Part IV : Chiral formulations of GR

This is the least familiar set of formalisms for GR, specific to four spacetime dimensions. At the same time, there is something very deep going on. One does not understand GR if one does not understand this chiral story. My personal feeling is that if any of the formulations has potential for "generalizations into unknown" then these are the chiral formulatio[ns]

The nature of soldering changes in these formalisms in a deep, not obvious way.

There are several (related) ways to see that something like a chiral formulation must be possible.

Lorentz group in 4D is not simple

At the level of Lie algebra

$$so(4) = su(2) \oplus su(2)$$
$$so(1,3) = sl(2,\mathbb{C}) \oplus \overline{sl(2,\mathbb{C})}$$
$$so(2,2) = sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$$

This means that thinking about the spin connection as the sum of its self-dual and anti-self-dual parts is a good idea

We already remarked that the full spin connection has too many "momentum" variables, some of them are eliminated by second class constraints. Here is the God-given

Decomposition of the Riemann tensor

In any dimension, Riemann tensor splits into different irreducible (with respect to the Lorentz group) components – Weyl, Ricci, scalar.

In four dimensions, there is another decomposition, which is into self- anti-self-dual pieces.

They encode the same information. But because Einstein equations can be stated as $\text{Ricci}\big|_{\text{tracefree}} = 0$ we get a completely new, specific only to 4D, way of encoding the Einstein condition

In details

Consider the Hodge operator (of a given metric)

$$* : \Lambda^2 \to \Lambda^2 \qquad *B_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} B_{\rho\sigma}$$

It squares to $\pm$ id, depending on the signature

$$*^2 = \sigma \cdot \mathrm{id} \qquad \sigma = \begin{cases} +1 & \text{Euclidean, Split} \\ -1 & \text{Lorentzian} \end{cases}$$

This means that the space of 2-forms on M splits into its self- and anti-self-dual parts

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

$$B^\pm \in \Lambda^\pm \qquad *B^\pm = \pm\sqrt{\sigma}\, B^\pm$$

In Lorentzian signature $\Lambda^\pm$ are complex!

Concretely let

$$P^{\pm} = \frac{1}{2}\left(\mathbb{1} \pm \frac{1}{\sqrt{\sigma}} *\right)$$

There is a deep relation between necessity of using complex numbers and Lorentz signature!

be the projectors on $\Lambda^{\pm}$

Then any 2-form gets represented as the sum of its self-dual anti-self-dual parts

$$B = B^+ + B^- \qquad B^{\pm} = P^{\pm} B$$

In Lorentzian signature, if B is real then $B^+ = (B^-)^*$
No relation between Λ^+ and Λ^- in the other signatures

Riemann curvature can be viewed as a map

$$\text{Riemann} : \Lambda^2 \to \Lambda^2 \qquad R_{\mu\nu}{}^{\rho\sigma}$$

can decompose $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ and thus get SD/ASD parts of the Riemann curvature

Lemma:

$$\overset{\text{SD - SD}}{A := P^+ \,\text{Riemann}\, P^+} = P_+\left(\text{Weyl} + \frac{R}{6}\mathbb{1}\right)P_+$$

$$\overset{\text{ASD - ASD}}{C := P^- \,\text{Riemann}\, P^-} = P_-\left(\text{Weyl} + \frac{R}{6}\mathbb{1}\right)P_-$$

$$\overset{\text{SD - ASD}}{B := P^+ \,\text{Riemann}\, P^-} \qquad B = 0 \iff \text{Ricci}\big|_{tf} = 0$$

In other words, Riemann can be thought of as consisting of 3x3 blocks

$$\text{Riemann} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

With A, C having equal traces – multiples of R and tracefree parts encoding Weyl$^{\pm}$, and B encoding the tracefree part of Ricci

Lemma: $B = P^{+}$ Riemann P^{-} vanishes if and only if Riemann commutes with the Hodge star

$$* \text{Riemann} = \text{Riemann} *$$

Corollary: Since B encodes the tracefree part of Ricci, a metric is Einstein if and only if Riemann tensor viewed as an endomorphism of Λ^2 commutes with the Hodge star.

Proof: $* \text{Riemann} = \text{Riemann} *$ is equivalent to

$$R\, \delta^{\alpha}_{[\mu} \delta^{\beta}_{\nu]} - 2\, \delta^{\alpha}_{[\mu} R^{\beta}_{\nu]} + 2\, \delta^{\beta}_{[\mu} R^{\alpha}_{\nu]} = 0$$

which is equivalent to $R^{\alpha}_{\beta} = (R/4)\, \delta^{\alpha}_{\beta}$

On the other hand P^{+} Rieman $P^{-} = 0 \iff$

$$* \text{Riemann} = \text{Riemann} *$$

Corollary: To impose Einstein condition, only need access to just one of the two rows of the matrix

$$\text{Riemann} = \begin{pmatrix} A & B \\ B^{T} & C \end{pmatrix}$$

I.e. only need the self-dual projection of Riemann with respect to one of the pair of indices

Proof of $\quad P^+ \text{Riemann}\, P^- = 0 \iff * \text{Riemann} = \text{Riemann} *$

$$P^+ \text{Rieman}\, P^- = \frac{1}{2}\left(\mathbb{1} + \frac{1}{\sqrt{\sigma}} *\right) \text{Riemann}\, \frac{1}{2}\left(\mathbb{1} - \frac{1}{\sqrt{\sigma}} *\right)$$

$$= \frac{1}{4}\left(\text{Riemann} + \frac{1}{\sqrt{\sigma}}\left(* \text{Riemann} - \text{Riemann} *\right) - \frac{1}{\sigma} * \text{Riemann} * \right)$$

Easiest to see when $\sqrt{\sigma} = i$

Then equating to zero real and imaginary parts

$$* \text{Riemann} = \text{Riemann} *$$

$$\text{Riemann} = \frac{1}{\sigma} * \text{Riemann} *$$

One implies the other because $*^2 = \sigma \cdot \mathbb{1}$

To put this to use we recall that the curvature of the spin connection (metric torsion-free) encodes all of Riemann.

This means that we only need

$$\overset{+}{R}{}^{IJ}_{\mu\nu}(\omega) = \frac{1}{2}\left(\delta^{I}_{[K}\delta^{J}_{L]} + \frac{1}{2\sqrt{\sigma}}\epsilon^{IJ}{}_{KL}\right)R^{KL}_{\mu\nu}(\omega)$$

But because Lie algebra of Lorentz group splits into two commuting SD/ASD parts we have

$$R^{+\,IJ}(\omega) = R^{IJ}(\omega^{+})$$

Self-dual part of the curvature of the spin connection equals the curvature of the self-dual part of the spin connection

We ever only need the self-dual part of the spin connection to impose the Einstein condition

Chiral Einstein-Cartan action

Because $*^2$ is a multiple of identity can always add to the Einstein-Cartan kinetic term

$$\int (ee) * R$$

a multiple of $\displaystyle\int (ee) *^2 R = \sigma \int e^I e^J R_{IJ} \sim \sigma \int d^{\omega}e^I d^{\omega}e_I$

vanishes (modulo surface term) when torsion is zero

Can adjust the coefficient in the added term so as to obtain the self-dual projector

Thus, modulo a surface term, get Einstein-Cartan equal to its chiral version

$$S_{chiral}[e,\omega] = \frac{\sqrt{\sigma}}{8\pi G}\int e_I e_J P^{+IJ}{}_{KL}\left(R^{KL}(\omega) - \frac{\Lambda}{6} e^K e^L\right)$$

$$= \frac{\sqrt{\sigma}}{8\pi G}\int e_I e_J R^{IJ}(\omega^+) - \frac{\Lambda}{24\sqrt{\sigma}}\epsilon_{IJKL}e^I e^J e^K e^L$$

We already know the conceptual explanation of why it is possible to only keep ω^+ in the action.

ω^+ has just 12 components - half of components of ω

The mismatch 16 components of e^I_μ

vs. 12 components of ω^{+IJ}_μ

is now just due to gauge.

No second class constraints in this chiral version of the Einstein-Cartan theory

Price to pay for this economy - the action is not manifestly real in Lorentzian signature

Remarks:

- Chiral Einstein-Cartan keeps all the good features of Einstein-Cartan while eliminating its main drawback - second class constraints, too many ω variables
- Leads to interesting and useful perturbative expansion

Comparison with chiral version of YM theory

$$\mathcal{L}_{YM} = \frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2$$

However, the term $\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$ is a total derivative - Pontryagin class

Can add any multiple of it to the Lagrangian without changing the theory

In particular, can choose this multiple to get

$$\frac{1}{2g^2} F^a_{\mu\nu} P^{+\mu\nu}{}_{\rho\sigma} F^{a\rho\sigma} = \frac{1}{2g^2}\left(F^{+a}_{\mu\nu}\right)^2$$

where $P^{+\mu\nu}{}_{\rho\sigma} = \frac{1}{2}\left(g^{[\mu}{}_{\rho}\, g^{\nu]}{}_{\sigma} + \frac{1}{2i}\epsilon^{\mu\nu}{}_{\rho\sigma}\right)$

A chiral Lagrangian, in particular not manifestly real in Lorentzian signature

Can also give a first-order version

$$S_{YM}[B,A] = \int B^{+a}_{\mu\nu} F^{a\mu\nu} - \frac{g^2}{2}\left(B^{+a}_{\mu\nu}\right)^2$$

self-dual 2-form with values in the Lie algebra

This gives a very useful cubic formalism for YM with nice perturbation theory

The chiral Einstein-Cartan is gravity analog of this formalism

Spinorial form of chiral Einstein-Cartan

SD/ASD projections in 4D are most clearly performed and kept track of via the spinor notations

Let us introduce spinors for the internal space $V = \mathbb{R}^4$ of appropriate signature. The two fundamental spinor representations of Lorentz are denoted by S_+ and S_-

$$\underbrace{\lambda_A}_{\text{unprimed spinor index}} \in S_+ \qquad \underbrace{\lambda_{A'}}_{\text{primed spinor index}} \in S_-$$

Then the vector representation is $V \sim S_+ \times S_-$

More generally, any irreducible representation of Lorentz is of the form $S_+^n \times S_-^k$ – symmetrised powers of S_+, S_-

The adjoint representation in this language is

$$so(1,3) = \underbrace{S_+^2}_{\text{SD part}} \oplus \underbrace{S_-^2}_{\text{ASD part}}$$

So, the spinor form of the spin connection is

$$\omega^{AA'BB'} = \epsilon^{A'B'}\omega^{AB} + \omega^{A'B'}\epsilon^{AB}$$

The coframe field is a 1-form with values in V

$$e^{AA'}$$

The spinor form of the chiral Einstein-Cartan action is

$$S_{chiral}[e,\omega] = \frac{\sqrt{6}}{8\pi G}\int \Sigma_{AB}(e)\, R^{AB}(\omega) - \frac{\Lambda}{6}\Sigma_{AB}(e)\,\Sigma^{AB}(e)$$

where $\Sigma^{AB}(e) := e^{AA'} e^{B}{}_{A'}$

the self-dual part of $e^I e^J$

In 4D, for reasons already mentioned, this is probably a more useful first order Lagrangian than its non-chiral version

Chiral pure connection formalism

As in the non-chiral case can decide to integrate out the coframe field. Possible when $\Lambda \neq 0$.

As in the non-chiral case, one gets a cubic equation for $e^{AA'}$, and explicit closed form solution is not available. However can solve perturbatively, around a constant curvature background.

This produces a remarkably simple linearized Lagrangian

$$L^{(2)} \sim \left(d^{w}_{(A}{}^{A'}\, a_{A'BCD)}\right)^2$$

Here $a_{AA'}{}^{BC} = \delta\omega_{AA'}{}^{BC}$

and spacetime indices converted to spinor ones using the background $e_\mu^{AA'}$

For the first time since we departed from the metric formalism the count of propagating modes is again readily available

$$a_{AA'}{}^{BC} \in S_+ \times S_- \times S_+^2 = S_+^3 \times S_- \oplus S_+ \times S_-$$

the connection contains two Lorentz irreducible components.

One easily finds that $\mathcal{L}^{(2)}$ is simply independent of the $S_+ \times S_-$ part of the linearised connection. This is diff invariance in this language.

The linearised Lagrangian is a function of 8 components of a in $S_+^3 \times S_-$.

Of them 3 are Lagrange multipliers for 3 constraints – Lorentz invariance

$$8 - 3 - 3 = 2 \text{ propagating DOF}$$

One gets remarkably simple perturbation theory in this chiral pure connection formalism

Plebanski chiral formulation of GR

As with non-chiral Einstein Cartan it was possible to introduce a new field B^{IJ} and then add Lagrange multipliers to guarantee that it is of the "metric" form, can do this for the chiral EC.

As in the non-chiral case this changes the nature of the object that solders the fibres to the base. However, in the chiral case the arising geometry is surprisingly beautiful and rich.

Let us first introduce Lagrange multipliers

$$S[B,A,\Psi] = \frac{\sqrt{\sigma}}{8\pi G}\int B_{AB}F^{AB} - \frac{1}{2}\left(\Psi^{ABCD} + \frac{\Lambda}{3}\epsilon^{A(C}\epsilon^{|B|D)}\right)B_{AB}B_{CD}$$

Here $\Psi^{ABCD} = \Psi^{(ABCD)}$ is the totally symmetric field of Lagrange multipliers

The general B^{AB} contains 6×3 components. The coframe contains 16 but when we form

$$\Sigma^{AB} = e^{AA'}e^{B}{}_{A'}$$

we loose 3 of them because Σ^{AB} is invariant under the ASD part of Lorentz

So we want $18-13=5$ constraints, which matches the number of components in Ψ^{ABCD}

It is often convenient to rewrite the chiral Lagrangian with constraints in $SO(3)$ rather than spinor notations

Replacing $(AB) \Leftrightarrow i \qquad i,j = 1,2,3$

$$S[B,A,\Psi] = \frac{\sqrt{\sigma}}{8\pi G}\int B_i F^i - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right) B_i B_j$$

where now $F^i = dA^i + \frac{1}{2}\epsilon^i{}_{jk} A^j A^k$

Chiral soldering form

We now have a vector bundle $\mathbb{R}^3 \longrightarrow S \to M$ over M with 3-dimensional fibres. We have A^i as an $SO(3)$ connection in this bundle. In the case of Lorentzian signature all objects must be complex-valued, with some reality conditions to be imposed later.

The object $B : \mathbb{R}^3 \to \Lambda^2$

is called the chiral soldering form

It solders the geometry of the fibres to the geometry of the base.

In the case of usual soldering with coframe the metric geometry of M arose from the metric in V directly by pulling back using the coframe map

Here the situation is much more subtle and beautiful

Hodge star in 4D determines the (conformal) metric

Lemma: Hodge star on middle degree forms in 4D is conformally invariant

Theorem: In 4D the knowledge of the Hodge star on middle forms, and thus the knowledge of the split $\Lambda^2 = \Lambda^+ + \Lambda^-$ is equivalent to the knowledge of the conformal metric

Conceptual proof: There is the wedge product (conformal) metric on Λ^2, existing independently of any metric on M

$$B_{1,2} \in \Lambda^2 \qquad (B_1, B_2)_\wedge = B_1 \wedge B_2 / v$$

where v is any top form

For any metric Hodge operator the spaces $\Lambda^\pm$ are orthogonal wrt the wedge product metric

The wedge product metric has signature $(3,3)$

The restriction of the wedge product metric to $\Lambda^\pm$ is signature dependent

Euclidean	$(\cdot,\cdot)_\wedge$ definite on $\Lambda^\pm$
Lorentzian	$(\cdot,\cdot)_\wedge$ complex on $\Lambda^\pm$ Λ^- is c.c. of Λ^+
Split	$(\cdot,\cdot)_\wedge$ is indefinite on $\Lambda^\pm$

Consider now, in each case, the Grassmanian of 3-planes in Λ^2 with the wedge product metric restricting to appropriate metric in each case

Euclidean	$O(3,3)/O(3)\times O(3)$
Lorentzian	$O(3,3)/O(3,\mathbb{C})$
Split	$O(3,3)/O(1,2)\times O(1,2)$

But have isomorphism $\quad SO(3,3) = SL(4,\mathbb{R})/\mathbb{Z}_2$

So, can replace the above cosets by

Euclidean	$SL(4,\mathbb{R})/SO(4)$
Lorentzian	$SL(4,\mathbb{R})/SO(1,3)$
Split	$SL(4,\mathbb{R})/SO(2,2)$

But these are exactly the cosets of conformal (modulo rescalings) metrics in each case !

Concrete expression for the metric

Let us assume that we know Hodge on 2-forms. This is the same as knowing which 2-forms are self-dual and which anti-self-dual, i.e. knowing how to decompose

$$B = B^+ + B^- \qquad \ast B = B^+ - B^-$$

(in Euclidean sign)
Split

Both $\Lambda^\pm$ are 3-dimensional.

Let us take a basis B^i $i=1,2,3$ in Λ^+

The knowledge of a triple of B's such that the wedge product metric between them is of the right type is equivalent to Hodge. The Λ^- forms arise as orthogonal complement to Λ^+ forms with respect the wedge product metric

Can now write an explicit conformal metric on M

$$(u,v)_g \sim i_u B^i \wedge i_v B^j \wedge B^k \epsilon_{ijk}$$

Urbantke metric

Can check that this is the metric that makes the triple of 2-forms B^i self-dual

Theorem: The signature of the Urbantke metric depends on the signature of the wedge product metric among the B^i

definite $\rightarrow$ Euclidean

indefinite $\rightarrow$ Split

complex with $B^i \wedge (B^j)^* = 0$ $\rightarrow$ Lorentzian

We can now complete the definition of the chiral soldering form

Definition: Let $S \to \underline{M}$ be a vector bundle with 3 dimensional fibres, equipped with either definite, indefinite, or complex metric in the fibres. $S \to M$ is required to be of the same topological type as the bundle of self-dual 2-forms on $\underline{M}$.

A map $B: S \to \underline{\Lambda}^2$ is called the chiral soldering form if the wedge product metric on $\underline{\Lambda}^2$ restricted to the image of this map is of the right type, i.e. definite, indefinite or complex

Moreover, in the complex case the chiral soldering form is required to satisfy $B^i \wedge (B^j)^* = 0$
"reality conditions"

As we saw, the chiral soldering form so defined defines a conformal metric of appropriate signature on which it is self-dual

Definition: A chiral soldering form is said to "satisfy the constraints" if the pullback of the wedge product metric via the map $B: S \to \underline{\Lambda}^2$ is a multiple of the fibre metric.

Concretely $B^i \wedge B^j \sim \delta^{ij}$

Theorem: When a chiral soldering form satisfies the constraints, it contains no more information than that of a metric, plus a choice of an orthonormal basis in Λ^+. In other words, there exists a coframe e^I such that $B = (e^I e^J)^+$

Theorem: When a chiral soldering form satisfies the constraints, the "torsion-free condition"

$$d^A B^i = 0$$

has a unique solution. The arising metric torsion-free connection coincides with the self-dual part of the metric torsion free spin connection for the coframe e^I determined by B^i

We now see how Einstein equations are reproduced by this chiral formalism

Plebanski formulation

$$S[B,A,\Psi] = \frac{\sqrt{\sigma}}{8\pi G}\int B_i F^i - \frac{1}{2}\left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right) B_i B_j$$

Euler-Lagrange equations are

Very useful in practice tool for evaluating Riemann

$$F^i = \left(\Psi^{ij} + \frac{\Lambda}{3}\delta^{ij}\right) B_j \quad \text{– Einstein equations}$$

$$d^A B^i = 0 \quad \text{– torsion-free condition}$$

$$B^i \wedge B^j \sim \delta^{ij} \quad \text{– constraints}$$

In this formalism the Einstein condition directly arises as the statement that the curvature of the SD part of the spin connection is SD as the 2-form

Chiral pure connection formulation

As in BF formalism with the full spin connection, can now use the BF type action as the starting point for a series of transformations. Very analogous to the non-chiral case, but with some surprises

Let us first integrate out all fields apart from the connection

$$S[B,A,M,\mu] = \frac{\sqrt{\sigma}}{8\pi G}\int B_i F^i - \frac{1}{2} M^{ij} B_i B_j + \frac{\mu}{2}\left(\mathrm{Tr} M - \Lambda\right)$$

$$F^i = M^{ij} B_j \quad \Rightarrow \quad B^i = (M^{-1})^{ij} F_j$$

$$S[A,M,\mu] = \frac{\sqrt{\sigma}}{16\pi G}\int (M^{-1})^{ij} F_i F_j + \mu\left(\mathrm{Tr} M - \Lambda\right)$$

M field equation

$$X^{ij} = F^i F^j$$

4-form with values in matrices

$$M^{-1} X M^{-1} = \mathbb{1} \cdot \mu$$

all matrices here are 3×3 symmetric matrices

Everything is as in the non-chiral case, apart from it being trivial to solve

$$M = \sqrt{\frac{X}{\mu}}$$

The constraint on M gives μ

$$\sqrt{\mu} = \frac{\mathrm{Tr}\sqrt{X}}{\Lambda}$$

and finally

$$M^{-1} = \frac{\mathrm{Tr}\sqrt{X}}{\Lambda} X^{-1/2}$$

Substituting into the action get

$$S[A] = \frac{\sqrt{\sigma}}{16\pi G\Lambda}\int\left(\mathrm{Tr}\sqrt{X}\right)^2$$

Similar to the non-chiral case, but simpler, no ϵ tensor is needed

Again passes all checks:

- Value on the maximally symmetric background $F^i = \frac{\Lambda}{3}B^i$ is $\frac{\Lambda}{8\pi G}\int e$

- Linearization around the maximally symmetric background gives precisely the expected second order Lagrangian.

- All signs are as in the non-chiral formalism. The $\Lambda > 0$ Euclidean action is non-negative, the Hessian on $F^i = \frac{\Lambda}{3}B^i$ is non-positive

- The class of "easy" backgrounds for this formalism is much larger than maximally symmetric. $F^i = \frac{\Lambda}{3}B^i$ just means that there is no self-dual part of the Weyl curvature. These are ASD Einstein metrics – instantons.

So, linearization is easy around all instantons in this formalism

Modified theories – Deformations of GR

Go back to formulation with Lagrange multiplier fields

$$S[B,A,M,\mu] = \frac{\sqrt{\sigma}}{8\pi G}\int B_i F^i - \frac{1}{2} M^{ij} B_i B_j + \frac{\mu}{2}\left(f_{GR}(M) - \Lambda\right)$$

where $f_{GR}(M) = \mathrm{Tr}\, M$

Can get very interesting modifications of GR by changing $f(M)$.

- $f_{SDGR} = \mathrm{Tr}(M^{-1})$

self-dual GR

Its only solutions are instantons

Integrating out B we get

$$S_{SDGR}[A,M,\mu] = \frac{\sqrt{\sigma}}{16\pi G}\int (M^{-1}) F_i F_j + \mu\left(\mathrm{Tr}\, M^{-1} - \Lambda\right)$$

fixes the trace part of M^{-1} to be constant

Can rewrite

$$S_{SDGR}[A,\Psi] = \frac{\sqrt{\sigma}}{16\pi G}\int \Psi^{ij} F_i F_j + \underbrace{\frac{\Lambda}{3}\delta^{ij} F_i F_j}$$

topological term, Pontryagin class

Varying wrt Ψ gives

$$F^i F^j \sim \delta^{ij}$$

This implies that $B^i = \frac{3}{\Lambda} F^i$ satisfies $B^i B^j \sim \delta^{ij}$ and $d^A B^i = 0$

$\Longrightarrow$ ASD Einstein instanton

This modification keeps the same "propagating DOF", but changes interactions - switches off some of the interactions present in GR

- More modifications possible. The biggest surprise is that an arbitrary modification of this type, i.e. $f(M)$ arbitrary $SO(3)$ invariant function of $\underline{M}$, does not change the number of "propagating DOF" in the theory. This is a complete surprise, because we know that GR is unique theory with this propagating content.

"Propagating DOF" are in quotes because the statement of this being unchanged by modification only really applies to the Euclidean and Split signatures. In these one counts DOF not by looking at $D\mathcal{L}=0$ type solutions, but by performing 3+1 split and analysing how many free functions one has to specify on the 3-surface to integrate field equations to the next slice. This notion of DOF makes sense in any signature

And what is the problem with Lorentzian signature?

Modifications seem to be incompatible with the reality conditions one wants to impose $B^i \wedge (B^j)^* = 0$

These guarantee that the metric constructed from B is real Lorentzian.

These constraints do not seem to commute with the evolution equations in the modified theories.

Either the "correct" reality conditions are something else, or there is no Lorentzian interpretation of the modified theories

Still, very interesting that can modify Split, Euclidean signature Einstein condition in an infinite parametric number of ways by changing $f(M)$ without changing the character of the theory. This is totally invisible in all other formulations.

Coming back to GR, some other transformations are possible, analogous to the non-chiral case.

Can first do field redefinitions $\quad B = G\tilde{B} + HF$

$$S[B,A,M,\mu] = \frac{\sqrt{6}}{8\pi G}\int B_i F^i - \frac{1}{2} M^{ij} B_i B_j + \frac{\mu}{2}\left(\mathrm{Tr}\left(\frac{M}{\mathbb{1}+2tM}\right) - \Lambda\right) + tF^iF_i$$

When $f=0$ get the original Lagrangian.

For $t\neq 0$ get a one-parameter family of Lagrangians of modified gravity type with

$$f = \mathrm{Tr}\left(\frac{M}{\mathbb{1}+2tM}\right)$$

Very interesting that $f(M)$ is modified in a non-trivial way, but the theory is still GR!

BF plus potential for the B field formulation

As in the non-chiral case can take the theory with $f(M)$ as above and integrate out M, μ. Get BF type formulation without Lagrange multipliers

The equation for M $\qquad\qquad X_B = B^i B^j$

$$(\mathbb{1}+2tM)\, X_B\, (\mathbb{1}+2tM) = \mu \cdot \mathbb{1}$$

Gives $\quad 2tM = \sqrt{\mu}\, X_B^{-1/2} - \mathbb{1}$

$$\Rightarrow \quad M(\mathbb{1}+2tM)^{-1} = \frac{1}{2t}\left(\mathbb{1} - \sqrt{\frac{X_B}{\mu}}\right)$$

Can now get μ from the constraint on M

$$\sqrt{\mu} = \frac{\mathrm{Tr}\sqrt{X_B}}{3-2t\Lambda} \quad \text{so} \quad M = \frac{1}{2t}\left(\frac{\mathrm{Tr}\sqrt{X_B}}{3-2t\Lambda}\, X_B^{-1/2} - \mathbb{1}\right)$$

Substituting into the action gives

$$V(B) = \mathrm{Tr}(M X_B) = \frac{1}{2t}\left(\frac{(\mathrm{Tr}\sqrt{X_B})^2}{3-2t\Lambda} - \mathrm{Tr}\, X_B\right)$$

So, the full BF action without Lagrange multipliers is

1708.07694

$$S[B,A] = \frac{\sqrt{6}}{8\pi G}\int B_i F^i + \frac{1}{4t} B^i B_i - \frac{1}{4t(3-2t\Lambda)}\left(\mathrm{Tr}\sqrt{B^i B^j}\right)^2$$

Surprisingly, this action describes GR for any t.

We are used to Lagrange multipliers being needed in an action of this type. But they are not here!

Can also solve $d^A B^i = 0$ for $A = A(B)$

A second order formulation with just B fields is possible!

KK 0911.4903 hep-th

Freidel 0812.3200 gr-qc

Summary

- A great variety of formulations of GR is possible. More than of any other physical theory (by far)

  May be there is a hint in this fact?

- Every formulation apart from metric uses some other bundle over spacetime, and then some form of soldering to relate this bundle to the tangent bundle.

  Different formalisms can be classified by the type of the soldering object.

  Possibilities we considered:

  - Co-frame $e: T_xM \to V$ tangent vectors to $\mathbb{R}^n$
  - Cartan connection $A: T_pP \to \mathfrak{g}$ tangent vectors to the total space of an H bundle over M to $\mathfrak{g}$
  - Pure spin connection formalism

    $$\omega : T_xM \to so(1,3)$$

    metric is constructed from the curvature of the spin connection

    $$R(\omega) : \underbrace{T_xM \times T_xM}_{\text{anti-symmetric}} \to so(1,3)$$

- BF formalism

$$B : T_xM \times T_xM \to so(1,3)$$
anti-symmetric

metric is constructed from this Lorentz Lie algebra valued 2-form field

- Plebanski formalism

$$B : so(3) \to \Lambda^2$$

metric is as the one that makes the image into self-dual 2-forms

- Chiral pure connection formalism

$$A : T_xM \to so(3)$$

metric is constructed from its curvature

$$F : T_xM \times T_xM \to so(3)$$
anti-symmetric

In all these formalisms (apart from may be MacDowell Mansouri) there is a second order description just based on the main object from the above list

- Any of the above descriptions of GR can be used as the starting point for "enlarging" the theory — unification. The idea is to make the bundle larger. In particular this means that what was a non-degenerate map becomes degenerate.

Review "Gravity and Unification" with Percacci 1712.03061

- Different formulations offer different prospects in practical applications. I would single out the chiral formalisms as having biggest hidden potential. The person who implements chiral Einstein-Cartan on the computer will be very pleasantly surprised how efficient it is

What next?

We don't know which of the geometries reviewed holds greatest potential (if any).

Gravity is gauge theory with soldering

Perhaps we need other pieces of the puzzle to get further clues on how to proceed.

Geometry of Standard Model fermions seems to give strong hints