Spinors and Geometric Structures

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Mathematics (geometry) motivation

Riemannian geometry – metric – is a reduction of the $GL(n,\mathbb{R})$ principal frame bundle to an $O(n)$ bundle.

Often one is interested in adding other geometric structures apart from the metric – e.g. orthogonal complex structure or several such structures (hyperkähler)

In many examples such structures can be encoded by a spinor on the manifold, and when the structure is integrable – parallel spinor. E.g. Calabi-Yau – parallel pure spinor

So, one is often interested in the setup

Metric + Spinor

The purpose of this talk is to explain that there is an equivalent viewpoint

Metric + Spinor $=$ Collection of differential forms on M

Sometimes, there is just a single object in the collection on the right-hand side.

Most of this talk is about algebra. But in all known examples, if one wants the geometric structure described by the spinor to be integrable $\Longleftrightarrow$ differential forms are closed

Physics (unification) motivations

There are 4 different types of fields that are used by physicists to describe the reality

So-called bosonic fields (Forces)

- Scalar fields (e.g. famous Higgs)
- Vector fields (electromagnetic potential, more generally gauge fields)
- Metric = gravity

Fermionic fields (Particles)

- Spinors = matter (particle) fields

"The idea of unification, in its simplest form, is to find a principle that realises different fields as components of some other fields

For bosonic fields such an idea is known for a very long time — dimensional reduction

E.g. if $M^n = X^k \times \mathbb{R}^{n-k}$

- gauge fields on M are gauge fields + scalars on X
- metric on M is metric on X + gauge fields + scalars on X

Spinors also behave very nicely under the dimensional reduction

$Spin(2n)$, S - spinor representation $S = S^+ \oplus S^-$

If take $Spin(2k) \times Spin(2(n-k)) \subset Spin(2n)$

then $S^+_{2n} = S^+_{2k} \otimes S^+_{2(n-k)} \oplus S^-_{2k} \otimes S^-_{2(n-k)}$

Spinors remain spinors under dimensional reduction

So, appears that one just needs a metric and a spinor in sufficiently high number of dimensions to encode all known fields

In this talk I want to explain that there is a large collection of examples where

Metric + Spinor on $\mathbb{R}^{2n}$ $\Longleftrightarrow$ Collection of differential forms on $\mathbb{R}^{2n}$

From the point of view of unification

— ultimate unification in objects of the same type.

Moreover, it appears that this encoding is always possible

Very little new in this talk, just the interpretation

Spinors and complex structures

For the first construction will stay in the more familiar territory of $Spin(2n)$ and complex structures.

Restriction to $U(n) \subset Spin(2n)$ arises if one chooses a complex structure on $\mathbb{R}^{2n}$

$J^2 = -\mathbb{1}$

$$\mathbb{R}^{2n}_{\mathbb{C}} = E^+ \oplus E^-$$

$$J E^{\pm} = \pm i E^{\pm}$$

eigenspaces of eigenvalue $\pm i$

$E^{\pm}$ are totally null $\quad (E^+, E^+) = 0$

Choosing a complex structure gives a very concerte and useful model for Cliff$(2n)$ and S

Consider $S := \Lambda(\mathbb{C}^n)$ where $\mathbb{C}^n \simeq E^+$

Define $a_i^+, a_i \quad i = 1, \dots, n$

fermionic creation-annihilation operators $\qquad \{a_i, a_j^+\} = \delta_{ij}$

Define $\gamma_i := a_i^+ + a_i \qquad \gamma_{i+n} := i(a_i^+ - a_i)$

Eeasy to check that generate Cliff$(2n)$

$$\gamma_I \gamma_J + \gamma_J \gamma_I = 2\delta_{IJ} \qquad I = 1, \dots, 2n$$

Canonical realisation of a_i^+, a_i on S

$$\Psi = \sum \Psi_{i_1 \dots i_k} dz_{i_1} \wedge \dots \wedge dz_{i_k}$$

$$a_i^+ \Psi = dz_i \wedge \Psi$$

$$a_i \Psi = i \, \partial/\partial z_i \, \Psi$$

preserves $S = \Lambda^+ \oplus \Lambda^-$

$Spin(2n)$ Lie algebra is generated by $\gamma_I \gamma_J$

$u(n)$ is generated by $a_i^+ a_j$

preserves the grading of $S = \Lambda(\mathbb{C}^n)$ by diff. form degree

<u>Charge conjugation</u>

$$\tilde{\Psi} = R(\Psi) := \gamma_1 \gamma_2 \cdots \gamma_n \Psi^*$$

complex conjugation

Either commutes or anti-commutes with all γ_I

so commutes with $spin(2n)$

When $R : S_+ \to S_+$ (n even)

squares to plus or minus identity

Invariant inner product in S

$$\langle \Psi_1, \Psi_2 \rangle = \widetilde{\Psi}_1 \wedge \Psi_2 \Big|_{\text{top form}}$$

$\widetilde{\Psi}$ differential form with all elementary 1-form factors written in the opposite order

$$\widetilde{dz} = dz$$

$$\widetilde{dz_1 \wedge dz_2} = - dz_1 \wedge dz_2$$

$$\widetilde{dz_1 \wedge dz_2 \wedge dz_3} = - dz_1 \wedge dz_2 \wedge dz_3$$

$$\widetilde{dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4} = dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4$$

Pure spinors

Definition $M(\Psi) \subset \mathbb{R}^{2n}_{\mathbb{C}}$ subspace spanned by all Clifford generators that annihilate Ψ

(empty inhomogeneous diff. form)

Example: $\Psi = 1$ is annihilated by all annihilation operators

$$a_i = \frac{1}{2}(\gamma_i + i\gamma_{i+n})$$

and so $M(1) = \mathrm{Span}(\gamma_i + i\gamma_{i+n}) = E^+$

When $J(\gamma_i) = -\gamma_{i+n}$
$J(\gamma_{i+n}) = \gamma_i$

Definition: A spinor whose annihilator subspace has the maximal dimension possible $- n -$ is called a pure spinor

Proposition: Lines of pure spinors (i.e. pure spinors up to scale) are in one-to-one correspondence with orthogonal complex structures

In one direction this is because one can declare $M(\Psi_{pure})$ to be E^+ of the complex structure. Thus pure spinors "parametrise" complex structures

Proposition: Pure spinors are Weyl spinors (i.e. $\Psi_{pure} \in S_\pm$)

Pure spinors in S_+, S_- encode complex structures giving different orientations of $\mathbb{R}^{2n}$

Proposition: Weyl spinors in dimensions $2n = 2, 4, 6$ are always pure.

The first impure spinors arise for $Spin(8)$ (and are related to octonions)

Geometric map
(or generalised Hopf map)

A Weyl spinor Ψ can be inserted between a set of γ-matrices, generating elements of $\Lambda_{\mathbb{C}}(\mathbb{R}^{2n})$

$$\langle \Psi, \gamma_{I_1} \cdots \gamma_{I_k} \Psi \rangle = B_{I_1 \cdots I_k} \equiv B_k(\Psi) \in \Lambda^k_{\mathbb{C}}(\mathbb{R}^{2n})$$

One can also use $\Psi, \bar{\Psi}$ for this purpose

$$\langle \bar{\Psi}, \gamma_{I_1} \cdots \gamma_{I_k} \Psi \rangle$$

The arising geometric objects are squares of Ψ, and this is why $\Psi = \sqrt{\text{geometry}}$

Talk by Atiyah
"What is a spinor"

Proposition (Cartan): A spinor Ψ is pure if and only if

$$B_k(\Psi) = 0 \quad \forall k < n$$

and $B_n(\Psi)$ is decomposable and given by the product of directions in $M(\Psi)$

Incidentally, this gives a way to see that spinors in 2, 4, 6 dimensions are pure

$2n = 2$ can only construct $\langle \Psi, \gamma_I \Psi \rangle$

$$\langle S_+, S_- \rangle \neq 0$$

$2n = 4$ $\langle S_+, S_+ \rangle \neq 0$ but anti-symmetric,

so $\langle \Psi, \Psi \rangle = 0$

$2n = 6$ $\langle S_+, S_- \rangle \neq 0$ but γ-matrices anti-symmetric

$$\langle \Psi, \gamma_I \Psi \rangle = 0$$

The first interesting case is $2n = 8$.

The obstruction to purity is $\langle \Psi, \Psi \rangle = B_0(\Psi)$

because $\langle \Psi, \gamma_I \gamma_J \Psi \rangle = 0$ identically

$S_+ \approx \mathbb{C}^8$ and so pure spinors – null quadric in $\mathbb{C}^8$

Example of $Spin(4)$ more closely $\qquad \Psi = \underbrace{\alpha + \beta dz_{12}}_{S_+} + \underbrace{\gamma dz_1 + \delta dz_2}_{S_-}$

$$Spin(4) = \underset{L}{SU(2)} \times \underset{R}{SU(2)}$$

S_+ – two-component columns, fundamental rep of $SU_L(2)$

$$S_+ \ni \Psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \alpha, \beta \in \mathbb{C}$$

$$R : S_+ \to S_+$$

$$\langle \bar{\Psi}, \Psi \rangle = |\alpha|^2 + |\beta|^2$$

$$\langle S_+, S_+ \rangle \neq 0$$

(but anti-symmetric)

The geometric objects one can construct from Ψ

$$\langle \bar{\Psi}, \gamma_I \gamma_J \Psi \rangle = i \Sigma^i V^i$$

$$V^i := (2\mathrm{Re}(\alpha^* \beta), 2\mathrm{Im}(\alpha^* \beta), |\alpha|^2 - |\beta|^2)$$

$$\Sigma^i = dx^4 dx^i - \frac{1}{2} \epsilon^{ijk} dx^j dx^k$$ self-dual 2-forms

$$(J_\Psi)_I{}^J := \frac{1}{|V|} V^i \Sigma^i{}_I{}^J$$ squares to $-\mathbb{1}$

complex structure corresponding to Ψ

$\langle \Psi, \gamma_I \gamma_J \Psi \rangle$ – decomposable, given by the product of two eigendirections of J_Ψ

Metric + Spinors $\longrightarrow$ Differential Forms

We have seen that a spinor (for simplicity assume $\dim M = 2n$ and $\Psi \in S^+$)

defines a set of differential forms
$$\langle \Psi, \gamma \cdots \gamma \Psi \rangle \quad \text{complex}$$
$$\langle \bar{\Psi}, \gamma \cdots \gamma \Psi \rangle \quad \text{real}$$

these will not be general, and satisfy algebraic constraints by their very origin

The question is to what degree the metric in M and the spinor Ψ are characterised by the differential forms they produce.

In the example $\dim M = 4$ the complex structure J can be recovered from either $\langle \Psi, \gamma_I \gamma_J \Psi \rangle$ or from $\langle \bar{\Psi}, \gamma_I \gamma_J \Psi \rangle$

E.g. when $\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \langle \bar{\Psi}, \gamma\gamma\Psi \rangle = i\Sigma^3 = i\omega$

$$\langle \Psi, \gamma\gamma\Psi \rangle = i(\Sigma^1 + i\Sigma^2) = i\Omega$$

raising the index of Σ^3 gives J directly,
alternatively $\Sigma^1 + i\Sigma^2$ is decomposable, and gives the eigenvectors of J

Thus, the knowledge of either of the two and the metric recovers the spinor, at least projectively

Alternatively, when know both $\langle \Psi, \gamma\gamma \Psi \rangle$ and $\langle \Psi, \gamma\gamma \Psi \rangle$ can recover both the spinor and the metric

Indeed, we have the familiar story of reductions of G- structures

two of the three give the third

Metric $O(2n)$

$J \Longleftrightarrow \Omega$ projectively

$\boxed{\omega}$

Kähler $U(n)$

Complex structure $GL(n,\mathbb{C})$

Symplectic structure $Sp(2n,\mathbb{R})$

The knowledge of $\langle \bar{\mathfrak{f}}, \gamma\gamma \mathfrak{f} \rangle$ gives the complex structure with its two decomposable factors as eigenvectors of J

The knowledge of $\langle \mathfrak{f}, \gamma\gamma \mathfrak{f} \rangle$ gives the Kähler 2-form

Together, the complex structure and the Kähler form recover the metric

Algebraic constraints that the data satisfies

$$\Omega \wedge \bar{\Omega} = 2\omega \wedge \omega$$
$$\Omega \wedge \omega = 0$$
$$\Omega \wedge \Omega = 0$$

Stabiliser of this data in $GL(4)$ is $SU(2)$

$$\dim GL(4)/SU(2) = 16 - 3 = 13$$

metric + const spinor

The formula for the metric explicitly

$\Sigma^1 + i\Sigma^2 = \Omega$

$\Sigma^3 = \omega$

$$g(\xi, \eta)\, V_g = \frac{1}{6} \epsilon^{ijk}\, i_\xi \Sigma^i \wedge i_\eta \Sigma^j \wedge \Sigma^k$$

$i, j, k = 1, 2, 3$

$\Sigma^i \wedge \Sigma^j \sim \delta^{ij}$

But can also rewrite

$$g(\xi, \eta)\, V_g = \frac{1}{2}\left(i_\xi \Sigma^1\, i_\eta \Sigma^2 + i_\eta \Sigma^1\, i_\xi \Sigma^2 \right) \Sigma^3$$

$$g(\xi, \eta)\, V_g = \left(\mathrm{Im}\; i_\xi \bar{\Omega} \wedge i_\eta \Omega \right) \omega$$

In this form generalises to arbitrary dimension $\quad \Omega, \omega \Rightarrow g$

This suggests the following general principle

The knowledge of all geometric objects constructed from a (Weyl) spinor	$\Longleftrightarrow$	The knowledge of both the metric and the spinor (possibly up to some ambiguity)

This holds for all pure spinors of Spin (2n), in the way we described

$$\Omega, \omega \Longrightarrow g, \Psi_{\text{pure}}$$

But this principle is more general and covers impure spinors as well

Impure spinors

These first arise for Spin(8).

In this dimension we also have $R : S_+ \to S_+$ and $R^2 = +\mathbb{1}$

So, meaningful to impose $R\Psi = \Psi$ Majorana condition

Majorana spinors of Spin(8) can be identified with octonions

$\langle \Psi_-, \gamma \Psi_+ \rangle$

map

$S_- \times S_+ \to \mathbb{R}^8$

$$S_{\pm}^{\text{Majorana}} = \mathbb{O}$$

Non-zero Majorana-Weyl spinors of Spin(8) are never pure, and so octonions give the "purest" impure spinors

$$\langle \Psi^M, \Psi^M \rangle = |q|^2$$

where q is the octonion representing Ψ^M

For Majorana spinors $R(\Psi) \equiv \bar{\Psi} = \Psi$,

so the only geometric object we can construct is

$$\langle \Psi, \gamma \cdots \gamma \Psi \rangle$$

In the case of Spin (8) the non-vanishing objects are

$$|\Psi|^2 \equiv \langle \Psi, \Psi \rangle \quad \text{and} \quad \langle \Psi, \gamma\gamma\gamma\gamma \Psi \rangle = B_4(\Psi)$$

According to our general principle, the knowledge of this 4-form should recover both the metric and the spinor

This indeed holds. The 4-form produced by an impure spinor is of a special algebraic type — Cayley form in $\mathbb{R}^8$

Its stabiliser in $GL(8, \mathbb{R})$ is $\mathrm{Spin}(7)$

$$\text{Cayley forms} = GL(8,\mathbb{R}) / Spin(7)$$

$$\text{dimension} = 64 - 21 = 43$$

Metric in $\mathbb{R}^8$ + unit Majorana spinor

$$\frac{8 \cdot 9}{2} + 7 = 43$$

This case is particularly striking, because the metric + spinor information is encoded by a single differential form rather than a collection of such forms

Cayley form can be described in several different ways
(depending on the subgroup of Spin (7) we restrict to).

If restrict to $Spin(6) = SU(4)$, adding a complex structure in $\mathbb{R}^8$

$$\varphi = -\frac{1}{2}\omega\wedge\omega + Re(\Omega)$$

The metric of φ can be recovered from the "area" metric

$$G(\xi_1, y_1 ; \xi_2, y_2) V = i_{\xi_1} i_{y_1}\varphi \wedge i_{\xi_2} i_{y_2}\varphi \wedge \varphi$$

For φ of the algebraic type of the Cayley form

$$G(\xi_1, y_1 ; \xi_2, y_2) = g(\xi_1, \xi_2) g(y_1, y_2) - g(y_1, \xi_2) g(\xi_1, y_2)$$

Higher dimensions

A general Weyl spinor can always be represented as a sum of pure spinors (non-uniquely)

Can define "parity" as the minimal # of pure spinors needed to represent a given spinor

The higher the dimension, the higher parities one gets.

Complex spinors of $Spin(2n, \mathbb{C})$ are understood and classified up to and including $2n = 14$

The case of $2n = 16$, where there are again Majorana-Weyl spinors of $Spin(16)$ presents new difficulties and only partially understood

It is quite interesting to consider Spin groups for which real spinors exist.

In this case there is a smaller number of differential forms that one can construct.

For example Spin(16)

$\underline{\Psi}_M \in S^+$ — there are many different orbits now (unlike the case of Spin(8))

But always has

$$\Psi_M \otimes \Psi_M = \mathbb{R} \oplus \Lambda^4(\mathbb{R}^{16}) \oplus \Lambda^8(\mathbb{R}^{16})$$

self-dual 8-form

$B_0 = \langle \Psi_M, \Psi_M \rangle$ $B_4 = \langle \Psi_M, \gamma\gamma\gamma\gamma \Psi_M \rangle$ $B_8 = \langle \Psi_M, \underbrace{\gamma \cdots \gamma}_{8\text{ times}} \Psi_M \rangle$

For example, there is a special orbit where B_8 does not add any new information and B_4 characterises metric + unit spinor in this orbit completely

In this case, $B_4 \in \Lambda^4(\mathbb{R}^{16})$ with very special algebraic properties

defines a metric + unit spinor (in a special orbit)

Would be interesting to understand other orbits from this point of view

Summary

- Fixing the metric and adding a spinor endows the space with additional geometric structures
  Encoded by certain differential forms constructed from the spinor ?

$$\text{Spinor} = \sqrt{\text{geometry}}$$

- Did not discuss this aspect, but easy to impose the condition that these geometric structures are integrable
  – simply require that differential forms are closed
- Complex and CY structures all fall into this pattern
  – pure spinors
- As one increases the dimension more and more interesting types of spinors appear

- In all known examples (where spinor gives access to at least two pure spinors)

All differential forms produced by a spinor $\Longleftrightarrow$ Metric + Spinor

Conjecture – this is always true.

It would be very interesting to work out the details and understand the types of geometry that can arise, e.g. in 16 II