Pattern formation in a two-dimensional simple chemical system with general orders of autocatalysis and decay.

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Abstract

In this paper we investigate pattern formation in a coupled system of reaction-diffusion equations in two spatial dimensions. These equations arise as a model of isothermal chemical autocatalysis with termination in which the orders of autocatalysis and termination, m and n respectively are such that $1 < n < m$. We build on the preliminary work by Leach and Wei (Physica D, 180 (3-4), 185-209, 2003) for this coupled system in one spatial dimension, by presenting rigorous stability analysis and detailed numerical simulations for the coupled system in two spatial dimensions. We demonstrate that spotty patterns are observed over a wide parameter range.

1 Introduction

In part I of this series of papers, Leach and Wei [4], (hereafter referred to as (I)), we considered pattern formation in a coupled system of reaction-diffusion equations in one spatial dimension. These equations arise as a simple chemical model of an isothermal, autocatalytic reaction scheme with termination. In general, the scheme may be represented formally by the two steps,

$$\mathrm{A} \rightarrow \mathrm{B} \qquad \text{rate} \quad k_1 a b^m, \quad \text{(autocatalytic step)} \tag{1}$$
$$\mathrm{B} \rightarrow \mathrm{C} \qquad \text{rate} \quad k_2 b^n, \quad \text{(decay step)} \tag{2}$$

Here a and b are the concentrations of the reactant A and the autocatalyst B respectively, $k_1 > 0$ is the rate constant of the autocatalysis, $k_2 > 0$ is the rate constant at which the autocatalyst B decays to the inert, stable product C, and m and n are the orders of the autocatalysis and decay (where we restrict attention to the case $m > n > 1$). In (I) it was supposed that the reaction was taking place within a one dimensional reactor, with the reactant A, whose concentration is fixed outside the reactor at a uniform, non-zero, concentration, being replenished inside the reactor by transport through the reactor walls from a well-stirred reservoir. The autocatalyst B which is initially introduced locally into the expanse of A (which initially is at uniform concentration), however is unable to pass through the reactor walls.

In this present paper we extend the preliminary work of (I) by considering a two dimensional spatial domain. A full description of the chemical system, derivation of the mathematical model and a list of relevant references can be found in (I) and is not repeated here for brevity. The equations governing the initial-value problem in two spatial dimensions are, in dimensionless form, from (I),

$$\alpha_t = \nabla^2\alpha - \alpha\beta^m + \mu(1-\alpha), \tag{3a}$$

$$\beta_t = D\nabla^2\beta + \alpha\beta^m - k\beta^n, \qquad x \in \mathbb{R}^2, t > 0, \tag{3b}$$

$$\alpha(x,0) = 1, \quad \beta(x,0) = \begin{cases} \beta_0 g(x) & |x| \leq \sigma, \\ 0 & |x| > \sigma, \end{cases} \tag{3c}$$

$$\alpha(x,t) \to 1, \beta(x,t) \to 0 \quad \text{as} \quad |x| \to \infty, \quad t \geq 0. \tag{3d}$$

Here α and β are the dimensionless concentrations of the reactant A and the autocatalyst B, respectively. The function $g(x)$ is positive and continuous in $|x| \leq \sigma$ with $\max(g(x)) = 1$, on $|x| \leq \sigma$. The initial value problem (3) has five positive dimensionless parameters, the specific details of which are discussed in (I). The parameter σ measures the spread of the initial input of the autocatalyst, k measures the strength of the termination step (2) relative to that of the autocatalytic step (1), β_0 measures the maximum concentration of the initial input of the autocatalyst, D measures the rate of diffusion of the autocatalyst relative to that of reactant whilst μ measures the strength of transport of reactant A relative to that of the autocatalytic step (1). In what follows we restrict attention to $D \ll 1$ and $1 < n < m < \infty$ with $k, \mu, \beta_0, \sigma > 0$.

In this paper we investigate the existence and stability of nontrivial patterns in $\mathbb{R}^2$ which can arise in initial value problem (3), when $\mu > 0$. These stationary, stable, symmetric **spotty** patterns in the nondimensional concentration of the autocatalyst β form in the wake of the advancing wave front in the nondimensional concentration of the reactant α.

Before we state our main results we first introduce some notation and basic results:

The uniform homogeneous stationary states of system (3), (α_s, β_s) are obtained from solving the pair of algebraic equations,

$$\alpha_s\beta_s^m = \mu(1-\alpha_s), \quad \alpha_s\beta_s^m = k\beta_s^n. \tag{4}$$

Clearly, (4) admits the trivial solution $(1, 0)$. The remaining uniform stationary states of (4) can be obtained as follows. On rewriting equations (4) in terms of β_s we obtain,

$$\beta_s^m - \frac{\mu}{k}\beta_s^{m-n} + \mu = 0, \tag{5}$$

where

$$\alpha_s = 1 - \frac{k}{\mu}\beta_s^n. \tag{6}$$

Clearly, (5) has two real positive roots, β_s^- and β_s^+, for $\mu > \mu^* = \frac{(km)^{m/(m-n)}}{(m-n)n^{n/(m-n)}}$, where,

$$0 < \beta_s^- < \left[\frac{\mu(m-n)}{km}\right]^{1/n} < \beta_s^+ < \left[\frac{\mu}{k}\right]^{1/n}, \tag{7}$$

one real positive root, $\beta_s^* = \left[\frac{mk}{n}\right]^{1/(m-n)}$ for $\mu = \mu^*$ and no real positive roots for $\mu < \mu^*$. We note that for $\mu \geq \mu^*$ that $0 < \alpha_s < 1$ with in particular that $0 < \alpha_s = \frac{n}{m} < 1$ when $\mu = \mu^*$.

We let w be the unique solution of the following problem:

$$\Delta_y w - w^n + w^m = 0 \qquad \text{in} \quad R^2, \tag{8a}$$

$$w(0) = \max_{y \in R^2} w(y), \; w(y) > 0 \qquad \text{for all} \quad y \in R^2, \tag{8b}$$

$$w(y) \to 0 \qquad \text{as} \quad |y| \to \infty. \tag{8c}$$

By the well-known result of Gidas-Ni-Nirenberg [2], any solution to (8a), $w(y)$, must be radially symmetric and nonincreasing, i.e., $w(y) = w(|y|), w_r(r) < 0$ for $r = |y| > 0$. The existence of a radial solution to (8a) can readily be established via a variational approach. The uniqueness of radial solution to (8a) is given in [6]. Here the fact that we are dealing with two-dimensional domain is vital as such solutions may not exist in three or more dimensions. We note that

$$w(y) = \frac{c_{m,n}}{|y|^{\frac{2}{(n-1)}}} + O\left(\frac{1}{|y|^{\frac{2(m-n+1)}{(n-1)}}}\right) \quad \text{as} \quad |y| \to \infty, \tag{9}$$

where $c_{m,n}$ is a constant which depends on m and n. We also need to introduce the following important parameters

$$L = \frac{D}{2\pi}\int_{R^2} w^m \quad k^{\frac{1}{m-n}} \log[(\mu D)^{\frac{1}{2}} k^{-\frac{m-1}{2(m-n)}}], \tag{10}$$

$$\gamma_0 = \frac{1}{m-n}, \tag{11}$$

and

$$L_0 = \frac{\gamma_0^{\gamma_0}}{(\gamma_0+1)^{(\gamma_0+1)}}. \tag{12}$$

If $0 < L < L_0$, then there are exactly two solutions to the following algebraic equation

$$\xi^{\gamma_0}(1-\xi) = L. \tag{13}$$

We denote these two solutions as ξ^s and ξ^l where $0 < \xi^s < \xi^l < 1$. Further, we note that

$$0 < \xi^s < \xi_0 = \frac{\gamma_0}{(\gamma_0+1)} < \xi^l < 1. \tag{14}$$

Associated with ξ^s and ξ^l we define

$$\epsilon^s = \left(\frac{\mu D}{k}\right)^{\frac{1}{2}} \left(\frac{\xi^s}{k}\right)^{\frac{(n-1)}{2(m-n)}}, \quad \epsilon^l = \left(\frac{\mu D}{k}\right)^{\frac{1}{2}} \left(\frac{\xi^l}{k}\right)^{\frac{(n-1)}{2(m-n)}}, \tag{15}$$

$$A^s = \left(\frac{k}{\xi^s}\right)^{\frac{1}{(m-n)}}, \quad A^l = \left(\frac{k}{\xi^l}\right)^{\frac{1}{(m-n)}}. \tag{16}$$

In the following statements, we drop the superscripts s and l if this leads to no confusion. The parameters (ϵ, A, ξ) satisfy the following relations

$$1 - \xi = \frac{1}{2\pi}\int_{R^2} w^m \quad \frac{A^m \epsilon^2 \log\frac{1}{\epsilon}\xi}{\mu}, \tag{17}$$

$$\frac{\mu D}{\epsilon^2} = kA^{n-1} = \xi A^{m-1}. \tag{18}$$

On rescaling equations (3), using the parameters introduced above, via

$$\bar{x} = \sqrt{\mu}x, \quad \bar{\alpha}(\bar{x}) = \alpha(x), \quad \bar{\beta}(\bar{x}) = \frac{\beta(x)}{A}, \quad \bar{t} = kA^{(n-1)}t, \quad \tau = \frac{kA^{(n-1)}}{\mu} = D\epsilon^2, \tag{19}$$

we obtain (on dropping the overbar for ease of notation),

$$\beta_t = \epsilon^2\beta_{xx} - \beta^n + \xi^{-1}\alpha\beta^m, \tag{20}$$

$$\tau\alpha_t = \alpha_{xx} + 1 - \alpha - \mu^{-1}A^m\alpha\beta^m, \tag{21}$$

which we will work with from now on.

The layout of the paper is as follows. We first consider the existence of two single-spot steady-state solutions.

Theorem 1 *If*

$$0 < L < L_0, \tag{22}$$

and

$$\epsilon \ll 1, \tag{23}$$

then there exists two single-spot steady-state solutions $(\alpha^s_\epsilon(x), \beta^s_\epsilon(x))$, $(\alpha^l_\epsilon(x), \beta^l_\epsilon(x))$ *to (20)-(21) such that*

(1) $\alpha^s(0) \sim \xi^s, \quad \alpha^l(0) \sim \xi^l$

(2) $\beta^s(x) \sim w\left(\frac{x}{\epsilon^s}\right), \quad \beta^s(x) \sim w\left(\frac{x}{\epsilon^l}\right)$

(3) $1 - \alpha(x) \to 0, \quad \beta \to 0 \quad as \quad |x| \to \infty$

Remark 2 *Unlike the Gray-Scott model where we have linear decay rate* $n = 1$, *here there are two scalings in space* $\epsilon^s < \epsilon^l$. *The scalings in space and in functions are interrelated.*

Next we consider the stability of $(\alpha^s_\epsilon, \beta^s_\epsilon)$ and $(\alpha^l_\epsilon, \beta^l_\epsilon)$. On linearizing (20)-(21) around $(\alpha^s_\epsilon, \beta^s_\epsilon)$ (or $(\alpha^l_\epsilon, \beta^l_\epsilon)$) we obtain the following eigenvalue problems

$$\epsilon^2 \Delta_x \phi - n\beta_\epsilon^{(n-1)}\phi + \xi^{-1} m\alpha_\epsilon \beta_\epsilon^{(m-1)}\phi + \xi^{-1}\psi\beta_\epsilon^m = \lambda_\epsilon \phi, \tag{24a}$$

$$\Delta_x \psi - \psi - \mu^{-1}A^m\psi\beta_\epsilon^m - \mu^{-1}A^m m\alpha_\epsilon\beta_\epsilon^{(m-1)}\phi = \tau\lambda_\epsilon\psi_\epsilon, \tag{24b}$$

where $\lambda_\epsilon \in \mathbb{C}$. Certainly $\lambda_\epsilon = 0$, $(\phi, \psi) = (\frac{\partial \beta_\epsilon}{\partial x_i}, \frac{\partial \alpha_\epsilon}{\partial x_i}), i = 1, 2$ are solutions to (24). We say that $(\alpha_\epsilon, \beta_\epsilon)$ is linearly stable

(1) If $\lambda_\epsilon = 0$ then $(\phi, \psi) \in \text{span}\,\{(\frac{\partial \beta_\epsilon}{\partial x_1}, \frac{\partial \alpha_\epsilon}{\partial x_1}), (\frac{\partial \beta_\epsilon}{\partial x_2}, \frac{\partial \alpha_\epsilon}{\partial x_2})\}$,

(2) If $\lambda_\epsilon \neq 0$ then $Re(\lambda_\epsilon) < 0$.

We say that $(\alpha_\epsilon, \beta_\epsilon)$ is linearly unstable if there exists an eigenvalue λ_ϵ to (24) such that $Re(\lambda_\epsilon) > 0$.

Before stating the stability result, we need to introduce the following definitions. Let $\xi = \xi^s$ or ξ^l and

$$\Gamma = 1 - \xi. \tag{25}$$

For each function $\phi \in C_0^\infty(R^2)$-the set of smooth functions with compact support, we introduce two norms:

$$\|\phi\|_{\mathcal{X}} = \left(\int_{R^2} (|\nabla\phi|^2 + |\phi^{'}|^2)\right)^{\frac{1}{2}}, \|\phi\|_{\mathcal{Y}} = (\int_{R^2} (|\phi|^2))^{\frac{1}{2}}. \tag{26}$$

We now define two spaces we shall work with later on: $\mathcal{X}$ is the completion of $C_0^\infty(R^2)$ under the norm $\|\cdot\||_{\mathcal{X}}$, and $\mathcal{Y}$ is the completion of $C_0^\infty(R^2)$ under the norm $\|\cdot\||_{\mathcal{Y}}$. It is easy to see that $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. (Here we do not take the obvious Hilbert space $H^1(R^2)$ as we have algebraic decay rate.) We also define

$$\mathcal{L}_0\phi = \Delta\phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi : \mathcal{X} \to \mathcal{Y}, \tag{27}$$

where $\mathcal{L}_0$ is invertible from $\mathcal{X}_s$ to $\mathcal{Y}_s$, where

$$\mathcal{X}_s = \mathcal{X} \cap \{\phi(y) = \phi(|y|)\}, \mathcal{Y}_s = \mathcal{Y} \cap \{\phi(y) = \phi(|y|)\},$$

thus $\mathcal{L}_0^{-1} w^{(m-1)}$ exists.

We are now ready to state the following theorem on stability.

Figure 1: Bifurcation diagram in the case $m = n + 1$. The dashed line representing instability, the solid part stable and the dotted section unknown.

Theorem 3 *Let $(\alpha_\epsilon, \beta_\epsilon) = (\alpha^s_\epsilon, \beta^s_\epsilon)$ or $(\alpha^l_\epsilon, \beta^l_\epsilon)$. Assume (22), (23) and further that*

$$\tau \sim 1. \tag{28}$$

Then we have

- *(a) (instability): $(\alpha^l_\epsilon, \beta^l_\epsilon)$ is linearly unstable.*
- *(b) (stability): Assume that*

  $\int_R w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)} > \frac{(2m-n+1-\frac{4}{\Gamma}(m-n))^2(m+1)\left(\int_R w^m\right)^2}{2m^2(2m-n+3)(m-n)\int_R w^{(m+1)}}$,

  then $(\alpha^s_\epsilon, \beta^s_\epsilon)$ is linearly stable.

Remark 4 *It is also possible to consider $\tau \sim \epsilon^\gamma$ for some $0 \leqslant \gamma < 2$, as was done in [11].*

We do not have an explicit formula for $\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)}$, for general m. However if $m = n + 1$, then we have

$$\int_{R^2} w^{(m-1)} \mathcal{L}_0 w^{(m-1)} = \frac{1}{m} \int_{R^2} w^m. \tag{29}$$

Hence we obtain the following corollary.

Corollary 5 *Assume that (22), (23) and (28) hold and let $\Gamma^s = 1 - \xi^s$. Then $(\alpha^s_\epsilon, \beta^s_\epsilon)$ is linearly stable provided that*

$$\Gamma^s > 4 - 2\sqrt{3}. \tag{30}$$

The bifurcation diagram when $m = n + 1$ is, via Corollary 5, given in figure 1.

2 Formal arguments and outline of the proofs

We first explain why we rescale to (20)-(21). On rewriting equations (3) in terms of $\bar{x} = \mu^{1/2} x$ we obtain,

$$\mu^{-1}\alpha_t = \Delta_{\bar{x}}\alpha - \mu^{-1}\alpha\beta^m + (1 - \alpha), \tag{31}$$

$$\beta_t = \mu D \Delta_{\bar{x}}\beta + \alpha\beta^m - k\beta^n. \tag{32}$$

The stationary-state solutions of (31),(32) are obtained from

$$\Delta_{\bar{x}}\alpha - \mu^{-1}\alpha\beta^m + (1 - \alpha) = 0, \tag{33}$$

$$\mu D \Delta_{\bar{x}}\beta + \alpha\beta^m - k\beta^n = 0. \tag{34}$$

Let us assume the following

$$\beta(\bar{x}) \sim A w\left(\frac{\bar{x}}{\epsilon}\right), \quad \epsilon \ll 1, \tag{35}$$

$$\alpha(\bar{x}) \sim \alpha(0), \tag{36}$$

and

$$y = \frac{\bar{x}}{\epsilon}.$$

Then we have from (33) ,(34) that

$$1-\alpha(0)=\frac{1}{2\pi\mu}\int_{R^2}\log\frac{1}{|\bar{x}|}\alpha\beta^m \sim \frac{1}{2\pi\mu}\alpha(0)A^m\epsilon^2\log\frac{1}{\epsilon}\int_{R^2}w^m \tag{37}$$

and

$$\frac{\mu D}{\epsilon^2}A\Delta_y w+\alpha(0)A^m w^m-kA^n w^n=0. \tag{38}$$

Hence we must have

$$1-\alpha(0)=\frac{1}{2\pi}\int_{R^2}w^m \quad \frac{A^m\epsilon^2\log\frac{1}{\epsilon}\alpha(0)}{\mu}, \tag{39}$$

$$\frac{\mu D}{\epsilon^2}=kA^{(n-1)}=\alpha(0)A^{(m-1)}. \tag{40}$$

We require a solution to (39)-(40) such that $\epsilon \ll 1$. ¿From (40), we have

$$\alpha(0)=kA^{(n-m)}, \quad A=\left(\frac{k}{\alpha(0)}\right)^{\frac{1}{(m-n)}} \tag{41}$$

and

$$\epsilon=\left(\frac{\mu D}{k}\right)^{\frac{1}{2}}A^{-\frac{(n-1)}{2}}. \tag{42}$$

On substituting (41) and (42) into (39), we obtain after some calculation that

$$1=\alpha(0)+L\alpha(0)^{-\gamma_0}, \tag{43}$$

where

$$L=\frac{D}{2\pi}\int_{R^2}w^m \quad k^{\frac{1}{m-n}}\log[(\mu D)^{\frac{1}{2}}k^{-\frac{m-1}{2(m-n)}}], \tag{44}$$

and

$$\gamma_0=\frac{1}{m-n}.$$

We now set $\alpha(0)=\xi$ where ξ must satisfy

$$\rho(\xi)=\xi^{\gamma_0}(1-\xi)-L=0. \tag{45}$$

We note that $\rho(0)=\rho(1)=-L<0$ and $\rho'(\xi_0)=0$ where $\xi_0=\frac{\gamma_0}{(\gamma_0+1)}$. Hence

$$\rho_{\max}=\rho(\xi_0)=\xi_0^{\gamma_0}(1-\xi_0)-L=L_0-L, \tag{46}$$

where L_0 is defined in (12). So $\rho(\xi)=0$ has a solution if and only if $L<L_0$. When $L<L_0$, there are two solutions $0<\xi^s<\xi_0<\xi^l<1$. We now rescale via (19) to obtain (20) and (21).

We now explain the basic ideas in proving Theorems 1 and 3. For existence, we use the implicit function theorem: consider the steady-state problem

$$\epsilon^2\Delta_x\beta-\beta^n+\xi^{-1}\alpha\beta^m=0, \tag{47}$$

$$\Delta_x\alpha+1-\alpha-\mu^{-1}A^m\alpha\beta^m=0. \tag{48}$$

For each fixed β, we can solve for α via (48). Let $\alpha=\mathcal{T}[\beta]$ and

$$x=\epsilon y, \tag{49}$$

then (47) becomes

$$\mathcal{S}[\beta]=\Delta_y\beta-\beta^n+\xi^{-1}\mathcal{T}[\beta(y)](\epsilon y)\beta^m(y)=0, \tag{50}$$

we construct solutions to (50) of the following form

$$\beta = w + \phi, \quad \|\phi\|_{\mathcal{X}} \quad \text{is small}, \tag{51}$$

where w is given in (8). On substituting (51) into (50) we obtain

$$\mathcal{S}[\beta] = \mathcal{S}[w+\phi] = \mathcal{S}[w] + \mathcal{S}'[\phi] + \mathcal{N}[\phi],$$

where $\mathcal{S}'[\phi]$ is the first order term and $\mathcal{N}[\phi]$ represents the higher order terms. We note that

$$\mathcal{T}[w](0) \approx \xi,$$
$$\mathcal{S}[w] = o(1),$$

and

$$\mathcal{S}'[\phi] \approx \Delta_y \phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi - m(1-\xi)\frac{\int_R w^{(m-1)}\phi}{\int_R w^m}\phi = \mathcal{L}[\phi].$$

So if we can show that $\mathcal{L}$ is invertible, then by the implicit function theorem, there is a solution to (50). To show that $\mathcal{L}$ is invertible, we have to study both $\mathcal{L}$ and $\mathcal{L}^*$ the conjugate operator of $\mathcal{L}$. For stability, we show in section 2, that if $\lambda_\epsilon \to \lambda_0$ then λ_0 satisfies

$$\Delta\phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi - m(1-\xi)\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}\phi = \lambda_0\phi. \tag{52}$$

We give a proof of the existence theorem 1 in section 4. In section 5 we use the results of section 3 to prove the stability theorem 3.

3 A study of a nonlocal eigenvalue problem

Let w be the solution of

$$\Delta w - w^n + w^m = 0 \quad \text{in} \quad R^2, \tag{53}$$

$$w(y) > 0 \quad \text{for all} \quad y \in R^2, \qquad w(y) \to 0 \quad \text{as} \quad |y| \to \infty, \tag{54}$$

$$w(0) = \max_{y \in R^2} w(y). \tag{55}$$

We state the following lemma on the properties of w.

Lemma 6 *There exists a unique solution, called w, to (53)-(55). Moreover w is radially symmetric and $w_r < 0$ for $r = |y| > 0$ and the kernel of the following linearized operator*

$$\mathcal{L}_0\phi := \Delta\phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi. \tag{56}$$

consists of linear combinations of $\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}$.

Proof: By the well-known theorem of Gidas-Ni-Nirenberg [2], the solution to (53)-(55) must be radially symmetric. Then the result of [6] shows that w is actually unique. Then the argument in Lemma 4.2 of [8] proves the last statement.

□

It is easy to see that

$$\mathcal{L}_0 w = (1-n)w^n + (m-1)w^m, \tag{57}$$

$$\mathcal{L}_0(y \cdot \nabla w) = 2w^n - 2w^m. \tag{58}$$

Multiplying (57) by $y \cdot \nabla w$, and (58) by w, and integrating over R^2, we obtain

$$\int_{R^2} w^{(m+1)} = \frac{(m+1)}{(n+1)}\int_{R^2} w^{n+1}. \tag{59}$$

From (57) and (58), we obtain the following key identities

$$\mathcal{L}_0\left(\frac{w}{(m-n)}+\frac{(m-1)}{2(m-n)}y\cdot\nabla w\right)=w^n, \tag{60}$$

$$\mathcal{L}_0\left(\frac{w}{(m-n)}+\frac{(n-1)}{2(m-n)}y\cdot\nabla w\right)=w^m. \tag{61}$$

We are now ready to study the following important nonlocal eigenvalue problem which is essential in the proofs of Theorem 1 and Theorem 3

$$\mathcal{L}\phi=\Delta\phi-nw^{(n-1)}\phi+mw^{(m-1)}\phi-m\Gamma\frac{\int_{R^2}w^{(m-1)}\phi}{\int_{R^2}w^m}w^m=\lambda\phi. \tag{62}$$

Remark 7 *In the case of $n=1, m=2$, problem (62) has been studied in [10] by functional analysis approach. Here such approach does not work here. Instead, we use a continuity approach, which is inspired by [14].*

In one dimensional case and $n=1, m=2$, problem (62) has been studied in [1] by a hypergeometric function approach. Such an approach is no longer working in higher dimensional case.

We now recall the following Lemma.

Lemma 8 *(Theorem 1.4 of [10].) Let $n=1, m=2$. Then we have*

$$\int_{R^2}\left(|\nabla\phi|^2+\phi^2\right)-2\int_{R^2}w\phi^2+2\frac{\int_{R^2}w\phi\ \int_{R^2}w^2\phi}{\int_{R^2}w^2}-\frac{\left(\int_{R^2}w\phi\right)^2}{\left(\int_{R^2}w^2\right)^2}\int_{R^2}w^3\geq c_0 d_{L^2(R^2)}(\phi,X_1), \tag{63}$$

where $X_1 := span\{w, \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

We define a quadratic form

$$Q_{m,n}[\phi]=\int_{R^2}\left(|\nabla\phi|^2+nw^{(n-1)}\phi^2\right)-m\int_{R^2}w^{(m-1)}\phi^2+m\frac{\int_{R^2}w^{(m-1)}\phi\int_{R^2}w^m\phi}{\int_R w^m}. \tag{64}$$

(63) implies, in particular that

$$Q_{m,n}[\phi]>0 \quad \text{for} \quad \phi\not\equiv 0, \qquad \text{when} \quad n=1, m=2,$$

since w is a continuous function of n and m (by the uniqueness of w). We now employ a continuation argument and vary (m,n). It is straightforward to see that

$$Q_{m,n}[\phi]=\left(-\int_{R^2}\left(\mathcal{L}_{m,n}\phi\right)\phi\right),$$

where

$$\mathcal{L}_{m,n}\phi=\Delta\phi-nw^{(n-1)}\phi+mw^{(m-1)}\phi-\frac{m\Gamma}{2}\frac{\int_{R^2}w^{(m-1)}\phi}{\int_{R^2}w^m}w^m-\frac{m\Gamma}{2}\frac{\int_{R^2}w^m\phi}{\int_{R^2}w^m}w^{(m-1)}. \tag{65}$$

Clearly,

$$Q_{m,n} \quad \text{is positive definite} \quad \iff \mathcal{L}_{m,n} \quad \text{has negative spectrum only.}$$

We begin with $(n,m)=(1,2)$ and vary (n,m). Suppose at some point (n,m), $L_{m,n}$ has a zero eigenvalue, that is there exists a $\phi\not\equiv 0$ such that

$$\mathcal{L}_{m,n}\phi=\Delta\phi-nw^{(n-1)}\phi+mw^{(m-1)}\phi-\frac{m\Gamma}{2}\frac{\int_{R^2}w^{(m-1)}\phi}{\int_{R^2}w^m}w^m-\frac{m\Gamma}{2}\frac{\int_{R^2}w^m\phi}{\int_{R^2}w^m}w^{(m-1)}=0, \tag{66}$$

which is equivalent to

$$\phi = \frac{m\Gamma}{2} \frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m} \mathcal{L}_0^{-1} w^m + \frac{m\Gamma}{2} \frac{\int_{R^2} w^m \phi}{\int_{R^2} w^m} \mathcal{L}_0^{-1} w^{(m-1)}. \tag{67}$$

Now let $c_1 = \int_{R^2} w^{(m-1)}\phi$ and $c_2 = \int_{R^2} w^m \phi$. Then we have

$$c_1 = \frac{m}{2} \frac{\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^m}{\int_{R^2} w^m} c_1 + \frac{m}{2} \frac{\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)}}{\int_{R^2} w^m} c_2, \tag{68}$$

$$c_2 = \frac{m}{2} \frac{\int_{R^2} w^m \mathcal{L}_0^{-1} w^m}{\int_{R^2} w^m} c_1 + \frac{m}{2} \frac{\int_{R^2} w^m \mathcal{L}_0^{-1} w^{(m-1)}}{\int_{R^2} w^m} c_2. \tag{69}$$

Since $c_1^2 + c_2^2 \neq 0$, we have

$$\begin{vmatrix} \frac{m\Gamma}{2} \frac{\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^m}{\int_{R^2} w^m} - 1 & \frac{m\Gamma}{2} \frac{\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)}}{\int_{R^2} w^m} \\ \frac{m\Gamma}{2} \frac{\int_{R^2} w^m \mathcal{L}_0^{-1} w^m}{\int_{R^2} w^m} & \frac{m\Gamma}{2} \frac{\int_{R^2} w^m \mathcal{L}_0^{-1} w^{(m-1)}}{\int_{R^2} w^m} - 1 \end{vmatrix} = 0, \tag{70}$$

which is equivalent to

$$\left(\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^m - \frac{2}{m\Gamma} \int_{R^2} w^m \right)^2 - \int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)} \int_{R^2} w^m \mathcal{L}_0^{-1} w^m = 0. \tag{71}$$

By (61),

$$\mathcal{L}_0^{-1} w^m = \frac{1}{(m-n)} \left(w + \frac{(n-1)}{2} y \cdot \nabla w \right),$$

$$\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^m = \frac{1}{(m-n)} \left(\int_{R^2} w^m - \frac{(n-1)}{m} \int_{R^2} w^m \right) = \frac{(m-n+1)}{(m-n)(m)} \int_{R^2} w^m, \tag{72}$$

and

$$\int_{R^2} w^m \mathcal{L}_0^{-1} w^m = \frac{1}{(m-n)} \left(\int_{R^2} w^{(m+1)} - \frac{(n-1)}{(m+1)} \int_{R^2} w^{(m+1)} \right) = \frac{(m-n+2)}{(m-n)(m+1)} \int_{R^2} w^{(m+1)}. \tag{73}$$

On substituting (72) and (73) into (71), we obtain that

$$\left(\frac{(m-n+1)}{m(m-n)} - \frac{2}{m\Gamma} \right)^2 \left(\int_{R^2} w^m \right)^2 - \frac{(m-n+2)}{(m-n)(m+1)} \int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)} \int_{R^2} w^{(m+1)} = 0. \tag{74}$$

We conclude that

Theorem 9 *If for $1 < n < m$, we have*

$$\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)} > \frac{\left((2/\Gamma - 1)n + (1 - 2/\Gamma)m + 1\right)^2 (m+1)}{m^2(m-n+2)(m-n)} \frac{\left(\int_{R^2} w^m\right)^2}{\int_{R^2} w^{(m+1)}}, \tag{75}$$

then

$$Q_{m,n}[u] > 0 \quad \textit{for} \quad u \not\equiv 0.$$

Proof: First, (75) holds for $n = 1, m = 2$. We vary (n, m), starting from $(n, m) = (1, 2)$. Suppose at some point $(n, m), Q_{m,n}$ is not positive definite. Then $\mathcal{L}_{m,n}$ must have a zero eigenvalue, which implies (74) must hold which is a contradiction to (75). □

It remains to compute $\int_{R^2} w^{(m-1)} \mathcal{L}_0^{-1} w^{(m-1)}$. In general, this is difficult. We consider the special case, where

$$m = n + 1. \tag{76}$$

Then we have by (60) that

$$\mathcal{L}_0^{-1}w^{(m-1)} = \mathcal{L}_0^{-1}w^n = \frac{1}{(m-n)}\left(w + \frac{(m-1)}{2}y\cdot\nabla w\right).$$

Hence

$$\int_{R^2} w^{(m-1)}\mathcal{L}_0^{-1}w^{(m-1)} = \frac{1}{m(m-n)}\int_{R^2} w^m = \frac{1}{m}\int_{R^2} w^m. \tag{77}$$

Note, via(59), that

$$\int_{R^2} w^m = \int_{R^2} w^{(n+1)} = \frac{(n+1)}{(m+1)}\int_{R^2} w^{(m+1)}. \tag{78}$$

On substituting (77) and (78) into (75), we have directly that (75) hold if and only if $4-2\sqrt{3} < \Gamma < 4+2\sqrt{3}$.

We now have the following corollary.

Corollary 10 *Let* $m = n+1$ *and* $4-2\sqrt{3} < \Gamma < 4+2\sqrt{3}$. *Then* $Q_{m,n}[\phi] > 0$ *for all* $\phi \not\equiv 0$.

On returning to the study of (62) we have the following lemma.

Lemma 11 *If* $Q_{m,n}[\phi] > 0$ *for all* $\phi \not\equiv 0$, *then for all nonzero eigenvalues* $\lambda_0 \neq 0$ *of (62), we have*

$$Re(\lambda_0) < 0.$$

Proof: Let $\lambda_0 = \lambda_R + \sqrt{-1}\lambda_I$ and $\phi = \phi_R + \sqrt{-1}\phi_I$. Then on multiplying (62) by $\bar{\phi} = \phi_R - \sqrt{-1}\phi_I$ and integrating over R^2, we obtain

$$-Q_{m,n}[\phi_R] - Q_{m,n}[\phi_I] = \lambda_R \int_{R^2}\left(|\phi_R|^2 + |\phi_I|^2\right), \tag{79}$$

which yields that $\lambda_R < 0$. □

Finally we consider (62) with $\lambda = 0$. We have the following Lemma.

Lemma 12

$$Kernel(\mathcal{L}) = Kernel(\mathcal{L}^*) = span\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}, \tag{80}$$

where $\mathcal{L}^*$ *is the conjugate operator of* $\mathcal{L}$, *namely*

$$\mathcal{L}^*\phi = \Delta\phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi - m\frac{\int_{R^2} w^m\phi}{\int_{R^2} w^m}w^{(m-1)}. \tag{81}$$

Proof: Let $\mathcal{L}\phi = 0$. Then

$$\mathcal{L}_0\phi = m\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}w^m = C(\phi)w^m,$$

$$\mathcal{L}_0\left(\phi - C(\phi)\mathcal{L}_0^{-1}w^m\right) = 0,$$

where

$$C(\phi) = m\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}.$$

Lemma 6 implies

$$\phi - m\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}\left(\frac{1}{(m-n)}\left(w + \frac{(n-1)}{2}yw'\right)\right) = a_1\frac{\partial w}{\partial y_1} + a_2\frac{\partial w}{\partial y_2}, \tag{82}$$

for some constants a_1 and a_2. Multiplying (82) by $w^{(m-1)}$ and integrating over R, we obtain

$$\left(1 - \frac{m}{(m-n)}\left(1 - \frac{(n-1)}{m}\right)\right)\int_{R^2} w^{(m-1)}\phi = 0, \tag{83}$$

which implies that

$$\int_{R^2} w^{(m-1)}\phi = 0,$$

and hence that $\phi \in \text{span}\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2})\}$. Next if $\mathcal{L}^*\phi = 0$, then

$$\mathcal{L}_0\phi = m\frac{\int_{R^2} w^m\phi}{\int_{R^2} w^m} w^{(m-1)}. \tag{84}$$

The argument proceeds, after minor modification, that given above with now

$$\mathcal{L}_0\left(\phi - m\frac{\int_{R^2} w^m\phi}{\int_{R^2} w^m}\mathcal{L}_0^{-1}w^{(m-1)}\right) = 0,$$

$$\phi - m\frac{\int_{R^2} w^m\phi}{\int_{R^2} w^m}\mathcal{L}_0^{-1}w^{(m-1)} = \beta w'.$$

Since $\int_{R^2} w^m\mathcal{L}_0^{-1}w^{(m-1)} = \int_{R^2} w^{(m-1)}\mathcal{L}_0^{-1}w^m$ we arrive at the following

$$\int_{R^2} w^m\phi = 0 \quad \text{and} \quad \phi = \beta_1\frac{\partial w}{\partial y_1} + \beta_2\frac{\partial w}{\partial y_2}.$$

□

Our last theorem concerns instability.

Lemma 13 *Let*

$$\Gamma < \Gamma_0 := \frac{(m-n)}{m-n+1}. \tag{85}$$

Then there exists an eigenvalue $\lambda_0 > 0$ *satisfying*

$$\Delta\phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi - m\Gamma\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}w^m = \lambda_0\phi. \tag{86}$$

Proof: First, we show that the operator $\mathcal{L}_0$ has a unique positive eigenvalue $\mu_1 > 0$ with the eigenfunction being radially symmetric and positive. In fact, we consider $\mathcal{L}_0$ on the space $\mathcal{X}_s$. It is well-known that for $n = 1$, $\mathcal{L}_0$ has a unique positive eigenvalue on $\mathcal{X}_s$. (See for example Theorem 1.1 of [7].) As we vary n, $\mathcal{L}_0$ will have only one positive eigenvalue until zero becomes an eigenvalue for $\mathcal{L}_0$ on $\mathcal{X}_s$, which is impossible. Thus for all $1 \leqslant n < m$, $\mathcal{L}_0$ will have a unique positive eigenvalue, which is the principal one, and hence the eigenfunction can be made positive. Solving (86) is equivalent to

$$\phi = m\Gamma\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}\int_{R^2}(\mathcal{L}_0 - \lambda)^{-1}w^m,$$

$$h(\lambda) = \int_{R^2} w^m - \Gamma m\int_{R^2} w^{(m-1)}(\mathcal{L}_0 - \lambda)^{-1}w^m = 0, \quad \lambda \in (0, \mu_1).$$

Note that

$$h(0) = \int_{R^2} w^m - \Gamma m\int_{R^2} w^{(m-1)}\mathcal{L}_0^{-1}w^m = \left(1 - \frac{\Gamma}{\Gamma_0}\right)\int_{R^2} w^m > 0.$$

On the other hand, for $\lambda \to \mu_1, \lambda < \mu_1$, we have

$$\int_{R^2} w^{(m-1)}(\mathcal{L}_0 - \lambda)^{-1}w^m \to +\infty \quad h(\lambda) \to -\infty.$$

By the mean-value theorem, there exists a $\lambda_0 \in (0, \mu_1)$, such that $h(\lambda_0) = 0$. Such a λ_0 will satisfy (86). □

We conclude this section with the following summary:

Theorem 14 *Consider the following eigenvalue problem*

$$\mathcal{L}\phi := \Delta\phi - nw^{(n-1)}\phi + mw^{(m-1)}\phi - m\Gamma\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m} w^m = \lambda\phi. \tag{87}$$

(1) If $\lambda = 0$ and $\Gamma \neq \Gamma_0$ then $\phi \in span\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$

(2) If $\lambda = 0$ and $\Gamma = \Gamma_0$ then $\phi \in span\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}, w + \frac{(n-1)}{2} y \cdot \nabla w\}$

(3) If $\Gamma < \Gamma_0$ then there exists a positive $\lambda_0 > 0$ satisfying (87).

(4) If $\Gamma > \Gamma_0$ and

$$\int_{R^2} w^{(m-1)}\mathcal{L}_0^{-1} w^{(m-1)} > \frac{((2/\Gamma - 1)n + (1 - 2/\Gamma)m + 1)^2(m+1)}{m^2(m-n+2)(m-n)} \frac{\left(\int_{R^2} w^m\right)^2}{\int_{R^2} w^{(m+1)}},$$

then for any non-zero eigenvalue λ of (87), $Re(\lambda_R) < 0$.

(5) If $m = n + 1$ and

$$4 - 2\sqrt{3} < \Gamma < 4 + 2\sqrt{3} \tag{88}$$

then for any non-zero eigenvalue λ of (87), $Re(\lambda) < 0$.

(6) If $\Gamma \neq \Gamma_0$ then Kernel($\mathcal{L}$) = Kernel($\mathcal{L}^*$)=span$\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.

Theorem 14 forms the basis for what follows in this paper. We remark that at $\Gamma = \Gamma_0$ the eigenfunction is given by

$$w_0(y) = w(y) + \frac{(n-1)}{2} y \cdot \nabla w(y).$$

We now consider the asymptotic behaviour of $w_0(y)$ for $|y| \gg 1$. To this end, we let

$$w(y) = A_0\gamma^{-\frac{2}{(n-1)}} + B_0\gamma^s, \quad \gamma = |y| \gg 1. \tag{89}$$

On substituting (89) into equation (8a), we obtain that

$$s = -\frac{2(m-n+1)}{(n-1)},$$

$$A_0 = \left(\frac{2}{(n-1)}\right)^{\frac{2}{(n-1)}},$$

$$B_0 = \frac{(2/(n-1))^{\frac{2(m-n+1)}{n-1}}}{n - (m-n+1)^2}.$$

Clearly, if

$$n > (m-n+1)^2 \tag{90}$$

then $B_0 > 0$. Note that this is possible if $m = n + 1$ provided $n > 4$. Clearly,

$$w_0(y) = B_0\left(1 + s\frac{(n-1)}{2}\right)\gamma^s < 0,$$

for $\gamma \gg 1$. The graph of $w_0(y)$ is given in figure 2. We note that $(-w_0(y))$ is a dent function and will be the eigenfunction responsible for the pulse-splitting for L near L_0.

Figure 2: Graph of $-w_0(y)$ against y

4 Construction of single-spot solutions

We fix $\xi = \xi^s$, the proof of the other case when $\xi = \xi^l$ follows that for $\xi = \xi^s$ and is not repeated here for brevity. In this section, we construct single-spot solutions to (20)-(21):

$$\epsilon^2 \Delta_x \beta - \beta^n + \xi^{-1} \alpha \beta^m = 0, \tag{91}$$

$$\Delta_x \alpha + 1 - \alpha - \mu^{-1} A^m \alpha \beta^m = 0, \tag{92}$$

$$\alpha, \beta > 0, \quad \beta(x), 1 - \alpha(x) \to 0 \quad \text{as} \quad |x| \to \infty. \tag{93}$$

We will follow the proofs given in Section 4 of [11]. Since the procedure is similar to that in [11], we shall highlight the steps and differences briefly. We shall write (91)-(92) as a nonlocal single equation. Before this, we first note the following Lemma.

Lemma 15 *Given $\beta = \beta(|x|)$ such that $\beta(x) \to 0$ as $|x| \to \infty$ then there exists a unique solution $\alpha := \mathcal{T}[\beta](x)$ such that*

$$\Delta_x \alpha + 1 - \alpha - \mu^{-1} A^m \alpha \beta^m = 0, \quad \alpha = \alpha(|x|), \quad \alpha(x) \to 1 \quad \text{as} \quad |x| \to \infty. \tag{94}$$

Proof: Follows that given in [9] for Lemma 4.1. □

Next, we study $\mathcal{T}[\beta]$, where w is the unique solution of (8). Then we have

$$1 - \mathcal{T}[w](x) = \int_{R^2} \frac{1}{2\pi} \log \frac{1}{|z-x|} \mu^{-1} A^m \mathcal{T}[w](z) w^m(\frac{z}{\epsilon}) dz$$

$$= (1 + o(1)) \epsilon^2 \mathcal{T}[w](0) \frac{\mu^{-1} A^m}{2} \int_{R^2} \log \frac{1}{|x - \epsilon y|} w^m(y) dy. \tag{95}$$

Hence

$$1 - \mathcal{T}[w](0) = (1 + o(1)) \epsilon^2 \log \frac{1}{\epsilon} \mathcal{T}[w](0) \frac{\mu^{-1} A^m}{2} \int_{R^2} w^m(y) dy.$$

giving, via (17), that

$$\frac{1 - \mathcal{T}[w](0)}{\mathcal{T}[w](0)} = \frac{(1-\xi)}{\xi}(1 + o(1)).$$

Hence, we have

$$\mathcal{T}[w](0) = (1 + o(1)) \xi. \tag{96}$$

For $x \neq 0$, we have

$$\mathcal{T}[w](x) - \mathcal{T}[w](0) = -(1 + o(1)) \epsilon^2 \mathcal{T}[w](0) \frac{\mu^{-1} A^m}{2} \int_{R^2} \left(\log \frac{\epsilon |z|}{|x - \epsilon z|} \right) w^m(z) dz$$

$$\mathcal{T}[w](\epsilon y) - \mathcal{T}[w](0) = (1 + o(1)) \epsilon^2 \mathcal{T}[w](0) \frac{\mu^{-1} A^m}{2} \left(\int_{R^2} \log(|y - z|) w^m(z) dz \right) = O(\epsilon^2 \log(1 + |y|)). \tag{97}$$

We now write (91)-(92) as a single equation. On letting $x = \epsilon y$, we obtain that

$$\Delta_y \beta - \beta^n + \xi^{-1} \mathcal{T}\left[\beta \left(\frac{x}{\epsilon} \right) \right] (\epsilon y) \beta^m(y) = 0. \tag{98}$$

We now use the implicit function theorem to solve (98) and to this end, we put

$$\beta = w(y) + \phi(y), \quad \phi \in X_s = X \cap \{\phi(y) = \phi(|y|)\}.$$

Then (91)-(92) is equivalent to the following

$$\Delta_y \beta - \beta^n + \xi^{-1}\mathcal{T}\left[\beta\left(\frac{x}{\epsilon}\right)\right](\epsilon y)\beta^m(y) = 0, \quad \beta = w + \phi, \quad \phi \in X_s. \tag{99}$$

We rewrite (99) as

$$\mathcal{S}_\epsilon[w+\phi] = \Delta_y(w+\phi) - (w+\phi)^n + \xi^{-1}\mathcal{T}[w+\phi](w+\phi)^m = 0. \tag{100}$$

Then we have that

$$\mathcal{S}_\epsilon[w] = \left(\xi^{-1}\mathcal{T}[w] - 1\right) w^m,$$

$$\mathcal{S}'_\epsilon[w](\phi) = \Delta_y \phi - n w^{(n-1)}\phi + \xi^{-1}\mathcal{T}[w] m w^{(m-1)}\phi + \xi^{-1}\mathcal{T}'[w](\phi) w^m,$$

where $\mathcal{T}'[w](\phi) = \psi$ satisfies,

$$\Delta\psi - \psi - \mu^{-1}A^m \psi w^m - \mu^{-1}A^m m\mathcal{T}[w] w^{(m-1)}\phi = 0.$$

We expand as

$$\mathcal{S}_\epsilon[w+\phi] = \mathcal{S}'_\epsilon[w](\phi) + \mathcal{S}_\epsilon[w] + \mathcal{N}_\epsilon[\phi], \tag{101}$$

where $\mathcal{N}_\epsilon[\phi]$ represents the higher order terms of ϕ. We note, via (97), that

$$\left(\xi^{-1}\mathcal{T}[w] - 1\right) w^m = O\left(\epsilon^2 \log(1+|y|) w^m\right) = o(1), \tag{102}$$

since

$$O\left(\log(1+|y|) w^m\right) = O\left(\frac{\log(1+|y|)}{(1+|y|)^{\frac{2m}{(n-1)}}}\right) = O(1).$$

Similarly to the calculations leading to (97), we have that

$$\xi^{-1}\mathcal{T}[w](\epsilon y) = \xi^{-1}\mathcal{T}[w](0) + O(\epsilon^2 \log(1+|y|)), \tag{103}$$

$$\xi^{-1}\mathcal{T}'[w](\phi)(\epsilon y) - \xi^{-1}\mathcal{T}'[w](\phi)(0) = O(\epsilon^2 \log(1+|y|)), \tag{104}$$

$$\xi^{-1}\mathcal{T}'[w](\phi)(0) = \psi(0) = -\mu^{-1}A^m \frac{1}{2\pi}\left(\int_{R^2} \log\frac{1}{|z|}\left(\psi w^m + m\mathcal{T}[w] w^{(m-1)} w^{(m-1)}\phi\right)\right)$$

$$= -\frac{1}{2\pi}\int_{R^2} w^m \epsilon^2 \log\frac{1}{\epsilon}\mu^{-1}A^m \psi(0) - \frac{1}{2\pi}\mu^{-1}\epsilon^2 \log\frac{1}{\epsilon}A^m m \int_{R^2} w^{(m-1)}\phi. \tag{105}$$

Hence

$$\xi^{-1}\mathcal{T}'[w](\phi)(0) = \frac{\frac{1}{2\pi}\mu^{-1}\epsilon^2 \log\frac{1}{\epsilon}A^m m \int_{R^2} w^{(m-1)}\phi}{1 + \frac{1}{2}\int_{R^2} w^m \epsilon^2 \log\frac{1}{\epsilon}\mu^{-1}A^m}$$

$$= m(1-\xi)\frac{\int_{R^2} w^{(m-1)}\phi}{\int_{R^2} w^m}. \tag{106}$$

Clearly, (99) is equivalent to

$$\mathcal{L}[\phi] + \mathcal{S}_\epsilon[w] + \mathcal{N}_\epsilon[\phi] + O\left(\epsilon \log(1+|y|) w^{(m-1)}|\phi| + \epsilon w^m \log(1+|y|)\right) = 0, \tag{107}$$

where

$$\mathcal{L}[\phi] = \Delta\phi - n w^{(n-1)}\phi + m w^{(m-1)}\phi - m(1-\xi)\frac{\int_R w^{(m-1)}\phi}{\int_R w^m} w^m, \tag{108}$$

$$\mathcal{S}_\epsilon[w] = O\left(\epsilon w^m \log(1+|y|)\right), \tag{109}$$

$$|\mathcal{N}_\epsilon[\phi]| \le O\left(|\phi|^{(1+\sigma)} + \|\phi\|_{\mathcal{X}}^{(1+\sigma)} w^m\right). \tag{110}$$

We are now in need of the following important lemma.

Lemma 16 *If $\Gamma = 1 - \xi \neq \Gamma_0$, then the operator $\mathcal{L}$ is an invertible map from X_s to Y_s. Moreover $\mathcal{L}^{-1}$ is a bounded operator from Y_s to X_s.*

Proof: By Fredholm alternatives, we just need to prove that

$$\text{Kernel}(\mathcal{L}) \cap X_s = 0, \quad \text{Kernel}(\mathcal{L}^*) \cap X_s = 0,$$

where $\mathcal{L}^*$ is the conjugate operator of $\mathcal{L}$. From theorem 14, Kernel($\mathcal{L}$)=Kernel($\mathcal{L}^*$) = span$\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$. But $\frac{\partial w}{\partial y_j} \notin \mathcal{X}_s, j = 1, 2$. □

By our assumption $\Gamma = 1 - \xi^s \neq \Gamma_0$ and Lemma 16, $\mathcal{L}$ is invertible from X_s to Y_s. Equation (107) is transformed to

$$\phi = G[\phi] = -\mathcal{L}^{-1}[\mathcal{N}_\epsilon[\phi] + \mathcal{S}_\epsilon[w] + O(\epsilon(1+|y|)w^{(m-1)}|\phi| + \epsilon w^m(1+|y|))]. \tag{111}$$

The existence of ϕ satisfying (111) now follows from a contraction mapping principle. More precisely, let

$$\Lambda_{\sqrt{\epsilon}} = \{\phi \mid \|\phi\|_{\mathcal{X}} \leq \epsilon^{\mu_0}\}, \quad \frac{1}{1+\sigma} < \mu_0 < 1,$$

Now we show that G is a contraction map from $\Lambda_{\sqrt{\epsilon}}$ to $\Lambda_{\sqrt{\epsilon}}$. In fact

$$\|G[\phi]\|_{\mathcal{Y}} \leq c(\|N_\epsilon[\phi]\|_{\mathcal{Y}} + O(\epsilon)) \leq c\epsilon^{(1+\sigma)\mu_0} + O(\epsilon) \leq c\epsilon < \epsilon^{\mu_0},$$

by (109)-(110). Similarly, we have that

$$\|G[\phi_1] - G[\phi_2]\|_{\mathcal{Y}} \leq c\epsilon^{\mu_0}\|\phi_1 - \phi_2\|_{\mathcal{X}} < \frac{1}{2}\|\phi_1 - \phi_2\|_{\mathcal{X}},$$

for $\epsilon \ll 1$. By the contraction mapping theorem, there exists a unique $\phi_\epsilon \in \Lambda_\epsilon \mu$ such that $\phi_\epsilon = G[\phi_\epsilon]$, which implies that there exists a solution to (107) and (99). It is now straightforward to see that $\beta_\epsilon = w + \phi_\epsilon, \alpha_\epsilon = \mathcal{T}[\beta_\epsilon]$ satisfies all the properties stated in theorem 1.

5 Stability Analysis:Proof of Theorem 3

In this section, we consider the following eigenvalue problems

$$\epsilon^2 \Delta_x \phi_\epsilon - n\beta_\epsilon^{(n-1)}\phi_\epsilon + \xi^{-1} m\alpha_\epsilon \beta_\epsilon^{(m-1)}\phi_\epsilon + \xi^{-1}\psi_\epsilon \beta_\epsilon^m = \lambda_\epsilon \phi_\epsilon, \tag{112a}$$

$$\Delta_x \psi_\epsilon - \psi_\epsilon - \mu^{-1} A^m \psi_\epsilon \beta_\epsilon^m - \mu^{-1} A^m m\alpha_\epsilon \beta_\epsilon^{(m-1)}\phi_\epsilon = \tau\lambda_\epsilon \psi_\epsilon, \tag{112b}$$

Here $(\alpha_\epsilon, \beta_\epsilon) = (\alpha_\epsilon^s, \beta_\epsilon^s)$ or $(\alpha_\epsilon^l, \beta_\epsilon^l)$, constructed in Theorem 1. Our key estimate is Theorem 14.

We follow the ideas in Section 5 and Section 6 of [11] and mention the necessary changes. We discuss two cases: **large eigenvalues**, $\lambda_\epsilon \to \lambda_0 \neq 0$, and **small eigenvalues**, $\lambda_\epsilon \to 0$. In the former case, we derive the nonlocal eigenvalue problem (68), while in the latter case, we show that $\lambda_\epsilon = 0$ and $(\phi_\epsilon, \psi_\epsilon) \in$ span $\{(\frac{\partial \beta_\epsilon}{\partial x_i}, \frac{\partial \alpha_\epsilon}{\partial x_i}), i = 1, 2\}$. This then finishes the proof of Theorem 3.

Let us first consider the case when $\lambda_\epsilon \to \lambda_0 \neq 0$ (λ_0 may be complex). From the equation (112b), similar to the derivation of (96), we see that

$$\psi_\epsilon(0) = \frac{1}{2\pi}\int_{R^2} \log \frac{1}{\sqrt{1+\tau\lambda_\epsilon}|x|} \times \left[-\mu^{-1}A^m \psi_\epsilon \beta_\epsilon^m - \mu^{-1} A^m m\alpha_\epsilon \beta_\epsilon^{(m-1)}\phi_\epsilon \right] dx$$

$$= \frac{\epsilon 2 \log \frac{1}{\epsilon}}{2\pi} \int_R (1 + O(\epsilon|y|)) \left[-\mu^{-1}A^m \psi_\epsilon(0) \beta_\epsilon^m - \mu^{-1} A^m m\alpha_\epsilon \beta_\epsilon^{(m-1)}\phi_\epsilon \right]$$

$$\sim \frac{\epsilon^2 \log \frac{1}{\epsilon}}{2\pi}(1+O(\epsilon))\left[-\mu^{-1}A^m\xi \int_R w^m - \mu^{-1}A^m \int_R w^{m-1}(y)\phi_\epsilon dy \right]$$

which yields, via (17), that

$$\psi_\epsilon(0) = \left(1 + \frac{(1-\xi)}{\xi}\right)^{-1} (-m(1-\xi)) \frac{\int_{R^2} w^{m-1}\phi_\epsilon}{\int_{R^2} w^m}$$

$$= \frac{-m\xi(1-\xi)}{\xi + 1 - \xi} \frac{\int_{R^2} w^{m-1}\phi_\epsilon}{\int_{R^2} w^m}. \tag{113}$$

Substituting (113) into (112a), and letting $\epsilon \to 0$, we obtain the following nonlocal eigenvalue problem

$$\Delta\phi_0 - nw^{(n-1)}\phi_0 + mw^{(m-1)}\phi_0 - m\Gamma \frac{\int_R w^{(m-1)}\phi_0}{\int_R w^m} w^m = \lambda_0\phi_0, \tag{114}$$

where

$$\phi_0 = \lim_{\epsilon\to 0} \phi_\epsilon, \Gamma = 1 - \xi$$

On letting $\tau = O(1)$, we have that $\Gamma = m(1-\xi) + o(1)$.

Problem (114) has been studied in Theorem 14 of Section 3. By Theorem 14, we finish the proof of stability and instability of Theorem 3 in the large eigenvalue case. (It is a delicate issue whether or not an instability of (114) implies an instability of (112). This can be shown along the lines of the proofs of (3) Theorem 6.1 in [13]. We omit the details here.)

Next, we consider the case when $\lambda_\epsilon = o(1)$. Certainly, $(\frac{\partial \beta_\epsilon}{\partial x_j}, \frac{\partial \alpha_\epsilon}{\partial x_j}), j = 1, 2$ is a solution to (112) with $\lambda_\epsilon = 0$. To show the uniqueness, we decompose $(\phi_\epsilon, \psi_\epsilon)$ into two parts:

$$(\phi_\epsilon, \psi_\epsilon) = \sum_{j=1}^{2} c_{j,\epsilon}(\frac{\partial \beta_\epsilon}{\partial x_j}, \frac{\partial \alpha_\epsilon}{\partial x_j}) + (\phi_\epsilon^\perp, \psi_\epsilon^\perp)$$

such that

$$c_\epsilon \in \mathcal{C}, \int_{R^2} \left[\frac{\partial \beta_\epsilon}{\partial x_j}(\phi_\epsilon^\perp) + \frac{\partial \alpha_\epsilon}{\partial x_j}(\psi_\epsilon^\perp) \right] = 0, j = 1, 2.$$

Then similar to the proof of (1) of Theorem 6.1 of [13], we show that $\psi_\epsilon^\perp = 0, \phi_\epsilon^\perp = 0$. The proof is almost the same. We omit the details here. Interested readers can refer to [9], [12] for similar arguments.

6 Numerical solutions

Finally, we present numerical solutions to the initial boundary value problem, given by (3), which support and illustrate the detailed analysis given in the previous sections. We restrict our attention to the case when $m = 3$, $n = 2$, $\mu = 0.02$ and $k = 0.1$, and consider two values of the parameter D, namely (a) $D = 0.2$ and (b) $D = 0.015$. The former value of D corresponds to the case when the stable stationary state is left behind the travelling wave front, while the latter corresponds to the case when the travelling wave front leaves behind a stable pattern of spots.

Equations (3) were solved using a standard linear Galerkin finite element method on a triangular mesh [3] for the spatial discretisation.

The temporal derivative is discretised using a combination of implicit time-stepping for the diffusion terms (to avoid the stiffness inherent in the discrete form of the second derivatives as the spatial mesh is refined) and explicit time-stepping for the reaction terms (which are not particularly stiff, but are nonlinear and so would be expensive to approximate implicitly). The backward and forward Euler methods have been employed in this work, using a constant time-step throughout each numerical experiment, leading to a scheme which is second order accurate in space and first order accurate in time. This could easily be improved, but is accurate enough to illustrate the types of solution that these equations can support. Simple Neumann conditions applied at the outer boundary of the computational domain, weakly imposing $\frac{\partial \alpha}{\partial n} = \frac{\partial \beta}{\partial n} = 0$, where n indicates the derivative normal to the boundary. This is valid for the numerical results presented here, but only because t is never large enough for the travelling wave front to reach the boundary.

The numerical solutions have been obtained on a uniform triangular mesh of 131585 nodes (262144 cells), covering a circular computational domain of radius 250. The initial data for the experiments is taken to be

$$\begin{aligned} \alpha(r,0) &= 1 \qquad\qquad\qquad r \geqslant 0 \\ \beta(r,0) &= \begin{cases} \left(\cos \frac{\pi r}{2}\right)^2 & 0 \leqslant r \leqslant 5 \\ 0 & r > 5 \end{cases} \end{aligned}$$

where $r = |x|$ and we have set $\sigma = 5$, $\beta_0 = 1$ and $g(x) = \left(\cos \frac{\pi |x|}{10}\right)^2$.

For these parameter values we have, via equations (10)–(12), $L_0 = \frac{1}{4}$ and

$$L = \frac{D}{20\pi} \log(\sqrt{2D}) \int_{R^2} w^3 \tag{115}$$

with, via (8a),

$$w(y) = \frac{4}{2 + y^2}$$

so that

$$L = 0.4\, D \log(\sqrt{2D}) \tag{116}$$

We now present numerical solutions for the two chosen values of D.

(a) $D = 0.2$, $L \approx -0.01592 < L_0$: a permanent form travelling wave solution develops for $t \gg 1$ which connects the unreacted state ahead of the wave, $(\alpha, \beta) = (1, 0)$, to the fully reacted stable stationary state (α_s, β_s) (where $\alpha_s \approx 0.2599$ and $\beta_s \approx 0.3847$ correspond, via (6) and (5), to α_s^+ and β_s^+ respectively) at the rear of the wave front, via a series of progressively weaker waves which are generated at the centre of the domain and follow the initial front. Figures 3 and 4 show the development of the pattern, giving the profiles of α and β at times $t = 600$, 1200, 1800, and 2400. The final snapshot clearly shows the strong initial travelling wave front two smaller peaks (of β) following it. The solution at the centre of the domain eventually settles down to the fully reacted stable stationary state.

(b) $D = 0.015$, $L \approx -0.00457 < L_0$: a permanent form travelling wave solution develops for $t \gg 1$ which leaves in its wake a stable pattern of spots. The early evolution of the solution is shown in close-up in Figure 5, which exhibits a similar pattern of spot replication as the solution shown in Figure 3 of [5], though the equations and the initial conditions differ slightly. The initial wave front breaks in to eight spots (instead of the four in [5]), each of which then splits again.

Figures 6 and 7 show the later development of the pattern, giving the profiles of α and β at times $t = 600$, 1200, 1800, and 2400. There is some initial repositioning as new spots are created: eight of the sixteen spots seen in the last snapshot in Figure 5 move back towards the centre of the domain, while the other eight continue to move out and split again.

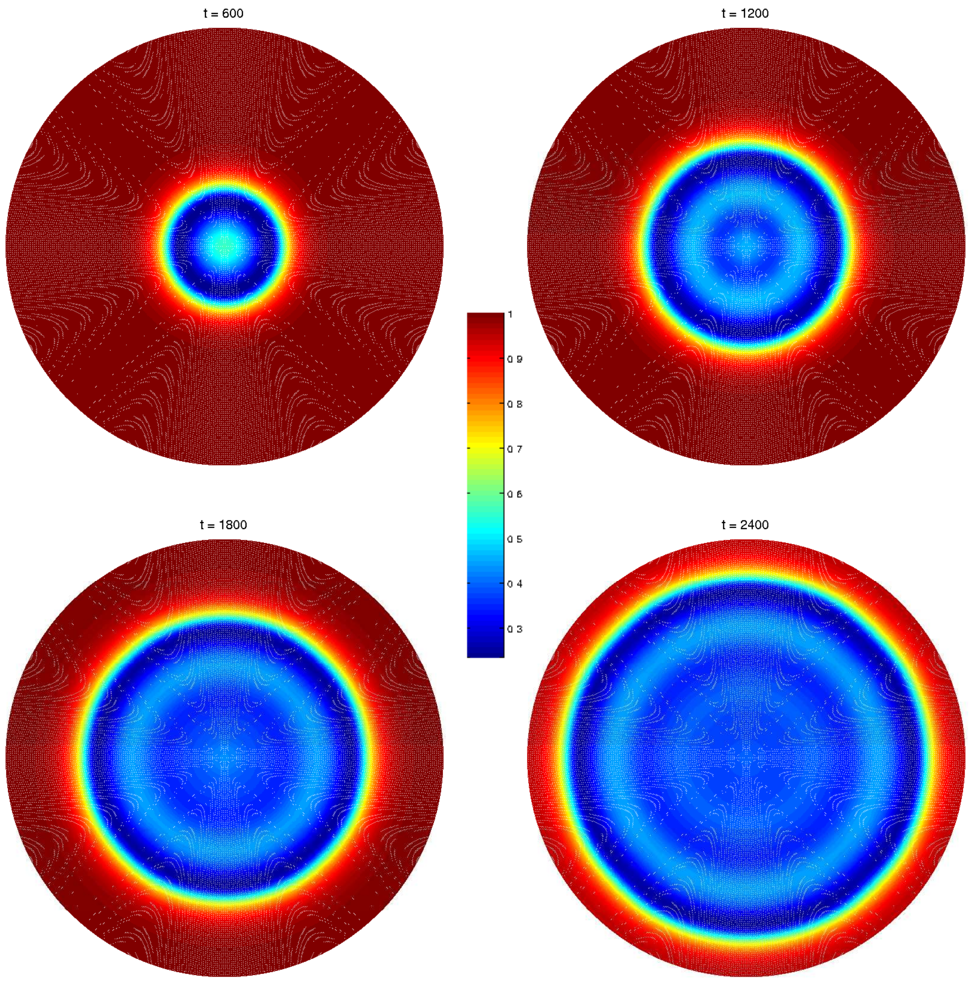

Figure 3: Plots of α for the travelling wave solutions at various times t, using parameter values $m = 3$, $n = 2$, $D = 0.2$, $\mu = 0.02$ and $k = 0.1$.

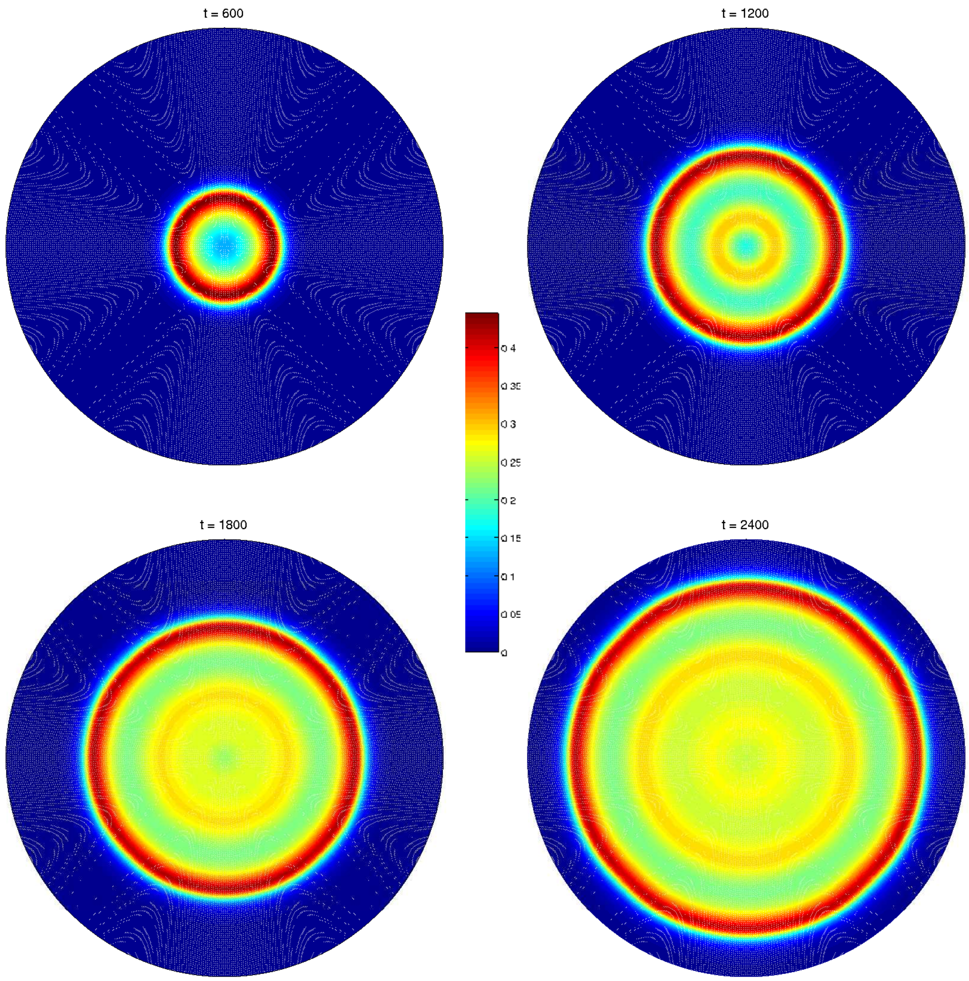

Figure 4: Plots of β for the travelling wave solutions at various times t, using parameter values $m = 3$, $n = 2$, $D = 0.2$, $\mu = 0.02$ and $k = 0.1$.

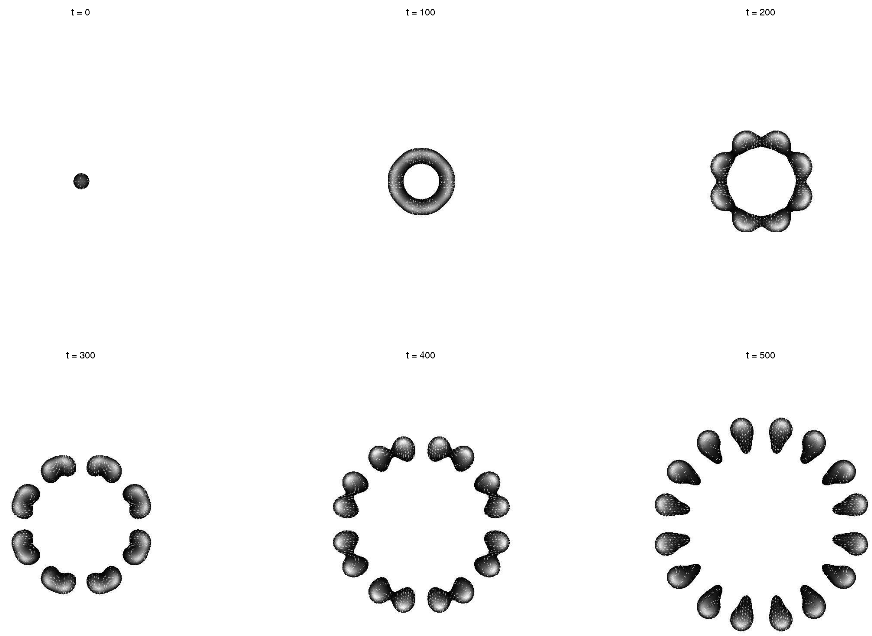

Figure 5: Close-up plots of β for the travelling wave solutions at various times t, using parameter values $m = 3$, $n = 2$, $D = 0.015$, $\mu = 0.02$ and $k = 0.1$. Only values of $\beta > 0.5$ are shaded − paler shading indicates higher solution values.

It is clear, though, by $t = 2400$ that the pattern behind the travelling wave front has settled down to a series of stationary spots which are approximately equidistributed over the fully reacted region of the domain. We note that the regularity of the pattern is entirely due to the regularity of the underlying computational mesh. This was constructed by dividing the circular domain in to quadrants and then recursively subdividing these quadrants into four. The pattern is dictated by the initial division into four, the effects of which can be seen clearly in all of the pictures. A different mesh would produce a different pattern but retain (approximately) the size and spacing of the spots. The reason for this is that the numerical problem is only approximately radially symmetric: the circular domain is being represented by a series of triangles and it is the asymmetries that occur in the representation of the initial conditions and the partial derivatives which trigger the pattern formation. Even if the problem was approximated on a two-dimensional radially symmetric mesh, similar patterns would arise through the effects of rounding error in the numerical calculations.

We note that as D increases to D_c (a bifurcation point which depends on the problem parameters) the amplitudes of the spots in α and β decrease until, at $D = D_c$, the stationary state (α_s^+, β_s^+) becomes stable and the permanent form travelling wave now leaves this stable stationary state behind the wave front (and an intermediate region of damped oscillations).

7 Summary

In this paper we have demonstrated, both by detailed stability analysis and by numerical integration, that spotty patterns develop in initial value problem (3) over a wide parameter range. The pattern we observe forms in the nondimensional concentration

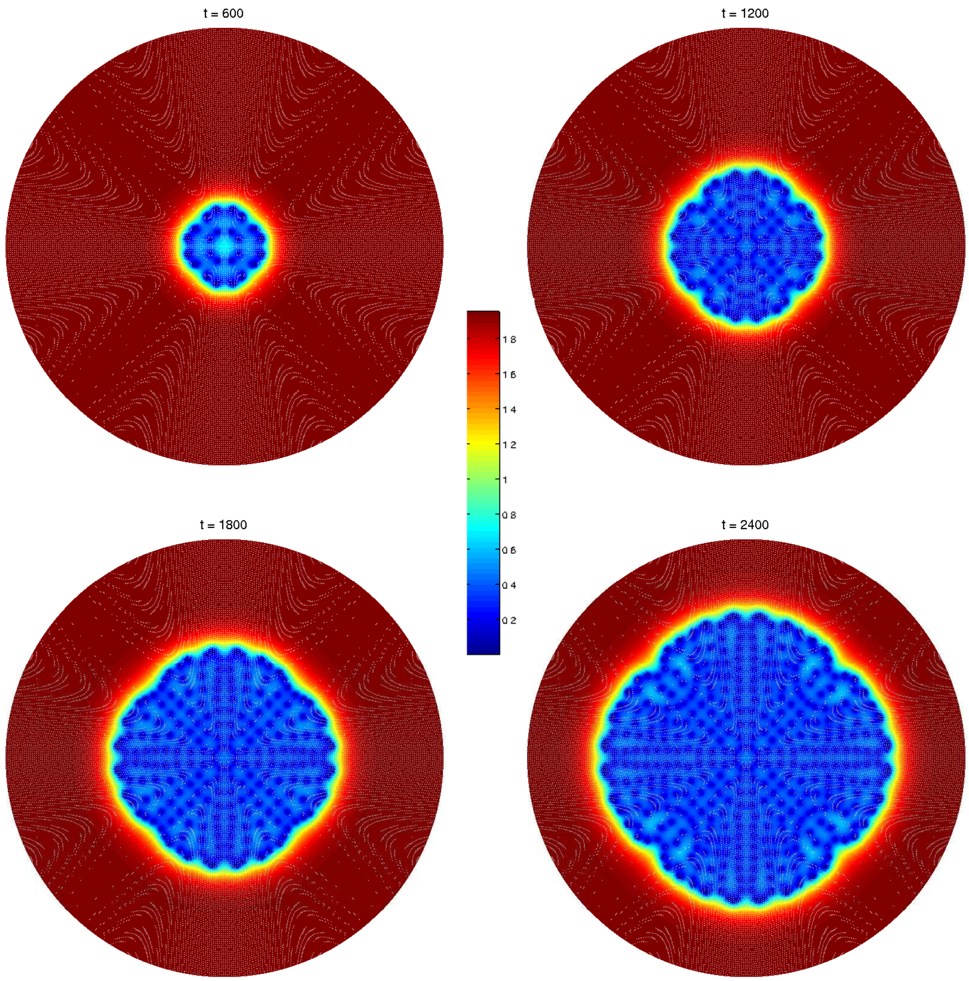

Figure 6: Plots of α for the travelling wave solutions at various times t, using parameter values $m = 3$, $n = 2$, $D = 0.015$, $\mu = 0.02$ and $k = 0.1$.

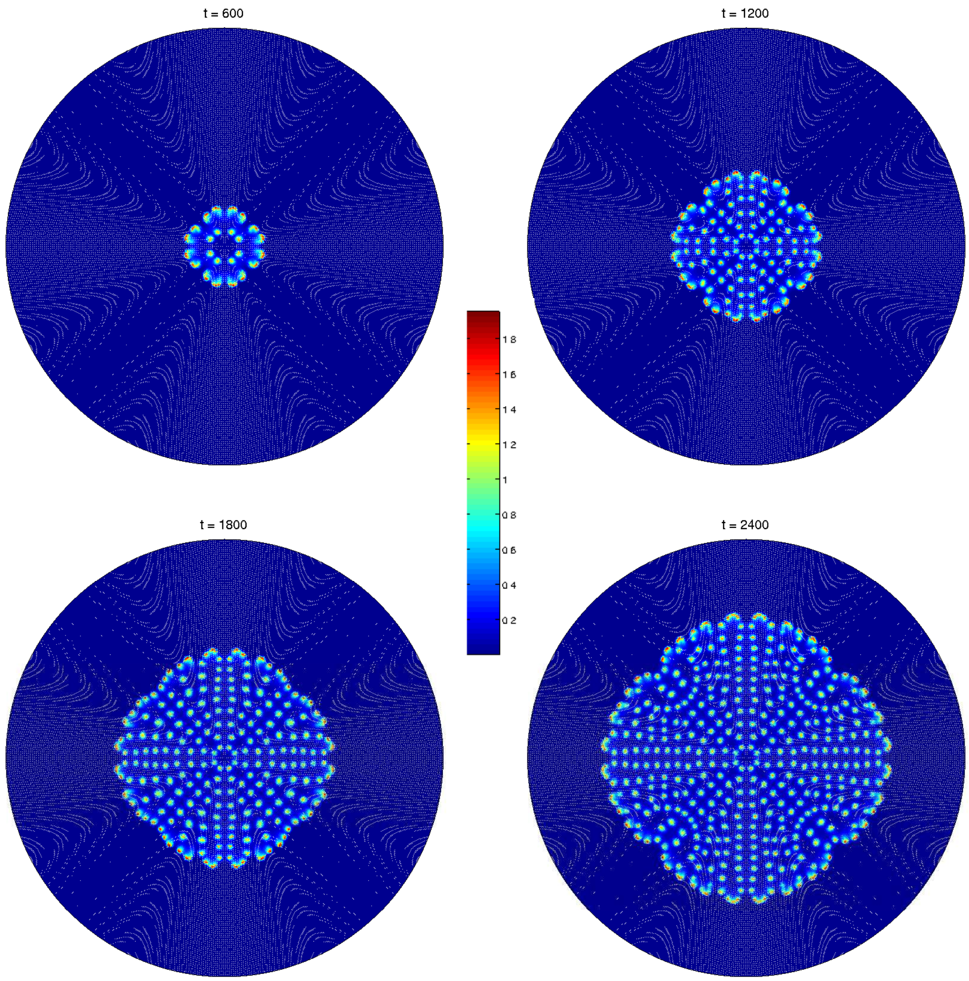

Figure 7: Plots of β illustrating the propagating spot solutions at various times t, using parameter values $m = 3$, $n = 2$, $D = 0.015$, $\mu = 0.02$ and $k = 0.1$.

of the autocatalyst β and forms in the wake of the travelling wave in the nondimensional concentration of the reactant α. This circular travelling wave in α propagates out from the initiation site, local to the origin in the two dimensional spatial domain, converting the unreacted state ($\alpha = 1$) ahead of the wave to the fully reacted state ($\alpha = \alpha_s$) at the rear of the wave. The situation for β is entirely different in that the initial distribution of β quickly divides into spots which grow and propagate outwards from the initiation site in the wake of the travelling wave in α. Each of these spots in turn divide to form two spots which grow and move away from each other. This process continues until the spots fill the spatial domain. Far enough away from the wave front the spots settle down into a regular, equidistant pattern. This process is very similar to cell division. Similar patterns have been observed in other mathematical models arising from chemical and biological systems and in experimental studies (see for example [5]).

Theoretically, by introducing the parameter L in (10) and the critical value L_0 in (10), it is shown that there are two single pulse steady states in the range of $L < L_0$ and $\epsilon^s < \epsilon^l << 1$. For $\tau = D\epsilon^2$ small, it is shown that one of the single-pulse solution is stable for large set of parameters (m, n), and the other single-pulse solution is always unstable. A nonlocal eigenvalue problem (62) was derived rigorously and theoretical rigorous results are obtained on the stability of such nonlocal eigenvalue problems. Such nonlocal eigenvalue problem is new and more difficult than the one derived in the Gray-Scott model. For example, the hypergeometric function method used in [1] does not work here. We discovered that in some parameter range of (m, n) (**??**), an eigenfunction corresponding to zero eigenvalue is found to have dent shape. We have shown numerically that this eigenfunction may be responsible for self-replicating patterns. Several important questions are left. For example, in the case of large τ, Hopf bifurcations can occur. It remains largely open to characterize the stability and instability of the nonlocal eigenvalue problem (62) for general parameter (m, n). We have not touched the problem of finite domain case.

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