Perturbative Quantum Gravity

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Review of some of modern developments in the subject

(designed for a mathematical audience)

Why mathematicians are an appropriate audience?

- (Perturbative) QG is a mess if approached in a brute force way
- Simplicity every time one gets through the mess

Everything this subject needs is some level of mathematical sophistication?

Why is it relevant to study this subject now?

- Abandoned in the 80's in favour of other approaches to quantum gravity, notably string theory
- Supersymmetry (needed by superstring) was widely expected to be discovered by the new generation of accelerators
- No SUSY was discovered at LHC
- Possible that the Standard Model of elementary particles + Gravity is everything there is up to the Planck scale

It is time to re-evaluate the status of the field theoretic version of quantum gravity

Outline of the talk

- Review of gravitational perturbation theory and quantization
- New beautiful results about "gravitons" in the last 10 years

  "On-shell methods"

  Simplicity points to some underlying structure yet to be discovered

- New gauge-theoretic formulation of gravity

  Simplifies perturbation theory

  Suggests a fresh look at old problems

Classical General Relativity

Spacetime dimension D=4

Consider "pure gravity"

World without material sources

dynamical theory of a spacetime metric

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G} \int \sqrt{-\det(g)}(R - 2\lambda)$$

Einstein-Hilbert action

R - scalar curvature

Λ - cosmological constant

G - Newton's constant

Einstein metrics are critical points of EH functional

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

Highly non-linear second order PDE on $g_{\mu\nu}$

Perturbation theory

Set $\Lambda = 0$ for now

Physically well-motivated because Λ is small

Minkowski metric

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

is a solution of field equations

$$\kappa^2 := 32\pi G$$

Expand $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$

set $\kappa = 1$ from now on

$$\mathcal{L}^{(2)} = -\frac{1}{2}(\partial_\rho h_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu h)^2 + (\partial^\nu h_{\mu\nu})^2 + h\partial^\mu\partial^\nu h_{\mu\nu}$$

where $h := \eta^{\mu\nu} h_{\mu\nu}$

Gravitons

Define $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$

Linearized field equations

where

$\Box := \partial^\mu \partial_\mu$

$$\Box \bar{h}_{\mu\nu} + \eta_{\mu\nu} \partial^\mu \partial^\nu \bar{h}_{\mu\nu} - 2\partial_{(\mu} \partial^\rho \bar{h}_{\nu)\rho} = 0$$

Everything is invariant under diffeomorphisms

$$\delta h_{\mu\nu} = 2\partial_{(\mu} \xi_{\nu)}$$

Can "fix the gauge"

$$\partial^\nu \bar{h}_{\mu\nu} = 0$$

Field equations become

$$\Box \bar{h}_{\mu\nu} = 0$$

solutions-
plane waves-
gravitons

Mode expansion

only two polarizations propagate

amplitudes of two different polarizations

$$h^k_{\mu\nu} = a^+_k \epsilon^+_{\mu\nu}(k) e^{ikx} + a^-_k \epsilon^-_{\mu\nu}(k) e^{ikx}$$

two polarization tensors

2+2 numbers per spatial point as initial data

Polarization tensors satisfy

$$(\epsilon^{\pm}_{\mu\nu}(k))^2 = 0$$

$$\epsilon^+_{\mu\nu}(k)\epsilon^{-\mu\nu}(k) = 1$$

(Positive / Negative) helicity == only (Self-dual / Anti-self-dual) part of Weyl is non-vanishing

Einstein gravity perturbatively: Nasty mess...

Expansion around an arbitrary background $g_{\mu\nu}$

quadratic order (together with the gauge-fixing term)

$$L_{g.f.} = -\sqrt{-g}\left(h^{\mu\nu}{}_{;\nu} - \frac{1}{2}h_{\nu}{}^{\nu;\mu}\right)\left(h^{\rho}{}_{\mu;\rho} - \frac{1}{2}h^{\rho}{}_{\rho;\mu}\right)$$

$$L_2 = \sqrt{-g}\Big\{-\frac{1}{2}h^{\alpha\beta}{}_{;\gamma}h_{\alpha\beta}{}^{;\gamma} + \frac{1}{4}h^{\alpha}{}_{\alpha;\gamma}h_{\beta}{}^{\beta;\gamma} + h_{\alpha\beta}h_{\gamma\delta}R^{\alpha\gamma\beta\delta} - h_{\alpha\beta}h^{\beta}{}_{\gamma}R^{\delta\alpha\gamma}{}_{\delta}$$
$$+ h^{\alpha}{}_{\alpha}h_{\beta\gamma}R^{\beta\gamma} - \frac{1}{2}h_{\alpha\beta}h^{\alpha\beta}R + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}R\Big\}.$$

from Goroff-Sagnotti "2-loop" paper

cubic order

$$L_3 = \sqrt{-g}\Big\{-\frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}{}_{;\alpha}h_{\gamma\delta;\beta} + 2h^{\alpha\beta}h^{\gamma\delta}{}_{;\alpha}h_{\beta\gamma;\delta} - h^{\alpha\beta}h^{\gamma}{}_{\gamma;\alpha}h^{\delta}{}_{\beta;\delta} - \frac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta\gamma;\delta}h_{\beta\delta;\gamma}$$
$$+ \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta\gamma;\delta}h_{\beta\gamma;\delta} - h^{\alpha\beta}h^{\gamma}{}_{\gamma;\delta}h^{\delta}{}_{\alpha;\beta} + \frac{1}{2}h^{\alpha\beta}h^{\gamma}{}_{\gamma;\alpha}h^{\delta}{}_{\delta;\beta} - h^{\alpha\beta}h_{\alpha\beta;\gamma}h^{\gamma\delta}{}_{;\delta}$$
$$+ \frac{1}{2}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta;\gamma}h^{\gamma\delta}{}_{;\delta} + h^{\alpha\beta}h_{\alpha\beta;\gamma}h_{\delta}{}^{\delta;\gamma} + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta;\gamma}h_{\delta}{}^{\delta;\gamma} - h^{\alpha\beta}h^{\gamma}{}_{\alpha;\delta}h_{\beta\gamma}{}^{;\delta}$$
$$+ h^{\alpha\beta}h^{\gamma}{}_{\alpha;\delta}h^{\delta}{}_{\beta;\gamma} + R_{\alpha\beta}(2h^{\alpha\gamma}h_{\gamma\delta}h^{\beta\delta} - h^{\gamma}{}_{\gamma}h^{\alpha\delta}h^{\beta}{}_{\delta} - \frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta}$$
$$+ \frac{1}{4}h^{\alpha\beta}h^{\gamma}{}_{\gamma}h^{\delta}{}_{\delta}) + R(-\frac{1}{3}h^{\alpha\beta}h_{\beta\gamma}h^{\gamma}{}_{\alpha} + \frac{1}{4}h^{\alpha}{}_{\alpha}h^{\beta\gamma}h_{\beta\gamma} - \frac{1}{24}h^{\alpha}{}_{\alpha}h^{\beta}{}_{\beta}h^{\gamma}{}_{\gamma})\Big\}$$

even in flat space, the corresponding vertex has about 100 terms!

quartic order

$$
\begin{aligned}
L_4 = \sqrt{-g}\Big\{&(h^\alpha{}_\alpha h^\beta{}_\beta - 2h^{\alpha\beta}h_{\alpha\beta})\Big(\frac{1}{16}h^{\gamma\delta;\sigma}h_{\gamma\delta;\sigma} - \frac{1}{8}h^{\gamma\delta;\sigma}h_{\gamma\sigma;\delta} + \frac{1}{8}h^\gamma{}_{\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} \\
&- \frac{1}{16}h^\gamma{}_{\gamma;\delta}h_\sigma{}^{\sigma;\delta}\Big) + h^\alpha{}_\alpha h^{\beta\gamma}\Big(-\frac{1}{2}h_{\beta\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} + \frac{1}{2}h_{\beta\gamma;\delta}h_\sigma{}^{\sigma;\delta} - \frac{1}{2}h^\delta{}_{\delta;\beta}h^\sigma{}_{\sigma;\gamma} \\
&+ \frac{1}{4}h^\delta{}_{\delta;\beta}h^\sigma{}_{\sigma;\gamma} + h^\delta{}_{\beta;\sigma}h^\sigma{}_{\delta;\gamma} - \frac{1}{4}h^{\delta\sigma}{}_{;\beta}h_{\delta\sigma;\gamma} - \frac{1}{2}h^\delta{}_{\beta;\sigma}h_{\delta\gamma}{}^{;\sigma} - \frac{1}{2}h^\delta{}_{\delta;\sigma}h^\sigma{}_{\beta;\gamma} \\
&+ \frac{1}{2}h_{\beta\delta;\sigma}h_\gamma{}^{\sigma;\delta}\Big) + h^\alpha{}_\beta h^{\beta\gamma}\Big(h^\delta{}_{\delta;\sigma}h^\sigma{}_{\alpha;\gamma} - h_{\alpha\gamma;\delta}h_\sigma{}^{\sigma;\delta} + \frac{1}{2}h^{\delta\sigma}{}_{;\alpha}h_{\delta\sigma;\gamma} \\
&- h^\delta{}_{\alpha;\sigma}h^\sigma{}_{\gamma;\delta} - 2h^\delta{}_{\alpha;\sigma}h^\sigma{}_{\delta;\gamma} + h_{\alpha\gamma;\delta}h^{\delta\sigma}{}_{;\sigma} + h^\delta{}_{\delta;\alpha}h^\sigma{}_{\gamma;\sigma} - \frac{1}{2}h^\delta{}_{\delta;\alpha}h^\sigma{}_{\sigma;\gamma} \\
&+ h^\delta{}_{\alpha;\sigma}h_{\gamma\delta}{}^{;\sigma}\Big) + h^{\alpha\gamma}h^{\beta\delta}\Big(h_{\alpha\gamma;\beta}h^\sigma{}_{\delta;\sigma} - h_{\alpha\gamma;\delta}h^\sigma{}_{\sigma;\beta} + \frac{1}{2}h_{\alpha\beta;\sigma}h_{\gamma\delta}{}^{;\sigma} \\
&- \frac{1}{2}h_{\alpha\gamma;\sigma}h_{\beta\delta}{}^{;\sigma} + h^\sigma{}_{\alpha;\beta}h_{\gamma\sigma;\delta} - h^\sigma{}_{\alpha;\beta}h_{\delta\sigma;\gamma} + h_{\alpha\beta;\delta}h^\sigma{}_{\sigma;\gamma} - 2h^\sigma{}_{\alpha;\beta}h_{\delta\gamma;\sigma} \\
&+ h_{\alpha\gamma;\sigma}h^\sigma{}_{\beta;\delta}\Big) + R_{\alpha\beta}\Big(-2h^{\alpha\gamma}h_{\gamma\delta}h^{\delta\sigma}h_\sigma{}^\beta + h^\gamma{}_\gamma h^{\alpha\delta}h_{\delta\sigma}h^{\sigma\beta} + \frac{1}{2}h^{\alpha\gamma}h_\gamma{}^\beta h^{\delta\sigma}h_{\delta\sigma} \\
&- \frac{1}{4}h^{\alpha\gamma}h_\gamma{}^\beta h^\delta{}_\delta h^\sigma{}_\sigma + \frac{1}{3}h^{\alpha\beta}h^{\gamma\delta}h_{\delta\sigma}h^\sigma{}_\gamma - \frac{1}{4}h^{\alpha\beta}h^\gamma{}_\gamma h^{\delta\sigma}h_{\delta\sigma} + \frac{1}{24}h^{\alpha\beta}h^\gamma{}_\gamma h^\delta{}_\delta h^\sigma{}_\sigma\Big) \\
&+ R\Big(-\frac{1}{192}h^\alpha{}_\alpha h^\beta{}_\beta h^\gamma{}_\gamma h^\delta{}_\delta + \frac{1}{16}h^\alpha{}_\alpha h^\beta{}_\beta h^{\gamma\delta}h_{\gamma\delta} + \frac{1}{4}h^{\alpha\beta}h_{\beta\gamma}h^{\gamma\delta}h_{\delta\alpha} \\
&- \frac{1}{16}h^{\alpha\beta}h_{\alpha\beta}h^{\gamma\delta}h_{\gamma\delta} - \frac{1}{6}h^\alpha{}_\alpha h^{\beta\gamma}h_{\gamma\delta}h^\delta{}_\beta\Big)\Big\}
\end{aligned}
$$

Imagine having to do calculations with these interaction vertices!

Perturbative quantization

Want to evaluate the "path integral" for the theory "perturbatively"

Compare to
$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}\alpha x^2 - \frac{1}{4!}\lambda x^4} = \int_{-\infty}^{+\infty} dx \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{4!}\lambda x^4\right)^n e^{-\frac{1}{2}\alpha x^2}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{4!}\lambda \left(\frac{d}{dy}\right)^4\right)^n \int_{-\infty}^{+\infty} dx\, e^{-\frac{1}{2}\alpha x^2 + xy}\Big|_{y=0}$$

$$= \sqrt{\frac{2\pi}{\alpha}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{4!}\lambda \left(\frac{d}{dy}\right)^4\right)^n e^{y^2/2\alpha}\Big|_{y=0}$$

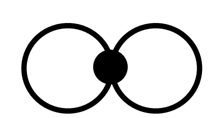
field theory in zero dimensions

graph combinatorics

$$= \sqrt{\frac{2\pi}{\alpha}} \left(\sum_{\text{4-valent graphs}} \frac{1}{\text{symmetry factor}} \prod_{\text{edges}} \frac{1}{\alpha} \prod_{\text{vertices}} (-\lambda) \right)$$

On the other hand, setting $\alpha = 1$

$$= \sqrt{\frac{3}{\lambda}}\, e^{\frac{3}{4\lambda}} K_{1/4}\left(\frac{3}{4\lambda}\right)$$



expanding $\frac{1}{\sqrt{2\pi}} \times \text{Integral} = 1 - \frac{\lambda}{8} + \frac{35\lambda^2}{384} - \frac{385\lambda^3}{3072} + \frac{25025\lambda^4}{98304} + O(\lambda^5)$

Perturbative quantization in field theory

Want to compute "correlation functions"

$$\langle h_{\mu_1\nu_1}(x_1)\dots h_{\mu_n\nu_n}(x_n)\rangle := \int \mathcal{D}h\, h_{\mu_1\nu_1}(x_1)\dots h_{\mu_n\nu_n}(x_n)e^{iS[h]}$$

interpreting this as a Gaussian integral plus perturbation

Similar sum over graphs with "Feynman rules"

$$\frac{1}{\alpha} \to G_{\mu\nu,\alpha\beta}(x-y)$$ "propagator"

satisfies $$\left(\delta^\rho_\mu\delta^\sigma_\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\right)\Box G_{\rho\sigma,\alpha\beta}(x-y) = \delta^4(x-y)$$

Need to multiply propagators, vertex contributions, then integrate over positions of vertices

Scattering amplitudes

Fourier transformed correlation functions

$$\langle h_{\mu_1\nu_1}(k_1)\dots h_{\mu_n\nu_n}(k_n)\rangle$$

Have simple poles at $k_i^2 = 0$

Residues at those poles are graviton scattering amplitudes

Projecting on polarization tensors

$$\mathcal{M}(1^{h_1}\dots n^{h_n}) := \epsilon^{h_1}_{\mu_1\nu_1}(k_1)\dots\epsilon^{h_n}_{\mu_n\nu_n}(k_n)\langle h^{\mu_1\nu_1}(k_1)\dots h^{\mu_n\nu_n}(k_n)\rangle$$

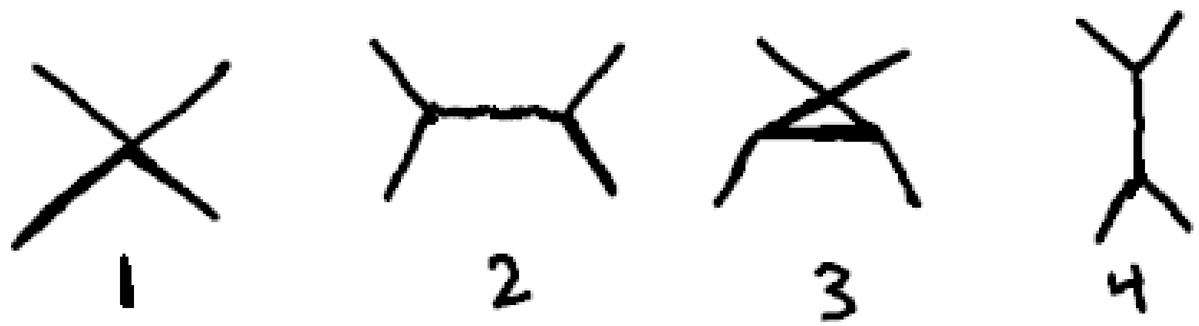
where $h_i = \pm$ "helicity"

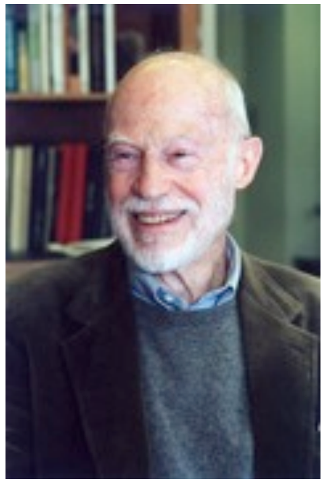
$\epsilon^{\pm}_{\mu\nu}(k)$ polarization tensors

Objects of main interest

First calculations

In 1963 I gave [Walter G. Wesley] a student of mine the problem of computing the cross section for a graviton-graviton scattering in tree approximation, for his Ph.D. thesis. The relevant diagrams are these:





Given the fact that the vertex function in diagram 1 contains over 175 terms and that the vertex functions in the remaining diagrams each contain 11 terms, leading to over 500 terms in all, you can see that this was not a trivial calculation, in the days before computers with algebraic manipulation capacities were available. And yet the final results were ridiculously simple.

From: Bryce DeWitt
arXiv:0805.2935
Quantum Gravity, Yesterday and Today

In modern notations

$$\mathcal{M}(1^-,2^-,3^+,4^+) = \frac{1}{4} s_{12} \frac{s_{12}}{s_{23}} \frac{s_{12}}{s_{24}}$$

where $\quad s_{ij} := (k_i + k_j)^2$

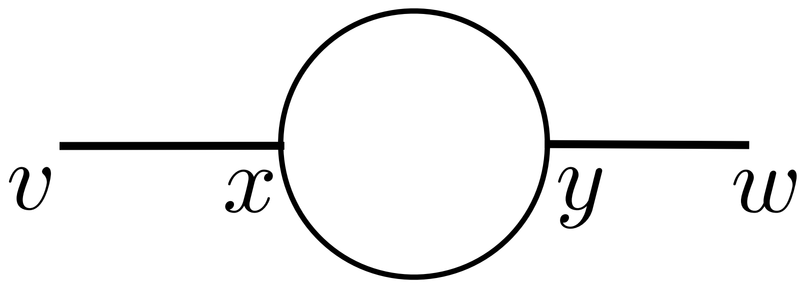
Amplitudes for more gravitons are very difficult to obtain

- too many diagrams to consider

Renormalization

In any field theory "loop diagrams" diverge

E.g. scalar field with interaction $\frac{\lambda}{3!}\phi^3$



$$\Delta(v-w) := \frac{1}{2}\int d^4x d^4y\, P(v-x)P(x-y)^2 P(y-w)$$

Propagator $P(x-y)$ is a distribution

$$\Box P(x-y) = \delta^4(x-y)$$

Product of propagators is ill-defined

Need to "regularize"

Dimensional regularization

Gelfand 50's

$$P(x) \sim \frac{1}{|x|^2}$$

Fourier transform

$$\int d^d x \, \frac{1}{|x|^{2\alpha}} e^{ikx} = \pi^{d/2} \frac{\Gamma\left(d/2-\alpha\right)}{\Gamma(\alpha)} \left(\frac{1}{4}k^2\right)^{\alpha-d/2}$$

This has poles when $\alpha - d/2 = n = 0, 1, \ldots$

$$\frac{1}{|x|^{2\alpha}} \sim \frac{1}{d/2-\alpha+n} \frac{\pi^{d/2}}{\Gamma(d/2+n)} \frac{1}{2^{2n} n!} \Box^n \delta^d(x)$$

E.g. $d = 4 - \epsilon$

$$P^2(x) \sim \frac{1}{\epsilon} \delta^4(x) + \mathrm{non-singular \ \ terms}$$

Renormalization

The singular "divergent" part of diagrams is "local", and can be absorbed into a "renormalization" of parameters (fields)

multiplicative renormalization

$$\phi \to Z_\phi \phi$$

$$\lambda \to Z_\lambda \lambda$$

$$Z = 1 + \frac{c}{\epsilon} + \dots$$

After this is done, finite parts are unambiguously computed in terms of renormalized parameters

Non-trivial mathematical structure: Connes and Kreimer Hopf algebra behind renormalization

This is enough in "renormalizable" field theories

Non-renormalizability of quantum gravity

"Dimensionful" coupling constant $\kappa \sim \sqrt{G} \sim \text{Length}$

Can do more interesting renormalizations

$$h_{\mu\nu} \to Z_h h_{\mu\nu} + \kappa^2 \left(\frac{\alpha}{\epsilon} R_{\mu\nu}(h) + \underbrace{\frac{\beta}{\epsilon}}_{\text{new}} \eta_{\mu\nu} R(h) \right) + \ldots$$

This is sufficient at 1-loop order

't Hooft, Veltman 74'

At two loops there is a divergence that cannot be removed by any renormalization

Goroff, Sagnotti '86
van de Ven '91

very difficult calculation: numerical work

$$\frac{\kappa^2}{\epsilon} \int (W_{\mu\nu}{}^{\rho\sigma})^3$$

$W_{\mu\nu\rho\sigma}$ - Weyl tensor

Need to add this term to the Lagrangian, and then an infinite number of other terms

Part II: New developments

Scattering amplitudes

Witten '03
twistor string

Consider tree-level graviton scattering amplitude $\mathcal{M}(1^{h_1} \dots n^{h_n})$

Can be seen to be a meromorphic function of k_i with $k_i^2 = 0$

with simple poles at $$\left(\sum_{i\in I} k_i\right)^2 = 0$$

I - some subset of momenta

Can show that vanishes when all h_i are plus or all are minus

At least one plus and at least one minus

Label $1^-, 2^+$

Choose

$$q : q^2 = 0, \quad k_{1,2} \cdot q = 0$$

q necessarily complex

Then helicities can be chosen to be

$$\epsilon^-_{\mu\nu}(k_1) = \epsilon^+_{\mu\nu}(k_2) = q_\mu q_\nu$$

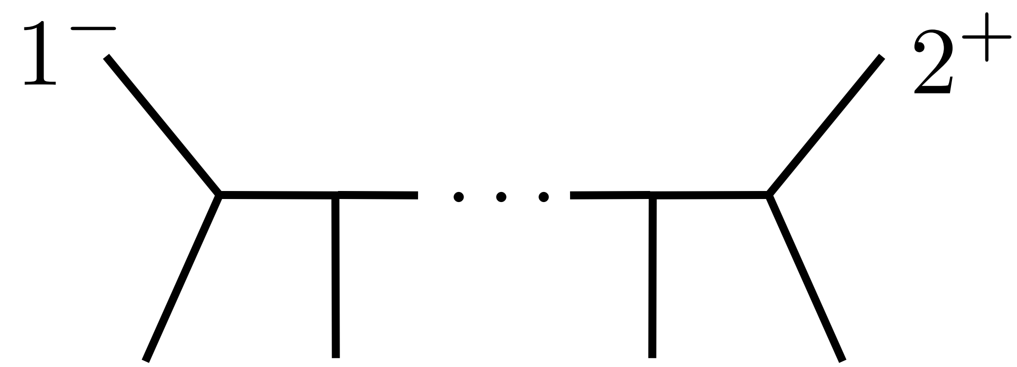
BCFW analytic continuation

Britto, Cachazo, Feng, Witten '05

Continue $k_1 \to k_1(z) = k_1 + zq$
$k_2 \to k_2(z) = k_2 - zq$

Clearly $\sum_i k_i = 0$ still holds

Also
$k_1(z)^2 = k_2(z)^2 = 0$



Consider $\mathcal{M}(z)$

Expect

$$\mathcal{M}(z) \sim \frac{z^{2(n-2)}}{z^{(n-3)}} \sim z^{n-1}$$

maximum $(n-3)$ propagators in between

$1/z$ for each propagator

$(n-2)$ vertices

z^2 for each vertex

In fact

$$\mathcal{M}(z) \sim 1/z^2$$

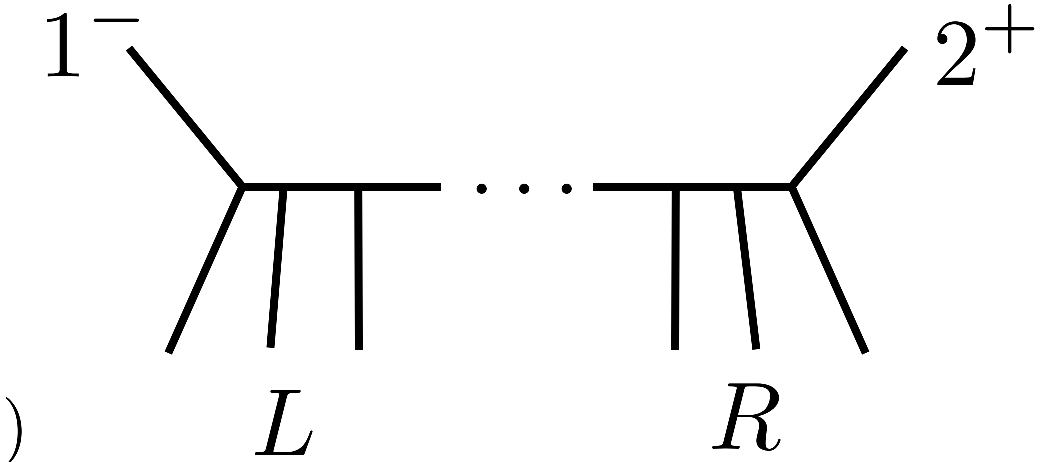
as $z \to \infty$

much "softer" high-energy behaviour than expected

Arkani-Hamed, Kaplan '08

"Softest" UV behaviour known among all QFT's

Recursion relation



One knows all poles of $\mathcal{M}(z)$

$$0 = \left(k_1 + zq + \sum_{i\in L} k_i\right)^2 = \left(q\cdot\sum_{i\in L} k_i\right)(z - z_L)$$

Residues are

$$\mathcal{R}(z_L) = \frac{1}{\left(q\cdot \sum_{i\in L} k_i\right)} \mathcal{M}_L(z_L)\mathcal{M}_R(z_L)$$

where

$$z_L = -\left(\sum_{i\in 1+L} k_i\right)^2 \Big/ \left(q\cdot\sum_{i\in L} k_i\right)$$

amplitudes for smaller number of particles

Have

$$\int_{|z|=\infty} \frac{dz}{z}\mathcal{M}(z) = 0 = \mathcal{M}(0) + \sum_{z_L} \frac{\mathcal{R}(z_L)}{z_L}$$

because $\mathcal{M}(z) \to 0$ as $z \to \infty$

$$\mathcal{M}(0) = \sum_L \frac{\mathcal{M}_L(z_L)\mathcal{M}_R(z_L)}{\left(\sum_{i\in 1+L} k_i\right)^2}$$

can get any amplitude recursively from the 3-graviton ones

Explicit formula for any $\mathcal{M}(1^{h_1},\ldots,n^{h_n})$

Cachazo, He, Yuan '13

Consider n equations on an n-punctured sphere

$$\sum_{j\neq i}\frac{s_{ij}}{z_i-z_j}=0$$

"scattering equations"

$z_i,\ i=1,\ldots,n$

puncture locations

$$s_{ij}=(k_i+k_j)^2$$

Using $\sum_i k_i=0,\quad k_i^2=0$ easy to show

- only (n-3) linearly independent
- $\mathrm{SL}(2,\mathbb{C})$ invariant

Can also show that $(n-3)!$ solutions

$$z_{ij}:=z_i-z_j$$

Claim

$$\mathcal{M}(\{k,h\})=\int\frac{\prod_i d^2z_i}{\mathrm{vol}\,\mathrm{SL}(2,\mathbb{C})}\prod_i\delta'\left(\sum_{j\neq i}\frac{s_{ij}}{z_{ij}}\right)E^2(\{k,h,z\})$$

measure

integrand

Measure can be shown to reduce to

$$\sum_{solutions} \frac{(z_{pq})(z_{qr})(z_{rp})(z_{ij})(z_{jk})(z_{ki})}{|\Phi|^{ijk}_{pqr}}$$

where $|\Phi|^{ijk}_{pqr}$ minor obtained by removing 3 rows 3 columns from

$$\Phi_{ij} = \begin{cases} s_{ij}/z_{ij}^2, & i \neq j \\ -\sum_{k\neq i} s_{ik}/z_{ik}^2, & i = j \end{cases}$$

sum of any row or column zero

compare matrix tree theorems

Integrand

$$E^2(\{k,h,z\}) = \frac{1}{z_{ij}^2}|\Psi|^{ij}_{ij}$$

$$\Psi = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}$$

anti-symmetric $2n \times 2n$ matrix

$$A_{ij} = \begin{cases} s_{ij}/z_{ij}, & i \neq j \\ 0, & i = j \end{cases} \qquad B_{ij} = \begin{cases} (q_i - k_j)^2/z_{ij}, & i \neq j \\ -\sum_{l\neq i}(q_i - k_l)^2/z_{il}, & i = j \end{cases}$$

$$C_{ij} = \begin{cases} (q_i - q_j)^2/z_{ij}, & i \neq j \\ 0, & i = j \end{cases}$$

where q's are "square roots" of polarization tensors $\epsilon^{\mu\nu}(k_i) = q_i^\mu q_j^\nu$

Scattering amplitudes summary

- At large (complex) momenta graviton scattering amplitudes are much better behaved than naive arguments suggest

- Gravity is best behaved QFT in this sense
  But it is also worst behaved - non-renormalizability

- Graviton scattering amplitudes can be obtained recursively

  BCFW recursion relations

- Closed formula for tree-level amplitudes is possible

This means that one can characterize the space of (perturbative) solutions of GR completely

Space of solutions = phase space

Quantization?

Gauge-theoretic formulation of GR

Given an SU(2) connection A^i

one can define a spacetime metric

connection as a "potential" for the metric

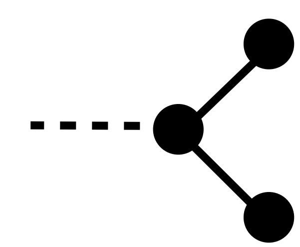
This metric owes its existence to the isomorphism

$$\mathrm{SO}(6,\mathbb{C}) \sim \mathrm{SL}(4,\mathbb{C})$$

Very important for twistor theory

Dynkin diagrams

$\mathfrak{sl}(n+1)$



$\mathfrak{so}(2n)$

Proof: Consider the 6-dimensional space Λ^2 of 2-forms in $\mathbb{R}$

The wedge product makes Λ^2 into a metric space

$$\Lambda^2 \ni U, V \to (U, V) = U \wedge V / d^4x \in \mathbb{R}$$

metric of signature (3,3) if over $\mathbb{R}$

$\mathrm{SL}(4, \mathbb{R})$ acts on Λ^2 $\qquad G_\mu^\nu \in \mathrm{SL}(4, \mathbb{R})$

$${}^G U_{\mu\nu} = G_\mu^\alpha G_\nu^\beta U_{\alpha\beta}$$

the wedge product metric is preserved

$$\Rightarrow \quad \mathrm{SL}(4, \mathbb{R}) \sim \mathrm{SO}(3, 3)$$

The isomorphism implies

$\mathrm{SL}(4)/\mathrm{SO}(4)$		$\mathrm{SO}(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$
	$\Leftrightarrow$	
conformal metrics on M		Grassmanian of 3-planes in Λ^2

Conformal metrics can be encoded into the knowledge of which 2-forms are self-dual

<u>Explicitly:</u> a triple of linearly independent 2-forms $B^i_{\mu\nu}$

$$\Rightarrow \quad g_{\mu\nu} \sim \tilde{\epsilon}^{\alpha\beta\gamma\delta}\epsilon^{ijk}B^i_{\mu\alpha}B^j_{\nu\beta}B^k_{\gamma\delta}$$

Urbantke formula

2-forms $B^i_{\mu\nu}$ are self-dual with respect to this metric

Definition of the metric:

Let A^i be an SU(2) connection

$$F^i = dA^i + (1/2)[A, A]^i$$

(SL(2,C) connection for Lorentzian signature)

$$F^i \wedge (F^j)^* = 0$$

reality conditions

declare F^i to be self-dual 2-forms $\Rightarrow$ conformal metric

To complete the definition of the metric need to specify the volume form

$$\Lambda \sim 1/L^2$$

dimensionful parameter

$$(\mathrm{vol}) := \frac{1}{\Lambda^2} f(F \wedge F)$$

Functions of the curvature

Let f be a function on $\mathfrak{g} \otimes_S \mathfrak{g}$

satisfying

$\mathfrak{g}$ - Lie algebra of G

$$f : X \to \mathbb{R}(\mathbb{C})$$

$$X \in \mathfrak{g} \otimes_S \mathfrak{g}$$

defining function

1) $$f(\alpha X) = \alpha f(X)$$ homogeneous degree 1

2) $$f(\mathrm{Ad}_g X) = f(X), \quad \forall g \in G$$ gauge-invariant

Then $f(F \wedge F)$ is a well-defined 4-form (gauge-invariant)

Choose a volume form and define X^{ij}

$$F^i \wedge F^j := X^{ij}(\mathrm{vol})$$

then $$f(F \wedge F) := (\mathrm{vol})\, f(X)$$

independent of choice of (vol)

To motivate a choice of f(X)

take an Einstein metric, consider

the Levi-Civita A^i connection on $\Lambda^+ \subset \Lambda^2$

the space of self-dual 2-forms

$$F^i = \left(\frac{\Lambda}{3} + W^+\right)^{ij} \Sigma^j$$

$\mathrm{Tr}(W^+) = 0$

where Σ^i is a basis of self-dual 2-forms

$\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

Then

$$\left(\mathrm{Tr}\sqrt{F \wedge F}\right)^2 = 2\Lambda^2 (\mathrm{vol})$$

This suggest that we take

$$(\mathrm{vol}) := \frac{1}{2\Lambda^2}\left(\mathrm{Tr}\sqrt{F \wedge F}\right)^2$$

this completes the definition of the metric from A^i

Variational principle

Consider a functional that is just a multiple of the volume

$$S[A] = \frac{\Lambda}{8\pi G} \int (\mathrm{vol})$$

$\Lambda \neq 0$ **KK PRL106:251103,2011**

related ideas for zero scalar curvature in early 90's Capovilla, Dell, Jacobson

Critical points

(*) $$d_A \left(\mathrm{Tr}\sqrt{X} (X^{-1/2})^{ij} F^j \right) = 0$$

second-order PDE's for the connection

Theorem:

For connections A^i satisfying (*)
the metric $g(A)$ is Einstein with non-zero scalar curvature Λ

In the opposite direction, the self-dual part of the Levi-Civita connection for an Einstein metric satisfies (*)

Caveat: only metrics with $\Lambda/3 + W^+$ invertible almost everywhere covered

examples not covered
$S^2 \times S^2$
Kahler metrics

Gauge-theoretic perturbation theory

Need a non-zero connection to expand about

Homogeneous isotropic connection

$$A^i = i\, a(t) dx^i$$

$$F^i \wedge F^j \sim \delta^{ij}$$

such a connection is a solution

The corresponding metric is de Sitter of cosmological constant Λ

(in flat slicing)

$$F^i = \frac{\Lambda}{3}\Sigma^i$$

where Σ^i is a basis of self-dual 2-forms

One gets the following linearised Lagrangian

square of a certain Dirac operator

$$L^{(2)} \sim P^{(2)}_{ijkl}(\Sigma^{i\mu\nu} d_\mu a^j_\nu)(\Sigma^{k\rho\sigma} d_\rho a^l_\sigma)$$

where

$$P^{(2)}_{ijkl} := \delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

a^i_μ connection perturbation

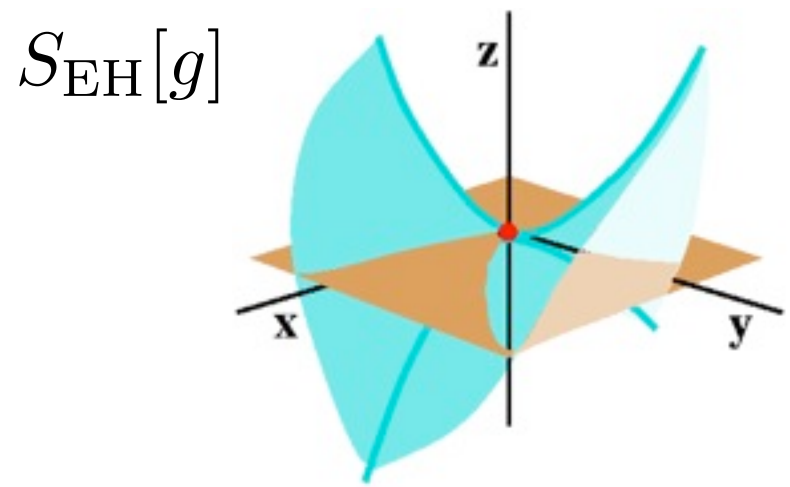
d_μ de Sitter covariant derivative

Easy to show that describes gravitons on de Sitter space

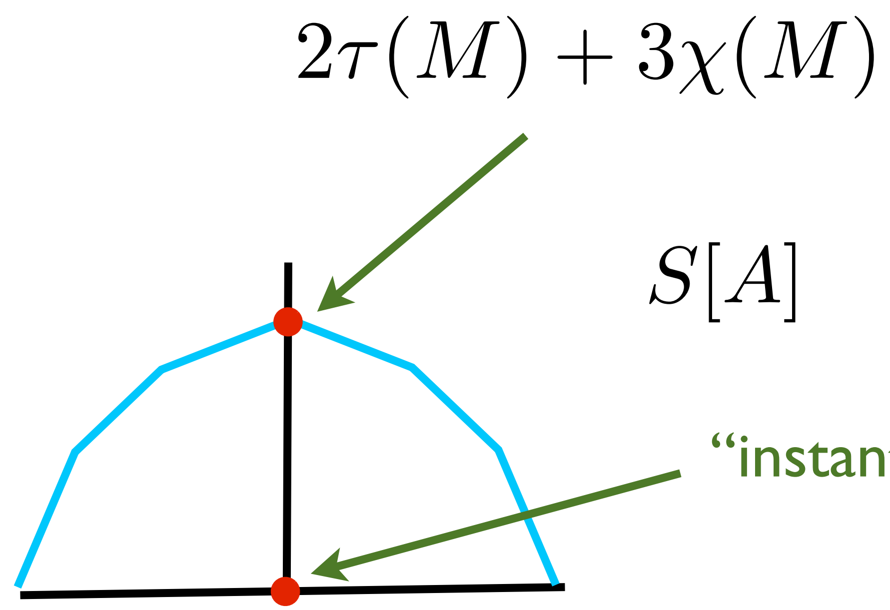
considerably simpler linearization than in the metric case

The new formulation is simpler than the metric-based GR

Concave action functional



space of metrics



SU(2) connections/gauge

Interactions

α, β - dimensionless parameters

$$\mathcal{L}^{(3)} = \frac{\alpha}{M^2} {\rm Tr}\left((\Sigma da)^3\right) + \frac{\beta}{M^2} (\Sigma da)^{ij} \left(\frac{1}{i} \epsilon^{\mu\nu\rho\sigma} d_\mu a^i_\nu d_\rho a^j_\sigma + M^2 \Sigma^{i\mu\nu} \epsilon^{jkl} a^j_\mu a^k_\nu \right)$$

with $(\Sigma da)_{ij} := P^{(2)}_{ijkl} (\Sigma^{k\mu\nu} d_\mu a^l_\nu)$ $\qquad M^2 := \Lambda/3$

compare with the mess in the metric formulation

Summary:

- After a long period of inactivity, perturbative quantum gravity is again at the cross-roads of many interesting developments
- Gravity has much better high-energy behaviour than was thought
- Powerful recursion relations for the amplitudes
- Tree-level amplitudes can be solved for in closed form
- GR can be described as an SU(2) gauge theory of a novel type: bounded from above Euclidean action
- Produces much simpler perturbative expansion than the usual description

Further interesting developments are guaranteed

Thank you!

Quantum Theory Hopes

Remark: no dimensionful coupling constants in any of these gravitational theories

(negative) dimension coupling constant comes when expanded around a background

Non-renormalizable in the usual sense

Hope: the class of theories {all possible f()} is large enough to be closed under renormalization

$$\frac{\partial f(F \wedge F)}{\partial \log \mu} = \beta_f(F \wedge F)$$

I.e. physics at higher energies continues to be described by theories from the same family

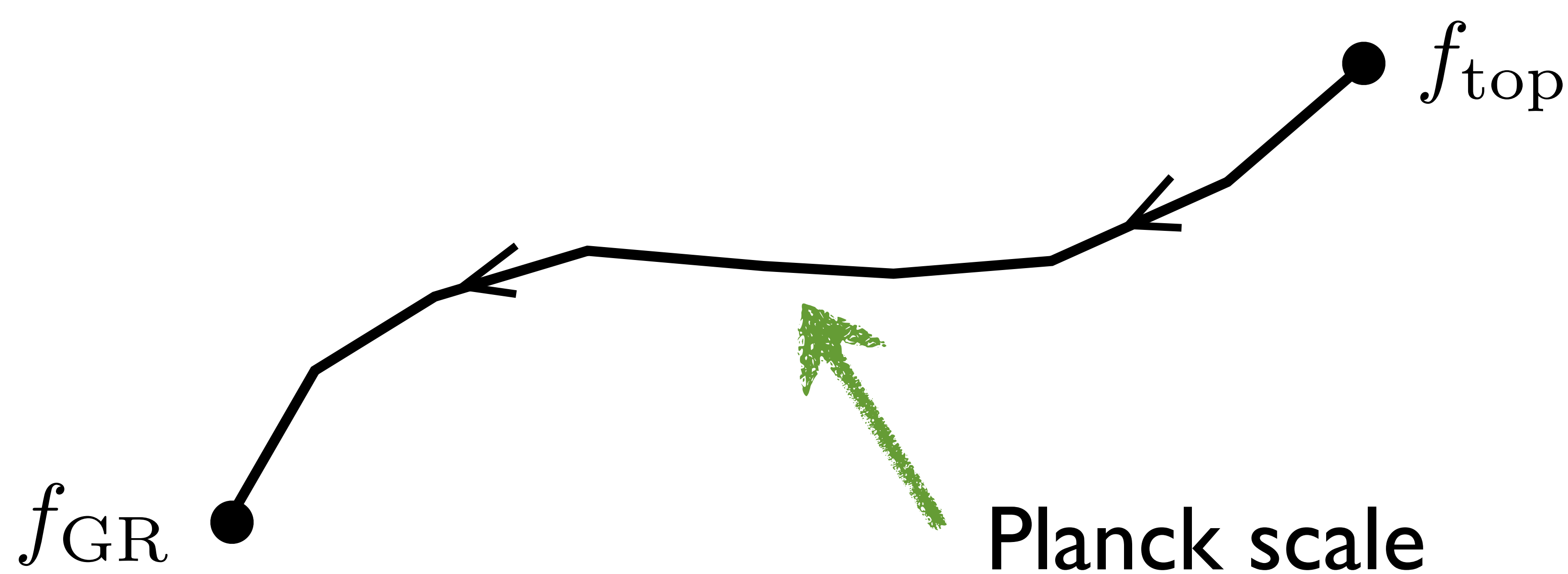
= no new DOF appear at Planck scale, just the dynamics changes

The speculative RG flow: topological theory ?

$$f_{\text{top}}(F \wedge F) = \text{Tr}(F \wedge F)$$

necessarily a fixed point of the RG flow

corresponds to a topological theory (no propagating DOF)



flow from very steep in IR towards very flat in UV potential