BALANCING SOURCE TERMS AND FLUX GRADIENTS IN FINITE VOLUME SCHEMES

M. E. HUBBARD
DAMTP, Silver Street, Cambridge, CB3 9EW, U.K.
Email: M.E.Hubbard@damtp.cam.ac.uk

AND

P. GARCIA-NAVARRO
Fluid Mechanics, CPS, University of Zaragoza, 50015, Spain.
Email: pigar@posta.unizar.es

1. Introduction

In the field of computational hydraulics the modelling can be dominated by the effects of source terms and in some cases, quantities which vary spatially but independently of the flow variables. This paper is concerned with the shallow water equations and how the additional terms should be discretised, given that Roe's scheme has been used to approximate the flux terms, extending recent research by other authors (Glaister, 1992; Vázquez-Cendón, 1999; Bermúdez and Vázquez, 1994). In each of these papers the discrete form of the source terms has been deliberately constructed along similar lines to the numerical fluxes. This is done to ensure that equilibria which occur in the mathematical model are retained by the numerical model, and that in the absence of additional terms, the conservative fluxes are retrieved for accurate modelling of discontinuous solutions. However, all previous work deals only with the first order scheme. In this paper the extension of these ideas to higher order Total Variation Diminishing (TVD) versions of Roe's scheme (using both flux limiting and slope limiting techniques) is described. It is then possible to construct a source term approximation which has each of the above properties on all types of regular and irregular grids in any number of dimensions; see (Hubbard and Garcia-Navarro, 1999) for details. Furthermore, following on from (Garcia-Navarro and Vázquez-Cendón, 1997), a new formulation is presented for the discretisation of the flux in the case where it depends on a spatially varying quantity which is independent of the solution. The one-dimensional shallow water equations have been chosen to demonstrate the effectiveness of these new techniques, by modelling the effects of a sloping bed and the inclusion of breadth variation in open channel flows.

2. The General Discretisation

The one-dimensional equations representing a general system of conservation laws with source terms may be written

$$\underline{U}_t + \underline{F}_x \;=\; \underline{S}\,, \tag{1}$$

where $\underline{U}$ is the vector of conservative variables, $\underline{F}$ is the conservative flux vector and $\underline{S}$ includes all of the source terms. The flux is assumed to depend not only on the conservative variables but also another independent, spatially varying quantity, denoted here by $B(x)$, *i.e.* $\underline{F} = \underline{F}(\underline{U}, B(x))$.

Using the standard (cell centre) finite volume approximation of the flux terms in (1) with forward Euler time-stepping gives

$$\underline{U}_i^{n+1} \;=\; \underline{U}_i^n - \frac{\Delta t}{\Delta x_i}\left(\underline{F}^*_{i+\frac{1}{2}} - \underline{F}^*_{i-\frac{1}{2}}\right) + \frac{\Delta t}{\Delta x_i}\,\underline{\mathbf{S}}^*_i\,, \tag{2}$$

in which $\underline{F}^*$ represents a numerical flux evaluated at an interface between cells and $\underline{\mathbf{S}}^* \approx \int \underline{S}\,\mathrm{d}x$ is a numerical source integral over the cell.

Roe's scheme (Roe, 1981) is used here to discretise the flux derivatives, with a minor modification to take into account the dependence of the flux on $B(x)$. This approximate Riemann solver splits the flux difference at an interface into independent components, giving

$$\begin{aligned}\Delta\underline{F}_{i+\frac{1}{2}} \;&=\; \left(\tilde{\mathbf{A}}\,\Delta\underline{U} + \underline{\tilde{V}}\right)_{i+\frac{1}{2}} \;=\; \left(\tilde{\mathbf{R}}\tilde{\mathbf{\Lambda}}\tilde{\mathbf{R}}^{-1}\,\Delta\underline{U} + \tilde{\mathbf{R}}\,\tilde{\mathbf{R}}^{-1}\underline{\tilde{V}}\right)_{i+\frac{1}{2}} \\ &=\; \left(\sum_{k=1}^{N_w}\tilde{\alpha}_k\tilde{\lambda}_k\underline{\tilde{r}}_k + \sum_{k=1}^{N_w}\tilde{\gamma}_k\underline{\tilde{r}}_k\right)_{i+\frac{1}{2}}\,,\end{aligned} \tag{3}$$

where $\underline{\tilde{V}} \approx \frac{\partial F}{\partial B}\Delta B$ (so it reduces to the standard Roe flux difference splitting when B is constant). $\Delta\underline{F}$ represents the jump in $\underline{F}$ across the edge of a grid cell, $\tilde{\mathbf{R}}$ is the matrix whose columns are the right eigenvectors $\underline{\tilde{r}}_k$ of $\tilde{\mathbf{A}}$, the approximate flux Jacobian, $\tilde{\mathbf{\Lambda}}$ is the diagonal matrix of eigenvalues $\tilde{\lambda}_k$ of $\tilde{\mathbf{A}}$, and the components of $\tilde{\mathbf{R}}^{-1}\,\Delta\underline{U}$ are the 'strengths' $\tilde{\alpha}_k$ associated with each component of the decomposition. Additionally, $\tilde{\gamma}_k$, the coefficients of the decomposition of the extra term, are the components of $\tilde{\mathbf{R}}^{-1}\underline{\tilde{V}}$. The final expression in (3) indicates how the flux difference is decomposed into N_w characteristic components, N_w being the number of equations in the system (1). Throughout $\tilde{\cdot}$ denotes the evaluation of a quantity at its Roe-average state (Roe, 1981), calculated specifically so that (3) is satisfied.

The numerical fluxes which are used in (2) are simply

$$\underline{F}^*_{i+\frac{1}{2}} = \frac{1}{2}\left(\underline{F}_{i+1} + \underline{F}_i\right) - \frac{1}{2}\left(\tilde{\mathbf{R}}|\tilde{\mathbf{\Lambda}}|\tilde{\mathbf{R}}^{-1}\,\Delta\underline{U} + \tilde{\mathbf{R}}\,\mathrm{sgn}(\mathbf{I})\tilde{\mathbf{R}}^{-1}\underline{\tilde{V}}\right)_{i+\frac{1}{2}}\,, \quad (4)$$

in the first order case, where $|\tilde{\mathbf{\Lambda}}| = \mathrm{diag}(|\tilde{\lambda}_k|)$ and $\mathrm{sgn}(\mathbf{I}) = \tilde{\mathbf{\Lambda}}^{-1}|\tilde{\mathbf{\Lambda}}|$. When a flux limited high resolution scheme is being used,

$$\underline{F}^*_{i+\frac{1}{2}} = \frac{1}{2}\left(\underline{F}_{i+1} + \underline{F}_i\right) - \frac{1}{2}\left(\tilde{\mathbf{R}}|\tilde{\mathbf{\Lambda}}|\mathbf{L}\tilde{\mathbf{R}}^{-1}\,\Delta\underline{U} + \tilde{\mathbf{R}}\,\mathrm{sgn}(\mathbf{I})\mathbf{L}\tilde{\mathbf{R}}^{-1}\underline{\tilde{V}}\right)_{i+\frac{1}{2}}\,, \quad (5)$$

in which, additionally, $\mathbf{L} = \mathrm{diag}(1 - L(r_k)(1 - |\nu_k|))$, where $\nu_k = \tilde{\lambda}_k \Delta t/\Delta x$ is the Courant number associated with the k^{th} component of the decomposition, L is a nonlinear flux limiter function, and $r_k = \tilde{\alpha}_k^{\text{upwind}}/\tilde{\alpha}_k^{\text{local}}$. If the high resolution scheme being used employs slope limiters then

$$\underline{F}^*_{i+\frac{1}{2}} = \frac{1}{2}\left(\underline{F}^{\mathrm{R}}_{i+\frac{1}{2}} + \underline{F}^{\mathrm{L}}_{i+\frac{1}{2}}\right) - \frac{1}{2}\left(\tilde{\mathbf{R}}|\tilde{\mathbf{\Lambda}}|\tilde{\mathbf{R}}^{-1}\,\Delta\underline{U} + \tilde{\mathbf{R}}\,\mathrm{sgn}(\mathbf{I})\tilde{\mathbf{R}}^{-1}\underline{\tilde{V}}\right)_{i+\frac{1}{2}}\,, \quad (6)$$

where the superscripts $\cdot^{\mathrm{R}}$ and $\cdot^{\mathrm{L}}$ represent evaluation on, respectively, the right and left hand sides of the interface indicated by the associated subscript, so the averages $(\tilde{\cdot})$ are now calculated from a linear reconstruction of the solution.

Following the work of (Glaister, 1992; Bermúdez and Vázquez, 1994; Garcia-Navarro and Vázquez-Cendón, 1997), the approximate source term integral associated with an edge of a cell is similarly projected on to the eigenvectors of the flux Jacobian, so that in its linearised form it becomes

$$\int_{x_i}^{x_{i+1}} \underline{S}\,\mathrm{d}x \approx \underline{\tilde{\mathbf{S}}}_{i+\frac{1}{2}} = \left(\tilde{\mathbf{R}}\,\tilde{\mathbf{R}}^{-1}\underline{\tilde{\mathbf{S}}}\right)_{i+\frac{1}{2}} = \left(\sum_{k=1}^{N_w} \tilde{\beta}_k \underline{\tilde{r}}_k\right)_{i+\frac{1}{2}}\,, \quad (7)$$

where $\tilde{\beta}_k$, the coefficients of the decomposition, are the components of the vector $\tilde{\mathbf{R}}^{-1}\underline{\tilde{\mathbf{S}}}$. $\underline{\mathbf{S}}^*_i$ will be constructed out of contributions from both ends of the cell, with consistency assured as long as the whole of each dual cell integral (7) is distributed.

In order to obtain the discrete balance which is required between the flux and source terms the numerical source term integral of (2) is approximated by

$$\underline{\mathbf{S}}^*_i = \underline{\mathbf{S}}^{*\,-}_{i+\frac{1}{2}} + \underline{\mathbf{S}}^{*\,+}_{i-\frac{1}{2}}\,, \quad (8)$$

in which the edge contributions are given by

$$\begin{aligned} \underline{\mathbf{S}}^{*\,-}_{i+\frac{1}{2}} &= \frac{1}{2}\left(\tilde{\mathbf{R}}(\mathbf{I}-\mathrm{sgn}(\mathbf{I}))\tilde{\mathbf{R}}^{-1}\underline{\tilde{\mathbf{S}}}\right)_{i+\frac{1}{2}}, \\ \underline{\mathbf{S}}^{*\,+}_{i-\frac{1}{2}} &= \frac{1}{2}\left(\tilde{\mathbf{R}}(\mathbf{I}+\mathrm{sgn}(\mathbf{I}))\tilde{\mathbf{R}}^{-1}\underline{\tilde{\mathbf{S}}}\right)_{i-\frac{1}{2}}. \end{aligned} \tag{9}$$

It is now simple to make high resolution corrections to the numerical source terms which will maintain the required balance. In the flux limiting case this leads to replacing (9) with

$$\underline{\mathbf{S}}^{*\,-}_{i+\frac{1}{2}} = \frac{1}{2}\left(\tilde{\mathbf{R}}(\mathbf{I}-\mathrm{sgn}(\mathbf{I})\mathbf{L})\tilde{\mathbf{R}}^{-1}\underline{\tilde{\mathbf{S}}}\right)_{i+\frac{1}{2}}, \tag{10}$$

and a similar expression for $\underline{\mathbf{S}}^{*\,+}_{i-\frac{1}{2}}$. When slope limiters are applied an appropriate correction to the numerical source within each cell is given by

$$\underline{\mathbf{S}}^{*}_{i} = \left(\underline{\mathbf{S}}^{*\,-}_{i+\frac{1}{2}} + \underline{\mathbf{S}}^{*\,+}_{i-\frac{1}{2}}\right) - \underline{\tilde{\mathbf{S}}}\left(\underline{U}^{\mathrm{L}}_{i+\frac{1}{2}}, \underline{U}^{\mathrm{R}}_{i-\frac{1}{2}}\right). \tag{11}$$

The first term on the right hand side is evaluated precisely as before, in (8), except that the interface values are now those of the linear reconstruction of the solution within each cell. $\underline{\tilde{\mathbf{S}}}$ is simply the source term integral approximated over the mesh cell and hence evaluated at the Roe-average of the left and right states of the linear reconstruction of the solution within that cell.

3. Shallow water flows

In one dimension, shallow water flow through a rectangular open channel of varying breadth and bed slope is modelled by the equations

$$\begin{pmatrix} bd \\ bdu \end{pmatrix}_t + \begin{pmatrix} bdu \\ bdu^2 + \frac{1}{2}gbd^2 \end{pmatrix}_x = \begin{pmatrix} 0 \\ \frac{1}{2}gd^2 b_x - gbdz_x \end{pmatrix}, \tag{12}$$

which, when compared with (1) to find $\underline{U}$, $\underline{F}$ and $\underline{S}$, ultimately leads to

$$\frac{\partial \underline{F}}{\partial \underline{U}} = \begin{pmatrix} 0 & 1 \\ gd - u^2 & 2u \end{pmatrix}, \quad \frac{\partial \underline{F}}{\partial B} = \begin{pmatrix} 0 \\ -\frac{1}{2}gd^2 \end{pmatrix}. \tag{13}$$

In these equations d is the depth of the flow, z is the height of the bed above a nominal zero level, $b = b(x)$ is the channel breadth, u is the flow velocity, and g is the acceleration due to gravity.

The characteristic decomposition (3) for the equations (12) and (13) is completely defined by

$$\begin{aligned}\tilde{\alpha}_1 &= \frac{\Delta(bd)}{2} + \frac{1}{2\tilde{c}}\left(\Delta(bdu) - \tilde{u}\,\Delta(bd)\right)\\ \tilde{\alpha}_2 &= \frac{\Delta(bd)}{2} - \frac{1}{2\tilde{c}}\left(\Delta(bdu) - \tilde{u}\,\Delta(bd)\right)\end{aligned}$$

$$\tilde{\lambda}_1 = \tilde{u} + \tilde{c}\ , \quad \tilde{\lambda}_2 = \tilde{u} - \tilde{c}\ , \quad \tilde{\gamma}_1 = -\frac{1}{4g}\tilde{c}^3\Delta b\ , \quad \tilde{\gamma}_2 = \frac{1}{4g}\tilde{c}^3\Delta b$$

$$\underline{\tilde{r}}_1 = \begin{pmatrix} 1 \\ \tilde{u}+\tilde{c} \end{pmatrix}, \quad \underline{\tilde{r}}_2 = \begin{pmatrix} 1 \\ \tilde{u}-\tilde{c} \end{pmatrix}, \tag{14}$$

and it is easily shown that (3) is satisfied exactly when

$$\tilde{u} = \frac{\sqrt{b^{\mathrm{R}}d^{\mathrm{R}}}u^{\mathrm{R}} + \sqrt{b^{\mathrm{L}}d^{\mathrm{L}}}u^{\mathrm{L}}}{\sqrt{b^{\mathrm{R}}d^{\mathrm{R}}} + \sqrt{b^{\mathrm{L}}d^{\mathrm{L}}}}\ , \quad \tilde{c}^2 = g\left(\frac{\sqrt{b^{\mathrm{R}}}d^{\mathrm{R}} + \sqrt{b^{\mathrm{L}}}d^{\mathrm{L}}}{\sqrt{b^{\mathrm{R}}} + \sqrt{b^{\mathrm{L}}}}\right), \tag{15}$$

which reduce to standard Roe-averages for one-dimensional shallow water flow in the absence of breadth variation (*i.e.* when $b^{\mathrm{R}} = b^{\mathrm{L}}$). The corresponding decomposition of the source terms (7) then leads to

$$\tilde{\beta}_1 = \frac{1}{4g}\tilde{c}^3\Delta b - \frac{1}{2}\tilde{b}\tilde{c}\Delta z = -\tilde{\beta}_2\,. \tag{16}$$

In order for (9) to maintain the correct balance when the flow is quiescent, $\tilde{b}$ is constructed so that it satisfies

$$\tilde{b}\Delta z = \Delta(bz) - \tilde{z}\Delta b \quad \text{where} \quad \tilde{z} = K - \frac{\sqrt{b^{\mathrm{R}}}d^{\mathrm{R}} + \sqrt{b^{\mathrm{L}}}d^{\mathrm{L}}}{\sqrt{b^{\mathrm{R}}} + \sqrt{b^{\mathrm{L}}}}\,, \tag{17}$$

K being the height of the still water surface above the nominal zero level.

4. Numerical results

The upwind source term discretisation described above maintains still water to machine accuracy for an indefinite period for any test case geometry for both first and higher order schemes (unlike most standard approximations) so no results of this type are presented here. Instead a 'tidal' flow is modelled in a channel of varying breadth and depth, and compared with an asymptotically exact solution, described fully in (Vázquez-Cendón, 1999). The comparison is made between first order, slope limited and flux limited schemes combined with pointwise and upwind source term discretisations: in all high resolution cases the Minmod limiter has been applied. The 'exact' and numerical solutions (all computed on the same regular 600 cell

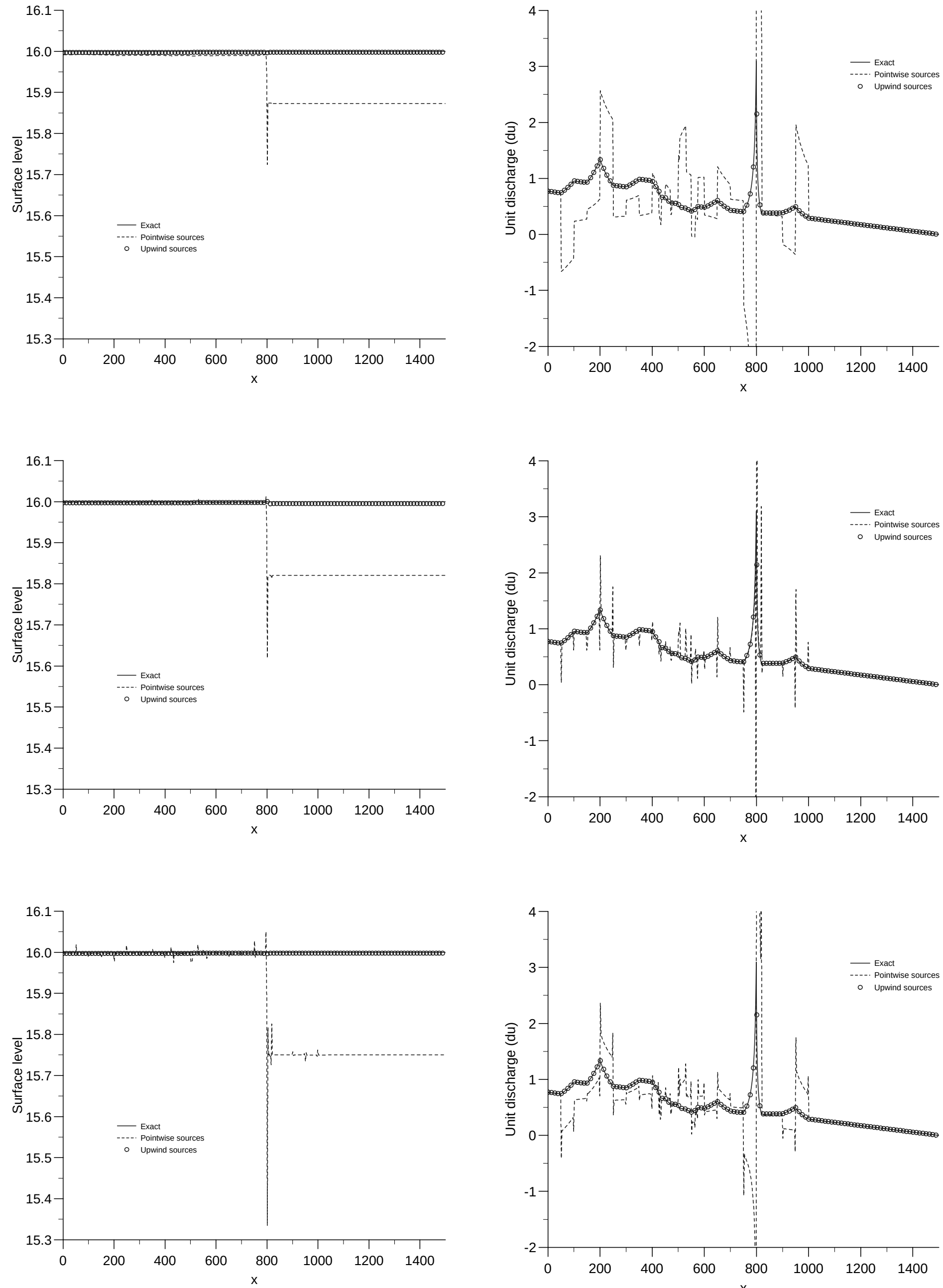

Figure 1. Water surface level and unit discharge for the tidal flow test case for first order (top) and high resolution slope limited (centre) and flux limited (bottom) schemes.

grid) to this problem when $t = 10800$ ('high tide') are compared in Figure 1. The agreement is very close, not only for the first order scheme, but also for both of the higher order schemes when the upwind source discretisation is used. However, as in the still water tests, the pointwise source discretisation gives, at best, only a reasonable approximation to the depth, and a very poor prediction of the flow velocity.

5. Conclusions

A new method has been presented for the discretisation of source terms which provide a balance with flux derivatives in nonlinear systems of conservation laws, extending the work of (Glaister, 1992; Bermúdez and Vázquez, 1994; Garcia-Navarro and Vázquez-Cendón, 1997) to high order TVD versions of Roe's scheme, using both flux and slope limiters. A technique for discretising fluxes which can vary independently of the flow variables is also suggested. The methods have been shown to be effective in the modelling of the one-dimensional shallow water equations (on the understanding that the TVD condition which underlies the limiting procedures is derived for homogeneous equations), and they also readily generalise for use on arbitrary polygonal meshes in any number of dimensions (Hubbard and Garcia-Navarro, 1999).

Acknowledgements

The work of the first author was funded by the EPSRC grant GK/K74616.

References

Bermúdez A and Vázquez M E (1994). Upwind Methods for Hyperbolic Conservation Laws with Source Terms. *Computers and Fluids*, **23(8)**, pp 1049-1071.

Garcia-Navarro P and Vázquez-Cendón M E (1997). Some Considerations and Improvements on the Performance of Roe's Scheme for 1D Irregular Geometries. Internal Report 23, Departamento de Matemática Aplicada, Universidade de Santiago do Compostela.

Glaister P (1992). Prediction of Supercritical Flow in Open Channels. *Comput. Math. Applic.*, **24(7)**, pp 69-75.

Hubbard M E and Garcia-Navarro P (1999). Flux Difference Splitting and the Balancing of Source Terms and Flux Gradients. Report NA-3/99, Department of Mathematics, University of Reading (submitted to *J. Comput. Phys.*).

Roe P L (1981). Approximate Riemann Solvers, Parameter Vectors, and Difference Schemes. *J. Comput. Phys.*, **43(2)**, pp 357-372.

Vázquez-Cendón M E (1999). Improved Treatment of Source Terms in Upwind Schemes for the Shallow Water Equations in Channels with Irregular Geometry. *J. Comput. Phys.*, **148(2)**, pp 497-526.