Fluctuation Distribution - An Alternative Viewpoint [1]

M.E.Hubbard

Numerical Analysis Report ?/95

Department of Mathematics
P.O.Box 220
University of Reading
Whiteknights
Reading
RG6 2AX
United Kingdom

[1]The work reported here forms part of the research programme of the Oxford/Reading Institute for Computational Fluid Dynamics and was supported by DRA Farnborough.

Abstract

This is a report on alternative ways of viewing fluctuation distribution schemes.

Contents

1 Introduction

This report is intended to give an alternative viewpoint of some of the fluctuation distribution schemes developed by Roe, Deconinck *et al*, and to point out some misapprehensions in the current theory.

2 An Alternative Viewpoint

The idea of this note is to try to look at the fluctuation distribution schemes of Roe, Deconinck et al in a slightly different light and, hopefully, to derive new schemes which may improve on the current ones and which become apparent from this alternative viewpoint.

In the current literature the schemes are chosen to satisfy 4 design criteria

a) Upwind - the fluctuation within a triangle is only distributed to its downstream vertices (those opposite inflow sides).

b) Positive - for a scheme of the form

$$u_i^{n+1} = \sum_j c_j u_j^n \tag{2.1}$$

all the c_j are positive. This ensures a local maximum/minimum principle is satisfied and imposes stability on the scheme. In fact, for much of the time, local positivity is assumed. This requires the contributions from each triangle taken individually to give a positive scheme. It is a stricter condition which implies positivity but, counterintuitively, gives a less restrictive limit on the time step. However, it is the global time step limit which must be satisfied just as it is in one dimension.

c) Linearity preserving - a zero fluctuation in a triangle sends zero contribution to all of its vertices. This ensures that a linear steady state solution is preserved by the scheme and gives second order accuracy on a suitable regular grid.

d) Continuous - the contributions to the vertices of a triangle vary continuously with the data, u_i, and the advection velocity, $\vec{a}$. This is a different definition to that given in the literature since that contradicts the statement that the PSI scheme is continuous as $\vec{a}$ varies (*show this*). The reason given for insisting on continuity is that switches are not introduced which inhibit convergence to a steady state by allowing limit cycling to occur. This new definition prohibit these switches but is slightly less restrictive than the previous one and includes the PSI and PSI2 schemes.

My intention is to use slightly different criteria which will lead to both the current schemes and a wide variety of new ones. Initially, my design criteria are

1) Upwind - precisely as before. I have yet to find a better or more convenient definition of upwind. It may be that the gradient dependent advection velocity can be used in the definition instead of $\vec{a}$. It has been suggested that this causes convergence problems because it can update upstream nodes but these problems may have been caused by the scheme not being continuous. Further investigation is needed as I only thought of that this very moment.

2) Satisfies a local maximum/minimum principle - this simply ensures that the maximum/minimum value in each triangle is not increased/decreased by the contributions from that triangle. This is implied by local positivity (but not vice versa) and again imposes stability on the scheme. The next stage in this working would be to impose a different maximum/minimum principle (implied by global positivity) which ensured that the value at a node is not increased/decreased above/below the maximum/minimum value on the patch of triangles surrounding it by the contributions from these triangles. Suitable time step limits would also have to be calculated.

3) Gives positive distribution coefficients - this implies linearity preservation since these coefficients sum to one and are therefore bounded above by one and below by zero as the fluctuation tends to zero. Note that this is true for the PSI scheme for ALL values of the data so it is ALWAYS linearity preserving, even at discontinuities contradicting what is said in the literature. The next stage here is to allow negative distribution coefficients but still retain linearity preservation but that is more complicated and all the current schemes fit into this definition.

4) Continuity - exactly as before.

Of course, the schemes must be nonlinear in order to accommodate all of these criteria but also any or all of them can be relaxed to produce others such as Lax-Wendroff, Jameson, etc.

First consider the upwind requirement. This already restricts us to a single target distribution for single inflow side triangles which satisfies both positivity and linearity preservation (local maximum/minimum principle and positive distribution coefficients). It therefore only remains to consider the two target case.

For this case assume 2) and 3) and consider a triangle where nodes 1 and 2 are the downstream vertices and $\phi > 0$. Obviously, in this case we can update node 1 if u_1 is not a maximum within the triangle and node 2 similarly. A similar thing is true when $\phi < 0$, leaving us with the following possibilities

Update node 1 if ($\phi > 0$ and ($u_1 < u_3$ or $u_1 < u_2$))
or ($\phi < 0$ and ($u_1 > u_3$ or $u_1 > u_2$)).
Update node 2 if ($\phi > 0$ and ($u_2 < u_3$ or $u_2 < u_1$))
or ($\phi < 0$ and ($u_2 > u_3$ or $u_2 > u_1$)).

It can easily be shown that the last of these options on each line is impossible. (Then show it!) so we are left with

Update node 1 if $\phi(u_1 - u_3) < 0$.
Update node 2 if $\phi(u_2 - u_3) < 0$.

So we have a one target scheme if only one of these is true and a two target scheme if both are true. The case where neither is true cannot occur by the same reasoning that makes the last options above impossible. Note that $\phi(u_i - u_j) = \vec{a}_m \cdot \vec{r}_{ji}$ give or take the odd cell area.

It now only remains to satisfy continuity, so we have to ensure firstly that the expressions for the coefficients themselves are continuous and secondly that they give 0 and 1 where the two target regions abut the single target regions *ie* when $k_i = 0$ and when $\vec{\nabla} u \cdot \vec{r}_{ji} = 0$. This is obviously done when the coefficients themselves contain these factors and the simplest scheme of this form turns out to be the PSI scheme where the two target coefficients to nodes 1 and 2, for example, are

$$\alpha_1 = \frac{k_1(u_1 - u_3)}{k_1(u_1 - u_3) + k_2(u_2 - u_3)}, \qquad \alpha_2 = \frac{k_2(u_2 - u_3)}{k_1(u_1 - u_3) + k_2(u_2 - u_3)}. \tag{2.2}$$

The PSI2 scheme is only slightly more complicated in that the distribution coefficients become

$$\alpha_1^* = k_1 \alpha_1, \qquad \alpha_2^* = k_2 \alpha_2, \tag{2.3}$$

but still satisfies these conditions. In fact, any two target distribution where the coefficients contain the factors $k_1(u_1 - u_3)$ and $k_2(u_2 - u_3)$ will be suitable - these two just happen to be a couple of the most obvious.

However, the coefficients need not take this form. As an example, consider the PSI scheme where the LDA scheme is used for the two target distribution instead of the N scheme. This scheme satisfies all the relevant criteria but has coefficients which are, at first glance, independent of the data. Of course this is impossible and the data dependence is incorporated in the decision as to whether the one or two target distribution should be used. (*more explanation needed*)

As always, such things are better explained pictorially. The case where a triangle is single target due to $\vec{a}$ is very straightforward so we only need to consider the two target case and we can assume that the two downstream vertices are nodes 1 and 2. This gives three distinct cases, shown in Figure 2.1. From this it is easy to see the available options for distributing the fluctuation. If we assume that the distribution coefficients, α_i, are positive then the possibilities are

For a)	$\phi < 0$	$\Rightarrow$	send to 1 and/or 2
	$\phi > 0$	-	impossible
For b)	$\phi < 0$	$\Rightarrow$	send to 2
	$\phi > 0$	$\Rightarrow$	send to 1
For c)	$\phi < 0$	-	impossible
	$\phi > 0$	$\Rightarrow$	send to 1 and/or 2

Allowing negative coefficients obviously gives much more freedom to the choice of distribution but positivity and linearity preservation still have to be guaranteed.

Figure 2.1: The three types of triangle possible with two downstream vertices.

Case b) we can see is already settled as a further situation in which the single target scheme would be used. This is where the original definition used for continuity gives a contradiction since there is a discontinuity in the coefficients where ϕ passes through zero, *ie* when $\vec{a} \cdot \vec{\nabla} u = 0$, which isn't there if we consider the continuity of the fluctuation distribution instead.

For cases a) and c) though, we can use any distribution which gives continuity. This simply requires that the coefficients themselves are continuous functions of the data and the advection velocity and give

$$\begin{array}{llll} \alpha_1 = 1 & \text{and} \quad \alpha_2 = 0 & \text{when} & \vec{a} \parallel \vec{r}_{31} \\ \alpha_1 = 0 & \text{and} \quad \alpha_2 = 1 & \text{when} & \vec{a} \parallel \vec{r}_{32}. \end{array}$$

This is true for the N scheme and leads to the PSI scheme and also true for the coefficients which lead to the PSI2 scheme. However, this is also satisfied by the LDA and LDB schemes so these could equally well be used to create new nonlinear schemes.

Another slightly different view of these schemes is to look at them in terms of flux limiters.

Further insight may also be gained by considering partial fluctuations and attempting to distribute them instead of considering the fluctuation as a whole.

3 Conclusions

The fluctuation distribution schemes of Roe, Deconinck *et al* can valuably be viewed in a subtly different light to give further insight into their workings and to suggest alternative schemes, although thus far none of the new schemes is significantly better than the originals. This may be rectified with further study.

Acknowledgements

M.E. Hubbard is supported financially by a contract funded by DRA Farnborough. He wishes to thank Prof. M.J. Baines of the Mathematics Department, University of Reading for his advice and assistance during this work.