# Discontinuous Residual Distribution Schemes for Time-Dependent Problems

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Abstract. This paper is concerned with the design of a new approach to numerical simulation of time-dependent hyperbolic PDEs within the residual distribution framework. The underlying representation of the solution is discontinuous-in-space which introduces extra flexibility into the design of new schemes. This is mainly due to edge-based residuals, which are necessary to impose communication between the cells and which are in addition to their cell-based counterparts. We show how to construct a scheme free of spurious oscillations and present numerical results for 2d scalar nonlinear equations on unstructured triangular grids to validate the method. The proposed framework leads to explicit time-stepping which is a promising alternative to more frequently applied implicit approaches.

## 1. Introduction

In this paper we consider a class of new numerical schemes for scalar hyperbolic conservation laws governing the evolution of an unknown quantity $u(\mathbf{x}, t)$, given by

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0 \qquad \text{in } \Omega \times [0, T], \tag{1.1}$$

where $\Omega$ is a closed subset of $\mathbb{R}^2$ and $\mathbf{f}$ is the flux of $u$. Equation (1.1) is equipped with appropriate boundary conditions and the initial conditions:

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}).$$

The framework we shall design our schemes in is that of Residual Distribution ($\mathcal{RD}$). Extension of the work presented here to systems, although possible with the aid of well-understood methods like the one in [**12**], is beyond the scope of this paper and will not be considered here.

Since their inception in [**15**], $\mathcal{RD}$ methods have proven to be reliable and accurate ways of discretizing hyperbolic equations. Their ability to carry out genuinely multidimensional upwinding enables them to capture more accurately the properties of the underlying physical problem which are incorporated within the mathematical model. This, at least in theory [**10**], means that they are capable of producing solutions of higher resolution then those produced by other known algorithms. For promising experimental observations on this matter see [**13**]. It was also demonstrated (see, for instance, [**4**] and [**5**]) that residual distribution methods are very

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robust and perform well when applied to complex problems arising in engineering and other applications.

In the case of steady state problems, at least for scalar equations, the $\mathcal{RD}$ framework has already reached a high level of sophistication and understanding (this was summarized in [**3**]). Even though further research is still being carried out, the emphasis is now mainly laid on the development of residual distribution methods for time-dependent problems. The main challenge is to design a scheme which retains all the properties of its steady counterpart(s), in particular positivity and linearity preservation [**11**] at the same time, and which is relatively efficient.

The space-time framework investigated in [**8**] (see also [**11**] and references therein) allows construction of discretizations with *all* the desired properties. Unfortunately, those methods are subject to a CFL-type restriction on the time-step, which is particularly disappointing when taking into account that they are, by construction, implicit. In the *two layer* variant [**9**] one couples two space-time slabs at a time and solves the equations simultaneously in both. On one hand the resulting system to be solved at each step is larger, but on the other the construction removes from one of the layers the restriction on the time-step. In theory this means that an arbitrarily large time-step can be used. For a full discussion see [**8**]. Recently, Hubbard and Ricchiuto [**1**] proposed to drive the height of one of the space-time slabs (and hence its associated time-step) to zero so that the scheme becomes discontinuous-in-time. The resulting formulation is simpler than the original whereas all of the properties are retained.

In this contribution we investigate a somewhat similar in spirit approach. We also assume that the discrete representation of the data is discontinuous, however we will consider discontinuities in space instead of time. Such an approach will enable us to construct explicit schemes with a localised system and a simple framework within which $h-$ and $p-$ adaptivity can be incorporated. This work extends previous work on discontinuous residual distribution schemes initiated by Hubbard [**14**] and Abgrall [**2**] in which only steady state problems were considered. We first apply those schemes to time-dependent equations and then show how to design positive schemes. Only first order methods will be considered. This paper also extends work of Abgrall and Shu [**6**] in the sense that it shows how ideas from the discontinuous Galerkin framework can be incorporated within residual distribution methods. This is, briefly speaking, done by considering flux differences (*edge residuals* in the RD nomenclature) instead of the fluxes themselves.

This paper is organised as follows. In the next section the residual distribution and discontinuous residual distribution methods are presented. A quick overview of available distribution strategies for the edges is given and one new distribution method (based on the discontinuous Galerkin method) is then introduced. Next, in Section 4, we present discontinuous-in-space residual distribution methods for time-dependent problems. This is followed by numerical results and conclusions.

## 2. Continuous Residual Distribution Schemes for Steady State Problems

The residual distribution framework (both in the continuous and discontinuous setting) was originally introduced for steady state problems:

$$\nabla \cdot \mathbf{f}(u) = 0 \quad \text{or} \quad \mathbf{a}(u) \cdot \nabla u = 0 \qquad \text{in } \Omega, \tag{2.1}$$

in which $\mathbf{a}(u) = \frac{\partial \mathbf{f}}{\partial u}$. Throughout the text we will assume that a triangulation of $\Omega$ is given and will denote it by $\mathcal{T}_h$ ($h$ being the mesh parameter). The numerical solution $u_h$ is assumed to be of the same form as in the case of finite-element-type approximations, i.e. given its nodal values $u_i = u_h(\mathbf{x}_i)(\mathbf{x}_i \in \mathcal{T}_h)$, it reads

$$u_h(\mathbf{x}) = \sum_{i \in \mathcal{T}_h} \psi_i(\mathbf{x}) u_i, \tag{2.2}$$

in which $\psi_i$ is the standard Lagrange basis function associated with $\mathbf{x}_i$. Cell interfaces will be denoted by $e$ and $\mathcal{D}_i$ will stand for the subset of triangles containing node $i$. The median dual cell, obtained by joining the gravity centres of triangles in $\mathcal{D}_i$ with the midpoints of the edges meeting at $i$, will be denoted by $S_i$ (illustrated in Figure 1).

To construct a set of linear equations for the nodal values of $u_h$ one first, for each cell $E \in \mathcal{T}_h$, calculates the cell residual

$$\phi^E = \int_E \nabla \cdot \mathbf{f}(u)\, d\Omega.$$

Next, fractions of this residual are distributed among the vertices of $E$ so that the resulting linear system reads:

$$\sum_{E \in \mathcal{D}_i} \beta_i \phi^E = 0 \qquad \forall i. \tag{2.3}$$

The distribution coefficients $\beta_i$ signify the fraction of the cell residual that is assigned to the node $i$ and are used to impose various properties of the scheme.

Here we will consider only **the N** (for Narrow) **scheme** [**3**] - the most successful linear positive distribution strategy. In order to introduce it, for each vertex $i \in E$, we define the so-called flow sensors:

$$k_i = -\frac{\mathbf{a}(u) \cdot \vec{\mathbf{n}}_i}{2} |e_i|, \quad k_i^+ = \max(0, k_i), \quad k_i^- = \min(0, k_i),$$

in which $\vec{\mathbf{n}}_i$ is the outward pointing unit normal vector to edge $e_i$ (opposite the $i$th vertex). We will also need the following quantities:

$$N = \left( \sum_{j \in E} k_j^+ \right)^{-1}, \quad u_{in} = -\sum_{j \in E} N k_j^- u_j.$$

Finally, the distribution itself reads:

$$\beta_i \phi^E = k_i^+ (u_i - u_{in}). \tag{2.4}$$

In practice, system (2.3) is solved with the aid of pseudo time-stepping:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|S_i|} \sum_{E \in \mathcal{D}_i} \beta_i \phi^E \qquad \forall i,$$

which is used to iterate to the steady state.

## 3. Discontinuous Residual Distribution Schemes For Steady State Problems

Discontinuous-in-space residual distribution schemes were introduced simultaneously by Abgrall [**2**] (based on a report written in French in 2007) and Hubbard [**14**]. In this framework the numerical solution is no longer assumed to be globally

continuous which means that, at least in each cell $E \in \mathcal{T}_h$, its discrete representation remains unchanged (cf. Equation (2.2)):

$$\forall \mathbf{x} \in E \qquad u_h(\mathbf{x}) = \sum_{i \in \mathcal{T}_h} \psi_i(\mathbf{x}) u_i.$$

Relaxing the continuity constraint means that one needs to impose some sort of communication between the cells. In both cases the authors suggest using the edge residuals:

$$\phi^e = \int_e [\mathbf{f}(u_h) \cdot \vec{\mathbf{n}}] \, d\Gamma$$

where $[\mathbf{f}(u_h) \cdot \vec{\mathbf{n}}]$ represents the jump of the function $\mathbf{f}(u_h) \cdot \vec{\mathbf{n}}$ across the edge, the sign of the difference being dictated by the direction chosen for $\vec{\mathbf{n}}$. To be more precise

$$\begin{aligned} \phi^e(u_h) = \int_e [\mathbf{f}(u_h) \cdot \vec{\mathbf{n}}] \, d\Gamma &= \int_e (\mathbf{f}_L \cdot \vec{\mathbf{n}}_L + \mathbf{f}_R \cdot \vec{\mathbf{n}}_R) \, d\Gamma \\ &= \int_e (\mathbf{f}_L - \mathbf{f}_R) \cdot \vec{\mathbf{n}}_L \, d\Gamma = \int_e (\mathbf{f}_R - \mathbf{f}_L) \cdot \vec{\mathbf{n}}_R \, d\Gamma, \end{aligned}$$

where the subscripts $_L$ and $_R$ mean that we take the value of a quantity in $E_L$ and $E_R$ respectively, the cells associated with $e$ (see Figure 1). Obviously $\phi^e$ is zero if

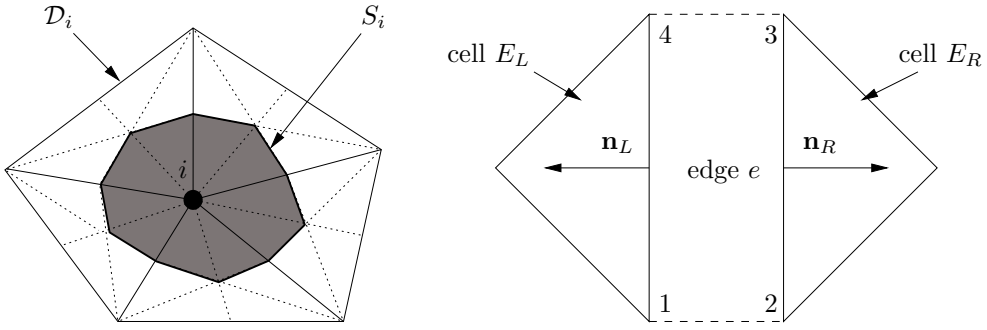


FIGURE 1. Edge $e$ and the two cells associated with it: $E_L$ and $E_R$.

$u_h$ is assumed to be globally continuous. As in the case of cell residuals, fractions of $\phi^e$ are *sent* to the vertices of $e$. Similar to continuous residual distribution, to find the numerical solution $u_h$ one first assembles signals sent to each degree of freedom $i$ and then solves the resulting linear system with the aid of pseudo time-stepping:

$$u_i^{n+1} = u_i^n - \frac{3\Delta t}{|E|} \left( \beta_i^E \phi^E + \alpha_i^{e_1} \phi^{e_1} + \alpha_i^{e_2} \phi^{e_2} \right) \qquad \forall i. \tag{3.1}$$

In analogy to the cell residual, $\alpha_i^{e_1}$ and $\alpha_i^{e_2}$ are the distribution coefficients for the degree of freedom $i \in E$ corresponding to the edges $e_1 \in E$ and $e_2 \in E$, respectively, adjacent to vertex $i$. Note that in the discontinuous setting each degree of freedom belongs to only one cell and two of its edges.

As far as cell residuals are concerned we will utilize only the aforementioned N scheme (2.4). It is the distribution strategy for the edges that is the main novelty within the discontinuous framework and we shall look into it in more detail. We point out that it gives us extra freedom in the design of new schemes compared to

the continuous case where only the cell distribution can be altered. Three natural choices that will be applied in this work are as follows.

**The mED scheme** of Hubbard [**14**] is linear, positive and linearity preserving. For a generic edge $e$ and its vertices $1, 2, 3$ and $4$ (see Figure 1) it is given by

$$\begin{aligned}
\phi_1^e &= \frac{1}{2}\left[\mathbf{a}_{12}\cdot\vec{\mathbf{n}}\right]^-(u_1-u_2) = \alpha_1\,\phi^e,\\
\phi_2^e &= \frac{1}{2}\left[\mathbf{a}_{12}\cdot\vec{\mathbf{n}}\right]^+(u_1-u_2) = \alpha_2\,\phi^e,\\
\phi_3^e &= \frac{1}{2}\left[\mathbf{a}_{43}\cdot\vec{\mathbf{n}}\right]^+(u_4-u_3) = \alpha_3\,\phi^e,\\
\phi_4^e &= \frac{1}{2}\left[\mathbf{a}_{43}\cdot\vec{\mathbf{n}}\right]^-(u_4-u_3) = \alpha_4\,\phi^e.
\end{aligned}$$

The $\mathbf{a}_{ij}$ are conservatively averaged values of the flux Jacobian defined as:

$$\mathbf{a}_{12} = \frac{1}{3}\left(\mathbf{a}_1+\mathbf{a}_2+\frac{\mathbf{a}_3+\mathbf{a}_4}{2}\right),\quad \mathbf{a}_{43} = \frac{1}{3}\left(\mathbf{a}_3+\mathbf{a}_4+\frac{\mathbf{a}_1+\mathbf{a}_2}{2}\right)$$

in which $\mathbf{a}_i$ $(i=1,\ldots,4)$ are the values of $\mathbf{a}$ at the vertices of $e$ and $\vec{\mathbf{n}} = |e|\vec{\mathbf{n}}_R$.

**The** (local) **Lax-Friedrichs distribution** for edges was proposed by Abgrall in [**2**] and is based on its counterpart for cells. It is defined as

$$\alpha_i\,\phi^e = \frac{\phi^e}{4} + \alpha^e(u_i-\bar{u}),\qquad i=1,\ldots,4,$$

with

$$\bar{u} = \frac{u_1+u_2+u_3+u_4}{4},$$

where $u_1, u_2, u_3.u_4$ are the values of $u_h$ at the vertices of $e$ (cf. Figure 1) and

$$\alpha^e \geq \max_{x\in e}\|\mathbf{f}'(u_h(x))\|.$$

It is also positive and linear, but not linearity preserving.

**The DG distribution** is simply the edge integral appearing in the strong formulation of the discontinuous Galerkin approximation of (2.1) (see [**7**] for a review of discontinuous Galerkin methods):

$$\begin{aligned}
\alpha_1^{DG}\,\phi^e &= \int_e \left(\mathbf{f}^*-\mathbf{f}\right)(u_h)\cdot\vec{\mathbf{n}}_L\,\psi_1^{E_L}\,d\Gamma,\\
\alpha_2^{DG}\,\phi^e &= \int_e \left(\mathbf{f}^*-\mathbf{f}\right)(u_h)\cdot\vec{\mathbf{n}}_R\,\psi_2^{E_R}\,d\Gamma,\\
\alpha_3^{DG}\,\phi^e &= \int_e \left(\mathbf{f}^*-\mathbf{f}\right)(u_h)\cdot\vec{\mathbf{n}}_R\,\psi_3^{E_R}\,d\Gamma,\\
\alpha_4^{DG}\,\phi^e &= \int_e \left(\mathbf{f}^*-\mathbf{f}\right)(u_h)\cdot\vec{\mathbf{n}}_L\,\psi_4^{E_L}\,d\Gamma,
\end{aligned}$$

in which the numerical flux $\mathbf{f}^*$ is defined as usual and $\psi_i^E$ is the Lagrange basis function in cell $E$ associated with edge vertex $i$. Now, since $\vec{\mathbf{n}}_R = -\vec{\mathbf{n}}_L$ it follows immediately that:

$$\sum_{i\in e}\alpha_i^{DG} = 1$$

which makes the above a valid distribution. It has yet to be tested on steady state problems, but in this paper we concentrate on transient problems and will not address that.

## 4. Discontinuous Residual Distribution Schemes for Transient Problems

To the authors' best knowledge there have been no (successful) attempts to apply discontinuous residual distribution methods to time-dependent problems. We start by assuming that the temporal domain $[0,T]$ is discretized into a set of $N$ discrete levels $\{t^n\}_{n=0,1\ldots,N-1}$ such that

$$t^0 = 0, \quad t^{N-1} = T, \quad t^n < t^{n+1} \quad \text{and} \quad \Delta t^n = t^{n+1} - t^n.$$

Given an appropriate initial condition $u_0(\mathbf{x})$, the method has the following form:

$$u_i^{n+1} = u_i^n - \frac{3\Delta t^n}{|E|}\left(\beta_i \phi^E(u_h^n) + \alpha_i\, \phi^{e_1}(u_h^n) + \alpha_i\, \phi^{e_2}(u_h^n)\right) \quad \forall E \in \mathcal{T}_h\ \forall i \in E \tag{4.1}$$

in which $u_h^n = u_h(\mathbf{x}, t^n)$ is the known numerical solution at time $t^n$. The numerical solution $u_h(\mathbf{x}, 0)$ is taken as the interpolation of the analytical initial condition $u_0$. The above is simply (3.1) applied to time-dependent problems which means that now $\Delta t$ has physical meaning. The limit on the time-step guaranteeing positivity is given by (cf. Equation (40) in [**14**]):

$$\Delta t \le \frac{|E|/3}{(k_i^E)^+ + (k_i^{e_1})^+ + (k_i^{e_2})^+} \qquad \forall E \in \mathcal{T}_h\ \forall i \in E,$$

in which cell $E$ and edges $e_1$ and $e_2$ are the only ones $i$ receives signals from.

We note that, since the time derivative was discretized with the aid of a first order approximation, the overall order of the approximation error should be expected to be no higher than one. Extension to higher order schemes is a subject of ongoing work.

## 5. Numerical Results

The two dimensional inviscid Burgers' equation,

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0 \qquad \text{in } \Omega \times [0,T],$$

with $\mathbf{f} = \left(\frac{u^2}{2}, \frac{u^2}{2}\right)$ is approximated over the square $\Omega = [-1,1]^2$ with the discontinuous initial condition:

$$u(\mathbf{x}, 0) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in (-0.5, 0) \times (-0.5, 0), \\ 0 & \text{otherwise.} \end{cases}$$

The problem is solved up to the final time $T = 1$. In all the experiments the N scheme was used to distribute the cell residuals and the three methods outlined in Section 3 were applied to the edges. In Figure 2 we present cross sections of the solutions and in Table 1 we show the corresponding minimal and maximal values.

The results show that both mED and LF lead to numerical approximations free of unphysical oscillations whereas the solution obtained with the aid of the DG-type approach exhibits some overshoots. This was expected as both mED and LF distributions were designed so that the resulting scheme is positive and in the case of discontinuous Galerkin methods extra limiting is usually applied to achieve similar results. We also note that the LF method is very diffusive and of all the considered approaches the mED algorithm performed the best.

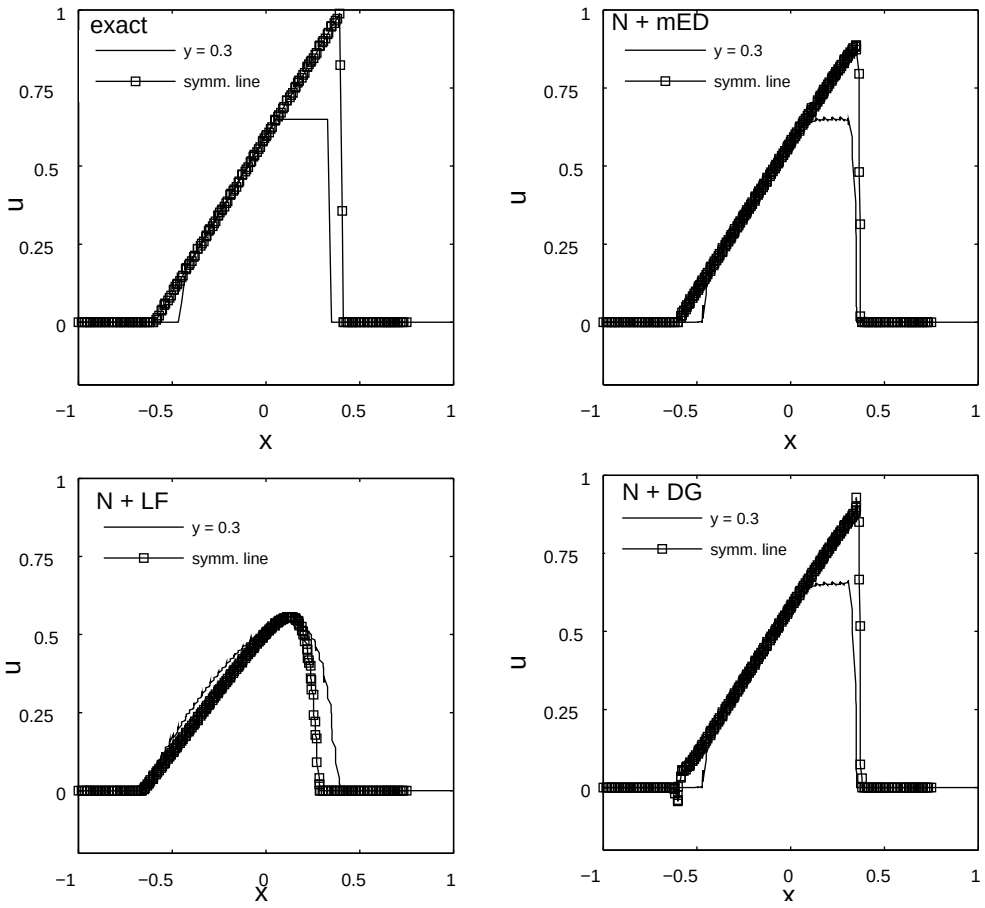


FIGURE 2. Solution along the line $y = 0.3$ and the symmetry line $y = x$ for the 2d Burgers' equation at time $t = 1$. Top left: the exact solution. Top right: N + mED. Bottom left: N + LF. Bottom right: N + DG.

| | **exact** | **N + mED** | **N + LF** | **N + DG** |
|---|---|---|---|---|
| $u_{min}$ | 0 | 0 | 0 | -1.026e-01 |
| $u_{max}$ | 1 | 8.833e-01 | 5.567e-01 | 9.319e-01 |

TABLE 1. Mininum and maximum values of the exact and numerical solution to the 2d Burgers' equation at time $t = 1$.

## 6. Conclusions

We proposed a new class of numerical approximations to time-dependent hyperbolic conservation laws. This new formulation was successfully applied to the inviscid Burgers' equation and three different methods for distributing the edge residuals were compared. The numerical results confirmed that as long as the cell

and edge residuals are distributed in a positive manner the resulting method is positive. All the presented schemes are first order accurate.

Our future work will be focused on the design of second order schemes. This can be achieved by, for instance, applying second order time-stepping and consistent mass matrices to discretize the time derivative. Incorporating discontinuities in time into our framework is also being investigated.

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