orbit in $\mathbb{R}^{1,4}$ or $\mathbb{R}^{2,3}$

Can integrate v, μ out!

$$\epsilon_{abcde} F^{ab} F^{cd} = \mu v_e$$

$$\Rightarrow \quad v_e = \frac{1}{\mu}\sqrt{\epsilon_{abcde} F^{ab} F^{cd}}$$

Constraint gives

$$\mu = \sqrt{|\epsilon_{abcde} F^{ab} F^{cd}|^2}$$

$$S_{SW}[A] = -\frac{3}{64\pi G\Lambda}\int \sqrt{|\epsilon_{abcde} F^{ab} F^{cd}|^2}$$

Unfortunately, not a useful starting point for perturbation theory, because wants to expand the square root around $F = 0$ configuration