Lorentzian Cayley Form

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Urbantke formula

In 4D metric can be recovered from (described by) a triple of 2-forms

$$\Sigma^{1,2,3} \in \Lambda^2(\mathbb{R}^4)$$

$$g(\xi,\eta)\, vol_g = \frac{1}{6}\, i_\xi \Sigma^i \wedge i_\eta \Sigma^j \wedge \Sigma^k \epsilon^{ijk}$$

Works with real 2-forms $\longrightarrow$ metrics of signature $++++$, $++--$

Can also make it work to get metrics of Lorentzian signature

Need complex 2-forms $\quad \Sigma^i \wedge \overline{\Sigma^j} = 0$

2-forms Σ^i become self-dual in the metric they define

Cayley form

A 4-form $\Phi \in \Lambda^4(\mathbb{R}^8)$ is called Cayley if there exists a compatible with it inner product (i.e. if it defines a metric in $\mathbb{R}^8$)

$$\left(g(\xi_1,\eta_1)\,g(\xi_2,\eta_2) - g(\xi_1,\eta_2)\,g(\xi_2,\eta_1)\right)\mathrm{vol}_g$$

$$= \frac{1}{6}\, i_{\xi_1} i_{\xi_2}\Phi \wedge i_{\eta_1} i_{\eta_2}\Phi \wedge \Phi$$

There are clear similarities between the 4D and 8D formulas.

Similar to 4D, one can obtain from a real Cayley form the two signatures – Riemannian
Split

One can recover the 4D formula from 8D by a variant of dimensional reduction

For this we need the notion of calibration

A 4-dimensional subspace H of $\mathbb{R}^8$ is called a calibration if for any orthonormal basis in H we have $\Phi(\xi_1, \xi_2, \xi_3, \xi_4) = \pm 1$

Or, in other words $\Phi|_H = \mathrm{vol}_H$

Choosing H, and a basis of self-dual bivectors $\sigma^{1,2,3} \in \Lambda^+(H)$, we can define $\Sigma^i = \sigma^i \lrcorner \Phi \Big|_{H^\perp} \in \Lambda^2(H^\perp)$

The metric g_Φ on $H^\perp$ is given by the Urbantke formula

Puzzle

Urbantke formula also works for Lorentzian signature (if $\Sigma^{1,2,3}$ are complex and $\Sigma^i \wedge \bar{\Sigma}^j = 0$)

Is there any version of the Cayley form in 8D that can give the Lorentzian Urbantke formula by the dimensional reduction?

Yes!

And this is what this talk is about

The answer comes from spinors

Cayley forms and spinors

A Cayley form $\Phi \in \Lambda^4(\mathbb{R}^8)$ is a differential form of a special algebraic type (can be characterised intrinsically, but not important for us)

Its stabiliser in $GL(8,\mathbb{R})$ is $Spin(7)$

This is also the stabiliser of a Majorana-Weyl spinor of $Spin(8)$

Cayley forms = Metrics + unit Majorana-Weyl spinors

$$\dim GL(8,\mathbb{R})/Spin(7) = 64 - 21 = 43$$

Metrics in 8D 36 + Unit spinors 7 = 43

More explicitly, given a metric that Φ defines, can construct the corresponding Clifford algebra $\gamma : \mathbb{R}^8 \to \mathrm{End}(S)$

$$S = S^+ \oplus S^- \qquad \dim_{\mathbb{R}} S^{\pm} = 8$$

Then $\exists\, \Psi \in S_+$ such that $\langle \Psi, \Psi \rangle = 1$ and

$$\Phi = \langle \Psi, \gamma_I \gamma_J \gamma_K \gamma_L \Psi \rangle \, e^I \wedge e^J \wedge e^K \wedge e^L$$

For every unit spinor there is a Cayley form whose compatible metric is the one we started from

In the opposite direction, every Cayley form determines a metric and the Clifford algebra in which it takes this form

Complex Cayley forms

Defn: A complex 4-form $\Phi \in \Lambda^4_{\mathbb{C}}(\mathbb{R}^8)$ is called a Complex Cayley form if it admits a compatible with it real metric

Example: Let us take the Clifford algebra for $\mathbb{R}^8$ with the usual Euclidean metric. Take a complex unit spinor Ψ_c. The form

$$\Phi_c = \langle \Psi, \gamma_I \gamma_J \gamma_K \gamma_L \Psi \rangle e^I e^J e^K e^L$$

is complex, but its compatible metric is real, and is just the metric on $\mathbb{R}^8$ used in the construction of the Clifford algebra γ_I

This construction works with complex unit spinors in any signature $\mathbb{R}^{p,q}$ $p+q=8$

But it is most interesting in signatures Riemannian and split, where we have link to octonions

Clifford algebras $C\ell(8)$, $C\ell(4,4)$ admit Majorana-Weyl spinors (i.e. meaningful to impose reality of $S_\pm$)

Majorana-Weyl spinors of $Spin(8)$ = Octonions

———— " ———— of $Spin(4,4)$ = Split octonions

To understand real and complex unit spinors of $Spin(8)$, $Spin(4,4)$ convenient to describe the Clifford algebras in octonionic terms

Octonionic models of $Cl(8)$, $Cl(4,4)$

These can be derived from other models of Clifford, for example from the model that is based on $S_{Spin(2n)} = \Lambda(\mathbb{C}^n)$

But for purposes of this talk can just postulate the model

$Cl(3)$ is generated by Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$X := x^i \sigma^i = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix} = \begin{pmatrix} t & \bar{x} \\ x & -t \end{pmatrix} \qquad \begin{matrix} t \in \mathbb{R} \\ x \in \mathbb{C} \end{matrix}$$

If replace $x \in \mathbb{C} \longrightarrow x \in \mathbb{H}$ or $\mathbb{O}$

get $Cl(5)$ or $Cl(9)$ respectively

Model of $C\ell(8)$

$$\Gamma_x = \begin{pmatrix} 0 & L_{\bar{x}} \\ L_x & 0 \end{pmatrix}$$

L_x - left multiplication by an octonion $x \in \mathbb{O}$

Spinor is a 2-component column $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ $\alpha, \beta \in \mathbb{O}$

We can now understand spinors of Spin(8) easily

Classification of spinors of Spin(8)

- Real (Majorana-Weyl) = octonions

  Stabiliser Spin(7)

  For example $\Psi^+ = \mathbb{1}$ identity octonion

- Complex null = complexified null octonion

  $\langle \Psi, \Psi \rangle = 0$ $\Psi^+ = \alpha_1 + i\alpha_2$ Stabiliser SU(4)

  Pure spinor $|\alpha_1|^2 = |\alpha_2|^2$ $(\alpha_1, \alpha_2) = 0$

- General complex spinor = complexified octonion

  $\langle \Psi, \Psi \rangle \neq 0$ Can always make unit by a complex rescaling

$\langle \Psi, \Psi \rangle = 1$ $|\alpha_1|^2 - |\alpha_2|^2 = 1$ $(\alpha_1, \alpha_2) = 0$

So, can parametrise $|\alpha_1| = \cosh \tau$

$|\alpha_2| = \sinh \tau$

$$\Psi^+ = \alpha_1 + i\alpha_2 = e^{\tau} \underbrace{\frac{1}{2}\left(\frac{\alpha_1}{|\alpha_1|} + i \frac{\alpha_2}{|\alpha_2|} \right)}_{\Psi_p} + e^{-\tau} \underbrace{\frac{1}{2}\left(\frac{\alpha_1}{|\alpha_1|} - \frac{i\alpha_2}{|\alpha_2|} \right)}_{\Psi_p^*}$$

a pair of complexified null octonions (pure spinors)

Stabiliser still $SU(4)$

(same as that of Ψ_p)

Complex Cayley Form

Can now compute Φ for a complex unit spinor.
But it is useful to first understand the usual real Cayley form in a similar way

Let $\Psi^+ = \alpha_1$ – real unit octonion

Choose $\alpha_2 \in \mathbb{O}$: $|\alpha_2| = 1$ and $(\alpha_1, \alpha_2) = 0$

Form $\Psi_p = \frac{1}{2}(\alpha_1 + i\alpha_2)$ – complex null octonion (pure spinor)

$$\Psi^+ = \Psi_p + \Psi_p^*$$

The choice of α_2 and thus Ψ_p is that of an $SU(4)$ subgroup

$$SU(4) \subset Spin(7)$$

stabiliser of Ψ_p stabiliser of Ψ^+

$$\Phi = \langle \Psi^+, \gamma\gamma\gamma\gamma \Psi^+ \rangle = \mathrm{Re}(\Omega) - \frac{1}{2}\omega \wedge \omega$$

$$\Omega = e^1 \wedge e^2 \wedge e^3 \wedge e^4 \quad \in \Lambda^{(4,0)}$$ for the complex structure that corresponds to the pure Spinor Ψ_p

$$\omega = e^1 \wedge \overline{e^1} + e^2 \wedge \overline{e^2} + e^3 \wedge \overline{e^3} + e^4 \wedge \overline{e^4}$$ Kähler form

One of the standard expressions for Φ once a complex structure in $\mathbb{R}^8$ is chosen

Only $SU(4)$ is manifest, while Φ is actually $Spin(7)$ invariant

Complex Cayley form is produced by a complex unit spinor

$$\Psi = e^{\tau}\Psi_p + e^{-\tau}\Psi_p^*$$

$$\Phi_c = \cosh(2\tau)\,\mathrm{Re}\,\Omega - \frac{1}{2}\omega\wedge\omega + i\sinh(2\tau)\,\mathrm{Im}(\Omega)$$

When $\tau = 0$ coincides with Φ – usual real Cayley

For $\tau \neq 0$ the stabiliser is $SU(4)$

Complex Cayley Forms $GL(8,\mathbb{R})/SU(4)$ = Riemannian Metrics + Complex unit Spinors

$\langle\Psi,\Psi\rangle = 1$

$\langle\Psi,\Psi^*\rangle = \cosh 2\tau$

$64 - 15 = 49 \qquad 36 \qquad + \; 13$

This complex Cayley form is interesting because it encodes an $SU(4)$ structure (ω, Ω) in a single complex 4-form Φ_c

Proposition: There is a one-to-one correspondence between integrable $SU(4)$ structures (g, ω, Ω) and closed complex Cayley forms $d\Phi_c = 0$

Similar to Φ can write Φ_c in a basis adapted to a calibration. Unlike Φ, the expression now involves not just self-dual 2-forms for $H, H^\perp$, but the full basis of one-forms Only $SO(4) \subset SU(4)$ is manifest in this way of writing

Lorentzian Cayley Forms

These arise as special types of complex Cayley forms for Spin(4,4)

Octonionic model for Cl(4,4)

$$P_x = \begin{pmatrix} 0 & L_{\bar{x}} \\ L_x & 0 \end{pmatrix}$$

L_x — left multiplication by a split octonion $x \in \mathbb{O}'$

split octonions

Types of spinors of Spin(4,4)

This is now much richer than in the case Spin(8)

There are now 3 different types of pure spinors

- real null split octonions   real index 4
- complex pure spinor (null split octonion) Ψ_p such that its complementary spinor $\langle \Psi_p, \overline{\Psi}_p \rangle \neq 0$

gives $\Psi_p + \overline{\Psi_p}$ real   real index 0

- real index 2 pure spinor

$$\langle \Psi_p, \overline{\Psi_p} \rangle \neq 0 \qquad \Psi_p + \overline{\Psi_p} \text{ complex}$$

It is this last case that is relevant for the Lorentzian Cayley form that we want to build. But can be described more directly

Definition: Let Ψ_L be a complex unit spinor of $Spin(4,4)$ such that $\langle \Psi_L, \Psi_L \rangle = 1$ and $\langle \Psi_L, \Psi_L^* \rangle = 0$

$$\Psi_L = \alpha_1 + i\alpha_2 \qquad |\alpha_1|^2 - |\alpha_2|^2 = 1 \qquad (\alpha_1, \alpha_2) = 0$$

$$|\alpha_1|^2 + |\alpha_2|^2 = 0$$

only possible for split octonions where $|\cdot|^2$ is not definite

Proposition: The stabiliser of Ψ_L is $SL(4,\mathbb{R})$, same as the stabiliser of a pair of real complementary pure spinors.

In fact, there exists a pair $\Psi_P, \overline{\Psi_P}$ of real pure spinors with $\langle \Psi_P, \overline{\Psi_P} \rangle = 1/2$

Such that
$$\Psi_L = \frac{(1+i)}{\sqrt{2}} \Psi_P + \frac{(1-i)}{\sqrt{2}} \overline{\Psi_P}$$

Definition: The Lorentzian Cayley form is the complex Cayley form for Ψ_L

$$\varphi_L := \langle \Psi_L, \gamma\gamma\gamma\gamma \Psi_L \rangle$$

Paracomplex structures

There exists a real analog of complex structures on $\mathbb{R}^8$ on $\mathbb{R}^{4,4}$

Definition: An orthogonal paracomplex structure on $\mathbb{R}^{4,4}$ is

$K: \mathbb{R}^{4,4} \to \mathbb{R}^{4,4}$: $\quad K^2 = +\mathbb{1}$ and $\quad g(K\xi, K\eta) = -g(\xi,\eta)$

Proposition: The eigenspaces of K are totally null (and real).

The subgroup of $O(4,4)$ that commutes with K is $GL(4,\mathbb{R})$

— the general linear group that acts on the four eigendirections of K

Proposition: A pair $\Psi_p, \bar{\Psi}_p$ of complementary $\langle \Psi_p, \bar{\Psi}_p \rangle \neq 0$ real pure spinors of $Spin(4,4)$ defines an orthogonal paracomplex structure on $\mathbb{R}^{4,4}$, as well as two 4-forms

$\omega_r = e_1\bar{e}_1 + e_2\bar{e}_2 + e_3\bar{e}_3 + e_4\bar{e}_4 \qquad \Omega_r = e_1e_2e_3e_4 \quad \bar{\Omega}_r = \bar{e}_1\bar{e}_2\bar{e}_3\bar{e}_4$

In other words, such a pair defines an $SL(4,\mathbb{R})$ structure $(\omega_r, \Omega_r, \bar{\Omega}_r)$ on $\mathbb{R}^{4,4}$

Proposition: The stabiliser of a Lorentzian Cayley form Φ_L is $SL(4,\mathbb{R})$. It defines an $SL(4,\mathbb{R})$ structure $(g, \omega_r, \Omega_r, \bar{\Omega}_r)$ in terms of which Φ_L has the following expression

$$\Phi_L = \frac{i}{2}\Omega_r - \frac{i}{2}\bar{\Omega}_r - \frac{1}{2}\omega_r \wedge \omega_r$$

Lorentzian Cayley forms $=$ Split signature metrics in 8D $+$ Complex unit spinors $\langle \Psi_L, \Psi_L \rangle = 1$ that are orthogonal to their complex conjugates $\langle \Psi_L, \Psi_L^* \rangle = 0$

$GL(8,\mathbb{R}) / SL(4,\mathbb{R})$

Lorentzian Cayley forms and calibrations

Definition: Given a complex Cayley form $\Phi_C \in \Lambda^4_{\mathbb{C}} TM$ we say $H \subset \underline{TM}$ is a calibration if for any orthonormal basis $\xi_1, \xi_2, \xi_3, \xi_4 \in H$ $\quad \Phi_C(\xi_1, \xi_2, \xi_3, \xi_4) = \text{const} \in \mathbb{C}$

Proposition: Lorentzian Cayley forms are calibrated by 4-dimensional subspaces with either Lorentzian or split metric

Given H, there exists an orthonormal basis $e^{0,1,2,3,4,5,6,7}$ such that

$$\Phi_L = -\frac{1}{6}\Sigma^i_L \Sigma^i_L - \frac{1}{6}\Sigma^{i'}_L \Sigma^{i'}_L - \Sigma^i_L \Sigma^{i'}_L$$

$$\Sigma^1_L = ie^{45} - e^{67} \qquad \Sigma^2 = ie^{46} - e^{75} \qquad \Sigma^3 = ie^{47} - e^{56}$$

$$\Sigma^{1'}_L = ie^{01} - e^{23} \qquad \Sigma^{2'}_L = ie^{02} - e^{31} \qquad \Sigma^{3'}_L = ie^{03} - e^{12}$$

two self-dual Lorentzian triples
for e^0, e^1, e^2, e^3 and e^4, e^5, e^6, e^7

Can also be written in the form that exhibits
$\mathbb{R}^{2,2} \oplus \mathbb{R}^{2,2}$ calibration

The metric has one sign on $e^{1,2,3,4}$
opposite sign on $e^{0,5,6,7}$

This proposition justifies the name "Lorentzian Cayley forms"

Proposition: Alternative expression for φ_L

$$\varphi_L = i e^0 \wedge C_L + C_L^*$$

$$C_L = e^{123} - e^1(ie^{45} - e^{67}) - e^2(ie^{46} - e^{75}) - e^3(ie^{47} - e^{56})$$

$$C_L^* = ie^{4567} + e^{23}(ie^{45} - e^{67}) + e^{31}(ie^{46} - e^{75}) + ie^{12}(ie^{47} - e^{56})$$

Canonical 3- and 4-forms for a G_2 structure, but built from Lorentzian self-dual 2-forms on $e^{4,5,6,7}$!

Proposition: (Dimensional reduction)

Given a Lorentzian calibration $\Phi|_H = i\,\mathrm{vol}_H$

let $\sigma^i \in \Lambda^+(H)$ be a (complex) triple of self-dual bivectors for H. Define $\Sigma^i := \sigma^i \lrcorner \Phi|_{H^\perp} \in \Lambda^2_{\mathbb{C}}(H^\perp)$

Then the metric g_Φ on $H^\perp$ is the Lorentzian Urbantke metric constructed from Σ^i

Reproduces the Lorentzian Urbantke formula as coming from dimensional reduction from 8D

Summary

- Usual Cayley forms are metrics + unit real spinors
- Can define the notion of complex Cayley forms

  real metrics + complex unit spinors

- This can be done for any signature in 8D, but most interesting when real Cayley forms exist
- There is a version of the complex Cayley form for the split signature that is calibrated by Lorentzian 4-dimensional subspaces
- This Lorentzian Cayley form is the sought 8D version of the Lorentzian 4D Urbantke formula