Diffeomorphism Invariant Gauge Theories

Kirill Krasnov
(University of Nottingham)

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Main message:

There exists a large new class of gauge theories in 4 dimensions

Lagrangian as a non-linear function of the curvature

No metric is used in the construction

The simplest non-trivial such theory describes gravity

could have discovered GR this way

Other gauge groups give gravity plus Yang-Mills fields

plus other stuff - still poorly understood

What are they good for?

They are very natural constructs

$\Rightarrow$ Must be good for something

Plan:

- Define the theories
- GR as a particular SO(3) diff. invariant gauge theory

recent talk by Joel Fine

- Linearisation (Hessian) around instantons

arbitrary SO(3) theory

- Parameterising connections by metrics

towards understanding a general SO(3) theory at a non-linear level

- Linearisation of a general theory and Yang-Mills

first attempt at understanding a general gauge group theory

- Summary

Diffeomorphism invariant gauge theories

(in 4 spacetime dimensions)

Dynamically non-trivial theories of gauge fields that use no external structure (metric) in their construction

"TQFT's" with local degrees of freedom

Let A be a G-connection

$$F = dA + (1/2)[A, A]$$

curvature 2-form

Can define a gauge and diffeomorphism invariant action

$$S[A] = \int f(F \wedge F)$$

no dimensionful coupling constants!

Functions of the curvature

Let f be a function on $\mathfrak{g} \otimes_S \mathfrak{g}$ satisfying

$\mathfrak{g}$ - Lie algebra of G

$f : X \longrightarrow \mathbb{R}(\mathbb{C})$ defining function

$X \in \mathfrak{g} \otimes_S \mathfrak{g}$

1) $f(\alpha X) = \alpha f(X)$ homogeneous degree 1

2) $f(\mathrm{Ad}_g X) = f(X), \quad \forall g \in G$ gauge-invariant

Then $f(F \wedge F)$ is a well-defined 4-form (gauge-invariant)

In practice: choose an auxiliary volume form (vol)

$$F \wedge F := X(\mathrm{vol}), \quad X \in \mathfrak{g} \otimes_S \mathfrak{g}$$

then $f(F \wedge F) := f(X)(\mathrm{vol})$

Field equations: $d_A B = 0$

where $B = \frac{\partial f}{\partial X} F$

Second-order (non-linear) PDE's

compare Yang-Mills equations: $d_A B = 0$

where $B = {}^* F$

$*$ - encodes the metric

Dynamically non-trivial theory with propagating DOF

apart from the point $f_{top} = \mathrm{Tr}(F \wedge F)$

Gauge symmetries:

$\delta_\phi A = d_A \phi$ gauge rotations

$\delta_\xi A = \iota_\xi F$ diffeomorphisms

Are there any such theories?

G=U(1) $F \wedge F$ is just a 4-form

$$\Rightarrow \quad f(F \wedge F) = F \wedge F$$ trivial dynamics

G=SU(2)~SO(3) $F \wedge F$ is a 3 x 3 symmetric matrix (times a 4-form)

$$X = O\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) O^T, \quad O \in \mathrm{SO}(3)$$

$$\Rightarrow \quad f(X) = f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 \chi(\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1})$$

homogeneity degree one function of 3 variables

invariant under $\lambda_1 \leftrightarrow \lambda_2$ etc.

no dimensionful couplings

Even for SU(2) the class of such theories is infinitely large

What are these theories about? Let us start with SU(2)

Given a theory, i.e. a function f(X), and a connection A
one can define a spacetime metric

This metric owes its existence to the "twistor" isomorphism

$$\mathrm{SO}(6,\mathbb{C}) \sim \mathrm{SL}(4,\mathbb{C})$$

The isomorphism implies

$$\mathrm{SL}(4)/\mathrm{SO}(4) \quad \Leftrightarrow \quad \mathrm{SO}(3,3)/\mathrm{SO}(3)\times\mathrm{SO}(3)$$

conformal metrics on M $\Leftrightarrow$ Grassmanian of 3-planes in Λ^2

Conformal metrics can be encoded into the knowledge of which 2-forms are self-dual

Definition of the metric:

Let A be an SU(2) connection

$$F = dA + (1/2)[A, A]$$

$\left(\begin{array}{c}\text{SL(2,C) connection for} \\ \text{Lorentzian signature}\end{array}\right)$

$$F \wedge (F)^* = 0$$

reality conditions

declare F to be self-dual 2-forms $\Rightarrow$ conformal metric

To complete the definition of the metric need to specify the volume form

$$(\text{vol}) := f(F \wedge F)$$

$$S[A] = \int_M (\text{vol})$$

$\Lambda \neq 0$ KK PRL106:251103,2011

Pure connection formulation of GR

$$f_{\mathrm{GR}}(X) = \left(\mathrm{Tr}\sqrt{X}\right)^2$$

related ideas for zero scalar curvature in early 90's Capovilla, Dell, Jacobson

Field equations: $$d_A\left(\mathrm{Tr}\sqrt{X}\left(X^{-1/2}\right)\circ F\right) = 0 \qquad (*)$$

second-order PDE's for the connection

Theorem:

For connections A satisfying (*) the metric $g(A)$ is Einstein with non-zero scalar curvature

In the opposite direction, the self-dual part of the Levi-Civita connection for an Einstein metric satisfies (*)

Caveat: only metrics with $s/12 + W^+$ invertible almost everywhere covered

examples not covered

$S^2 \times S^2$

Kahler metrics

Details were given in Joel Fine's talk on Nov 4th

Analogy to Hitchin's action for stable forms

$\Omega \in \Lambda^3$ 3-forms in 6 dimensions, symplectic space

given a vector field $i(v)\Omega \wedge \Omega \in \Lambda^5 \sim \Lambda^6 \otimes V$

define $K_\Omega : V \to V \otimes \Lambda^6$ with $K_\Omega(v) = i(v)\Omega \wedge \Omega$

Stable 3-forms $GL(6)/SL(3) \times SL(3)$

K_Ω -moment map for the action of $SL(6)$

$\mathrm{Tr}(K_\Omega) = \Omega \wedge \Omega = 0$

$$S[\Omega] = \int_M \sqrt{\mathrm{Tr}(K_\Omega^2)}$$

homogeneity degree 2 functional in Ω

Back to connections:

curvature as a 2-form with values $F \in \Lambda^2 \otimes \mathfrak{so}(3)$

"Stable" objects in $\Lambda^2 \otimes \mathfrak{so}(3)$ form a quotient space

$$\frac{SL(4) \times GL(3)}{SL(2) \times SL(2)} = \{\text{conformal metrics}\} \times GL(3)$$

similar to the case of 3-forms, homogeneous space as an open set in a vector space

again similar, the action is a homogeneity degree 2 functional in $F \in \Lambda^2 \otimes \mathfrak{so}(3)$

Is there any natural moment map interpretation?

What do other choices of f(X) correspond to?

Can understand by linearising the theory

Let us take a constant curvature space to expand about

Corresponds to a connection with $F^i \wedge F^j \sim \delta^{ij}$ $\qquad i = 1, 2, 3$

Such a connection is a solution for any f(X) - instanton

For concreteness, can take the self-dual part of Levi-Civita for S^4 or its Lorentzian signature analog de Sitter space

for such a connection, the curvature can be identified with the orthonormal basis of self-dual 2-forms

holds more generally - for any instanton

$F^i = \Sigma^i$ where Σ^i is a basis of self-dual 2-forms

One gets the following linearised Lagrangian

$i = 1, 2, 3$

μ spacetime index

$$\mathcal{L}^{(2)} \sim \frac{\partial^2 f}{\partial X^{ij} \partial X^{kl}} \left(\Sigma^{i\mu\nu} \nabla_\mu a^j_\nu\right) \left(\Sigma^{k\rho\sigma} \nabla_\rho a^l_\sigma\right) + \frac{\partial f}{\partial X^{ij}} \left(\epsilon^{\mu\nu\rho\sigma} \nabla_\mu a^i_\nu \nabla_\rho a^j_\sigma + \Sigma^{i\mu\nu} [a_\mu, a_\nu]^j\right)$$

a^i_μ connection perturbation

∇_μ de Sitter covariant derivative

Easy to show that for any f(X)

$$\left.\frac{\partial f}{\partial X^{ij}}\right|_{X=\mathrm{Id}} \sim \delta^{ij}$$

second term is a total derivative

$$\left.\frac{\partial^2 f}{\partial X^{ij} X^{kl}}\right|_{X=\mathrm{Id}} \sim P^{(2)}_{ijkl} := \delta_{i(k}\delta_{l)j} - \frac{1}{3}\delta_{ij}\delta_{kl}$$

point f(X)=Tr(X) is singular

Linearised Lagrangian is the same for any f(X)

Easy to show that describes spin 2 particles on de Sitter space

Any of SU(2) theories is a gravity theory!

Spinorial description of the Hessian around an instanton:

$$\mu \to AA' \quad i \to (AB)$$

$$a^i_\mu \to a_{AA'}{}^{BC} \qquad \in S_+ \otimes S_- \otimes S_+^2 = S_+^3 \otimes S_- \oplus S_+ \otimes S_-$$

$$\mathcal{L}^{(2)} \sim \left(\nabla^{(A}_{A'} a^{BCD)A'}\right)^2$$

$$a_{A'}^{(ABC)}$$

$$a_{A'E}{}^{E}{}_{A}$$

pure gauge (diffeomorphisms) part

explicitly non-negative (Euclidean signature) functional

only depends on the $S_+^3 \otimes S_-$ part of $a_{AA'}{}^{BC}$

$\dim(S_+^3 \otimes S_-) = 8$ (per point)

The Hessian is elliptic modulo gauge

$$\mathcal{L}^{(2)} \sim \left(\nabla^{A}_{A'} a^{(BCD)A'}\right)^2$$

gauge-fixing

$$\nabla_{AA'} a^{(ABC)A'} = 0$$

The Hessian as a square of the Dirac operator

$\not{\partial}: S^3_+ \times S_- \to S^3_+ \times S_+$ Dirac operator

$$\text{Hessian} = (\not{\partial})^* \not{\partial}$$

There is no formula of comparable simplicity in the metric formulation!

Weitzenbock formula

$$\text{Hessian} = \nabla^* \nabla + \frac{s}{4}$$

Immediately implies that there are no Einstein deformations of positive scalar curvature instantons

To analyse the negative scalar curvature case, need a different rewriting of the Hessian

For f_{GR} the gauge-fixed Hessian can be rewritten in another form

$$\mathcal{L}^{(2)} \sim \left(\nabla^{E}_{(A'} a_{E}{}^{AB}{}_{B')} \right)^2 - \frac{s}{12} (a)^2$$

here the Dirac operator is used in another way

define

$$d_- : \Lambda^1 \otimes \mathfrak{so}(3) \to \Lambda^- \otimes \mathfrak{so}(3) \sim \Lambda^- \otimes \Lambda^+$$

can restrict to $S^3_+ \otimes S_- \subset \Lambda^1 \otimes \mathfrak{so}(3)$

$$d_- : S^3_+ \otimes S_- \to S^2_+ \otimes S^2_-$$

symmetric tracefree tensors

$$\mathrm{Hessian} = d_-^* d_- - \frac{s}{12}$$

Immediately implies that there are no Einstein deformations of <u>negative</u> scalar curvature instantons

General f(X) SO(3) theories at non-linear level

Need to develop new tools (also useful for GR)

Lemma: Given $B \in \Lambda^2 \otimes \mathfrak{so}(3)$ there exists a pair

$b \in \mathrm{End}(\mathfrak{so}(3))$ $\Sigma \in \Lambda^2 \otimes \mathfrak{so}(3)$ with properties

- $\Sigma \wedge \Sigma \sim \mathrm{Id}$ or $\Sigma : \mathfrak{so}(3) \to \Lambda^2$ is an isometry

  in other words, Σ is a basis in Λ^+ as defined by B
- b is self-adjoint $(bX, Y) = (X, bY), \quad \forall X, Y \in \mathfrak{so}(3)$
- $B = b\Sigma$

This pair is unique modulo "conformal" rescalings

$$\Sigma \to \Omega^2 \Sigma$$
$$b \to \Omega^{-2} b$$

b can have eigenvalues of both signs

the idea is to parameterise connections by pairs (Σ, b)

Lemma: Given $B \in \Lambda^2 \otimes \mathfrak{so}(3)$

there is a unique SO(3) connection $A(B)$ satisfying $d_{A(B)} B = 0$

Lemma: When Σ is an orthonormal basis in Λ^+

the connection $A(\Sigma)$ is the Levi-Civita connection

$$A(\Sigma) = \nabla \quad \text{on} \quad \Lambda^+$$

Lemma: For a general $B = b\Sigma \in \Lambda^2 \otimes \mathfrak{so}(3)$

$$A(B) = \nabla + \rho(b) \quad \text{where}$$

$$\rho(b) = \frac{bT^{-1}b\,(\Sigma\nabla b)}{\det(b)}$$

with $T : \Lambda^1 \otimes \mathfrak{so}(3) \to \Lambda^1 \otimes \mathfrak{so}(3)$

being a map of eigenvalue $+1$ on $S^3_+ \times S_-$, -2 on $S_+ \times S_-$

and Σ viewed as a map

$$\Sigma : \Lambda^1 \otimes \mathfrak{so}(3) \to \Lambda^1 \quad \text{so that}$$

$$(\Sigma\nabla b) \in \Lambda^1 \otimes \mathfrak{so}(3)$$

Parameterising connections by pairs (b, Σ)

Given $F = dA + (1/2)[A, A]$ can represent $F = b\Sigma$

Then Bianchi identity implies $d_A(b\Sigma) = 0$

and thus $A = \nabla + \rho(b)$

<u>Theorem</u>: pairs (b, Σ) satisfying

$$R(\nabla) + \nabla\rho(b) + (1/2)[\rho(b), \rho(b)] = b\Sigma \qquad (*)$$

are in one-to-one correspondence with SO(3) connections.

There are only 6 independent equations in (*),

with 12 relations being $d_{\nabla+\rho(b)}(LHS - RHS) = 0$

Interpretations of (*)

- Given a metric and the corresponding Σ can view (*) as a PDE on the b

  The corresponding $A = \nabla + \rho(b)$ are SO(3) instantons for this conformal class

second order elliptic PDE's

essentially the same method for finding instantons as physicists use

- Given b can view (*) as Einstein's equations with a non-trivial source as prescribed by b

It is the <u>second interpretation</u> that is used when we add the Euler-Lagrange equations for our gauge theory

first order PDE's on b

$$\nabla\left(\frac{\partial f}{\partial X} b\right)\Sigma + [\rho(b), \frac{\partial f}{\partial X} b\Sigma] = 0 \qquad (**)$$

When $f = f_{\rm GR}$ have $\frac{\partial f_{\rm GR}}{\partial X} b = const$ and (**) reduce to

$$T\rho(b) = 0 \quad \Rightarrow \quad \rho(b) = 0$$

Then (*) become usual Einstein equations with $b = s/12 + W^+$

and $\rho(b) = 0$ following from the Bianchi identity

Relevance to quantum gravity

When $f \neq f_{\mathrm{GR}}$ need to be solving (*) and (**) simultaneously

If viewed as equations for the metric only, these are non-local as would arise from a Lagrangian

$$\mathcal{L} = -2\Lambda + R + \alpha (W^+)^2 + \ldots$$

a series that does not terminate

These are precisely types of Lagrangians that arise by adding to the Einstein-Hilbert Lagrangian the "quantum corrections"

Making sense of such Lagrangians is the problem of quantum gravity

We have shown that the corresponding Euler-Lagrange equations have an equivalent description in terms of second order PDE's for an SO(3) gauge field

Deformations of GR

the open problem is to show that this class is large enough, in that it includes all possible quantum corrections

Other gauge groups. Unification

Consider a larger gauge group $G \supset \mathrm{SU}(2)$

No longer can define any natural metric

Can still understand the theory via linearisation

A class of backgrounds is classified by how SU(2) embeds into G

$$F = e(\Sigma)$$

e an embedding of SO(3)~SU(2) into G

Only the SO(3) part of the curvature is "on"

The corresponding connection is a solution of the EL equations for any f(X)

Generically, there is a part of G that commutes with the SU(2)

E.g. $$\mathrm{SU}(3) \ni \begin{pmatrix} \mathrm{SU}(2) & * \\ * & * \end{pmatrix}$$

other stuff (higher spin)

The SU(2) part of the connection continues to describe gravitons

The part that commutes with SU(2) describes YM gauge fields

Linearized Lagrangian

$$L^{(2)} \sim \frac{\partial^2 f}{\partial X^{i\alpha} X^{k\beta}} (\Sigma^{i\mu\nu}\partial_\mu a^\alpha_\nu)(\Sigma^{k\rho\sigma}\partial_\rho a^\beta_\sigma) \sim (\partial_{[\mu} a^\alpha_{\nu]})^2$$

because $$\frac{\partial^2 f}{\partial X^{i\alpha} X^{k\beta}} \sim \delta_{ij} g_{\alpha\beta}$$

linearized Yang-Mills

Summary:

- Dynamically non-trivial diffeomorphism invariant gauge theories
- The simplest non-trivial such theory G=SU(2) - gravity
- GR can be described in this language:

  bounded from above Euclidean action

  stunningly simple Hessian at instantons   Dirac squared
- Other theories in case G=SO(3) are deformations of GR

  potentially relevant for quantum gravity
- Can "unify" gravity with Yang-Mills in this framework

Are these theories Yang-Mills theories 20 years before they were used in physics and maths?

Thank you