Some explicitly solvable quantum stochastic differential equations.
Dedicated to the memory of Slava Belavkin.

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Quantum probability is a noncommutative generalisation of the classical theory of probability, which is made necessary because the observables, or random variables, which occur in quantum mechanics have a noncommutative multiplication.

For example, the momentum $p$ and the position $q$ of a one-dimensional particle satisfy the Heisenberg commutation relation $pq - qp = i\hbar$, where Planck's constant $\hbar = 6.6261 \times 10^{-34}$ m$^2$kg/s, or $pq - qp = i$ at the Planck scale, which is very small.

My life in quantum probability has been built around ramifications of this commutation relation.
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- My life in quantum probability has been built around ramifications of this commutation relation.

- Slava felt very strongly that quantum probability originates in physics.
The Feynman-Kac formula

can be stated abstractly as:

\[
e^{-t\left(\frac{1}{2}p^2 + V(q)\right)} = (\text{id} \otimes \mathbb{E}) \left[ \exp\left( \int_0^t V(q \otimes l + l \otimes Q(s)) \, ds \right) e^{ip \otimes Q(t)} \right]
\]

where \( Q \) is the standard position Brownian motion in the Fock space \( \mathcal{F} \left( L^2 \left( \mathbb{R}_+ \right) \right) \) and \( \mathbb{E} \) is the Fock vacuum expectation.
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Heuristic proof. Start with the unperturbed case \( e^{-\frac{t}{2}p^2} = (\text{id} \otimes \mathbb{E}) \left[ e^{ip \otimes Q(t)} \right] \) from which it follows that \( e^{-\frac{t}{2n}p^2} = (\text{id} \otimes \mathbb{E}) \left[ e^{ip \otimes \left( Q \left( \frac{jt}{n} \right) - Q \left( \frac{(j-1)t}{n} \right) \right)} \right] \)

using the stationarity of Brownian motion. Then by independence of increments and the commutation relation \( e^{ipx} q = (q + x) e^{ipx} \), the Trotter product formula approximates \( e^{-t\left(\frac{1}{2}p^2 + V(q)\right)} \) as

\[ \left( e^{-\frac{t}{2n}p^2} e^{-\frac{t}{n}V(q)} \right)^n = \prod_{j=1}^n (\text{id} \otimes \mathbb{E}) \left[ e^{ip \otimes \left( Q \left( \frac{jt}{n} \right) - Q \left( \frac{(j-1)t}{n} \right) \right)} e^{-\frac{t}{n}V(q)} \right] \]
\[ (\mathrm{id} \otimes \mathbb{E}) \prod_{j=1}^{n} \left[ e^{ip \otimes \left( Q\left( \frac{j}{n} \right) - Q\left( \frac{(j-1)}{n} \right) \right)} e^{-\frac{t}{n} V(q)} \right] \]

\[ = (\mathrm{id} \otimes \mathbb{E}) \left[ e^{\frac{1}{n} \sum_{j=0}^{n-1} V(q \otimes l + l \otimes Q\left( \frac{j}{n} \right))} \prod_{j=1}^{n} e^{ip \otimes \left( Q\left( \frac{j}{n} \right) - Q\left( \frac{(j-1)}{n} \right) \right)} \right] \]

\[ \xrightarrow{n \to \infty} (\mathrm{id} \otimes \mathbb{E}) \left[ \exp\left( \int_{0}^{t} V(q \otimes l + l \otimes Q(s)) \, ds \right) e^{ip \otimes Q(t)} \right]. \]

- The key fact is that, for any interval \([a, b]\), \(e^{ip \otimes (Q(b) - Q(a))}\) is the solution at \(b\) of the stochastic differential equation

\[ dX = X[ip \otimes dQ + \frac{1}{2} p^2 \otimes dT]; \quad X(a) = 1. \]
\[
(id \otimes E) \left[ \prod_{j=1}^{n} \left[ e^{ip \otimes \left( Q\left( \frac{jt}{n} \right) - Q\left( \frac{(j-1)t}{n} \right) \right)} e^{-\frac{t}{n} V(q)} \right] \right.
\]

\[
= \left( id \otimes E \right) \left[ e^{\frac{1}{n} \sum_{j=0}^{n-1} V(q \otimes I + l \otimes Q(\frac{jt}{n}))} \prod_{j=1}^{n} e^{ip \otimes \left( Q\left( \frac{jt}{n} \right) - Q\left( \frac{(j-1)t}{n} \right) \right)} \right]
\]

\[
\overset{n \to \infty}{\sim} (id \otimes E) \left[ \exp \left( \int_{0}^{t} V(q \otimes I + l \otimes Q(s)) ds \right) e^{ip \otimes Q(t)} \right]. \square
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\[
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- In the language of product integrals,

\[
e^{ip \otimes (Q(b) - Q(a))} = \prod_{a}^{b} \left( 1 + ip \otimes dQ - \frac{1}{2} p^2 \otimes dT \right)
\]
\[
\begin{align*}
\prod_{j=1}^{n} \left[ e^{i p \otimes \left( Q \left( \frac{j t}{n} \right) - Q \left( \frac{(j-1)t}{n} \right) \right)} e^{-\frac{t}{n} V(q)} \right] \\
\prod_{j=0}^{n-1} \frac{1}{n} V(q \otimes l + l \otimes Q \left( \frac{j t}{n} \right)) \\
= (\text{id} \otimes \mathbb{E}) \left[ e^{i p \otimes \left( Q \left( \frac{it}{n} \right) - Q \left( \frac{(j-1)t}{n} \right) \right)} \prod_{j=1}^{n} e^{-\frac{t}{n} V(q)} \right] \\
\prod_{j=0}^{n-1} \frac{1}{n} V(q \otimes l + l \otimes Q \left( \frac{j t}{n} \right)) \end{align*}
\]

\[
\overset{n \to \infty}{\longrightarrow} (\text{id} \otimes \mathbb{E}) \left[ \exp \left( \int_{0}^{t} V(q \otimes l + l \otimes Q(s)) \, ds \right) e^{i p \otimes Q(t)} \right].
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- The key fact is that, for any interval \([a, b] \), \(e^{i p \otimes (Q(b) - Q(a))} \) is the solution at \( b \) of the stochastic differential equation

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e^{i p \otimes (Q(b) - Q(a))} = \prod_{a}^{b} \left( 1 + i p \otimes dQ - \frac{1}{2} p^2 \otimes dT \right)
\]

- Similarly, \(e^{-i q \otimes (P(b) - P(a))} = \prod_{a}^{b} \left( 1 - i q \otimes dP - \frac{1}{2} q^2 \otimes dT \right)\).
A Feynman-Kac formula for the oscillator semigroup.

**Theorem**

For real $\lambda$

\[ e^{-t \frac{\lambda^2}{2} (p^2+q^2-1)} = \left( \text{id} \otimes \mathbb{E} \right) \left[ \prod_{0}^{t} \left( 1 + i\lambda \left( p \otimes dQ - q \otimes dP \right) \right) \right] \]
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**Corollary**

$$e^{-t \left(\frac{1}{2} \left(p^2 + q^2 - 1\right) + V(q)\right)} = (\text{id} \otimes \mathbb{E}) \left[ \exp\left(\int_{0}^{t} V(q \otimes I + I \otimes Q(s)) \, ds\right) \right] \left[ \prod_{0}^{t} (1 + i \left(p \otimes dQ - q \otimes dP\right)) \right]$$
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Corollary

$$e^{-t \left( \frac{1}{2} (p^2 + q^2 - 1) + V(q) \right)} = (\text{id} \otimes \mathbb{E}) \left[ \exp \left( \int_{0}^{t} V(q \otimes I + I \otimes Q(s)) ds \right) \right]$$

$$\prod_{0}^{t} (1 + i (p \otimes dQ - q \otimes dP))$$

Towards an explicit construction of a product integral.

- The product integral notation suggests novel ways of constructing solutions of (quantum, stochastic) differential equations.
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For example

\[
\prod_{0}^{t} (1 + i\lambda (p \otimes dQ - q \otimes dP)) \simeq \prod_{j=1}^{m} \left(1 + i\lambda \sqrt{\frac{t}{m}} (p \otimes q_j - q \otimes p_j)\right)
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where \( p_j = \sqrt{\frac{m}{t}} \left(P \left(\frac{jt}{m}\right) - P \left(\frac{(j-1)t}{m}\right)\right) \) and

\( q_j = \sqrt{\frac{m}{t}} \left(Q \left(\frac{jt}{m}\right) - Q \left(\frac{(j-1)t}{m}\right)\right) \)
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\[ q_j = \sqrt{\frac{m}{t}} \left( Q \left( \frac{j}{m} t \right) - Q \left( \frac{(j-1)}{m} t \right) \right) \]

- These *normalised increments* are canonical pairs (for \( \frac{h}{2\pi} = 1 \), satisfying
\[
[p_j, q_k] = -i\delta_{j,k}, [p_j, p_k] = [q_j, q_k] = 0.
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- We can interpret \(p \otimes q_{j} - q \otimes p_{j}\) as an angular momentum.
Towards an explicit construction of a product integral.

- The product integral notation suggests novel ways of constructing solutions of (quantum, stochastic) differential equations.

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\[
\prod_{0}^{t} \left( 1 + i \lambda (p \otimes dQ - q \otimes dP) \right) \simeq \prod_{j=1}^{m} \left( 1 + i \lambda \sqrt{\frac{t}{m}} (p \otimes q_{j} - q \otimes p_{j}) \right)
\]

where

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p_{j} = \sqrt{\frac{m}{t}} \left( P \left( \frac{j}{m} t \right) - P \left( \frac{(j-1)}{m} t \right) \right)
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q_{j} = \sqrt{\frac{m}{t}} \left( Q \left( \frac{j}{m} t \right) - Q \left( \frac{(j-1)}{m} t \right) \right)
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- For large \(m\), \(1 + i \lambda \sqrt{\frac{t}{m}} (p \otimes q_{j} - q \otimes p_{j}) \simeq e^{i \lambda \sqrt{\frac{t}{m}} (p \otimes q_{j} - q \otimes p_{j})}\), a second quantized rotation, through the small angle \(\lambda \sqrt{\frac{t}{m}}\).
A second quantized product of rotations.

- We realise $p$ and $q$ as $\frac{i}{\sqrt{2}} (a^+ - a)$ and $\frac{1}{\sqrt{2}} (a^+ + a)$ where $a^+$ and $a$ are the standard creation and annihilation operators in $l^2$, which we regard as the Fock space $\mathcal{F}(\mathbb{C})$. 

Thus our approximate solution becomes 

$$
\prod_{\lambda} (1 + i \lambda (p dQ + q dP)) \Gamma = \prod_{j=1}^{\mathcal{F}} \Gamma R(j)\, m \prod_{j=1}^{\mathcal{F}} R(j)\, m
$$

(Institute)
A second quantized product of rotations.

- We realise $p$ and $q$ as $\frac{i}{\sqrt{2}} (a^\dagger - a)$ and $\frac{1}{\sqrt{2}} (a^\dagger + a)$ where $a^\dagger$ and $a$ are the standard creation and annihilation operators in $l^2$, which we regard as the Fock space $\mathcal{F}(\mathbb{C})$.

- Thus the action takes place in

$$\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2(\mathbb{R}_+)) = \mathcal{F}((\mathbb{C} \oplus L^2(\mathbb{R}_+))$$
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- $e^{i\lambda \sqrt{\frac{t}{m}}} (p \otimes q_j - q \otimes p_j)$ is then the second quantization $\Gamma\left(R_m^{(j)}\right)$ of the rotation $R_m^{(j)}$ through the angle $\lambda \sqrt{\frac{t}{m}}$ in the plane in $\mathbb{C} \oplus L^2(\mathbb{R}_+)$ spanned by $1 \in \mathbb{C}$ and the normalised indicator function

$$\sqrt{\frac{m}{t}} \chi_{[\frac{i-1}{m} t, \frac{i}{m} t]}.$$


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- Thus our approximate solution becomes

$$\prod_{0}^{t} \left(1 + i\lambda (p \otimes dQ - q \otimes dP)\right) \simeq \prod_{j=1}^{m} \Gamma\left(R_m^{(j)}\right) = \Gamma\left(\prod_{j=1}^{m} R_m^{(j)}\right).$$
Limit of a product of rotations.

Thus, writing
\[
\begin{bmatrix}
\cos \lambda \sqrt{\frac{t}{m}} & - \sin \lambda \sqrt{\frac{t}{m}} \\
\sin \lambda \sqrt{\frac{t}{m}} & \cos \lambda \sqrt{\frac{t}{m}}
\end{bmatrix}
= \begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix},
\]
we are interested in the product of \((1 + m) \times (1 + m)\) matrices

\[
\prod_{j=1}^{m} R_{m}^{(j)} \sim \prod_{j=1}^{m} \begin{bmatrix}
\alpha & 0 & \cdots & (1+j) \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\gamma & 0 & \cdots & \delta
\end{bmatrix}.
\]

By induction on \(m\) this is the \((1 + m) \times (1 + m)\) matrix
\[
\begin{bmatrix}
\alpha^m & \beta & \alpha\beta & \alpha^2\beta & \alpha^3\beta & \ldots & \alpha^{m-1}\beta \\
\gamma^{m-1}\alpha & \delta & \gamma\beta & \gamma\alpha\beta & \gamma\alpha^2\beta & \ldots & \gamma\alpha^{m-2}\beta \\
\gamma^{m-2}\alpha & 0 & \delta & \gamma\beta & \gamma\alpha\beta & \ldots & \gamma\alpha^{m-3}\beta \\
\gamma^{m-3}\alpha & 0 & 0 & \delta & \gamma\beta & \ldots & \gamma\alpha^{m-4}\beta \\
\gamma^{m-4}\alpha & 0 & 0 & 0 & \delta & \ldots & \gamma\alpha^{m-5}\beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma & 0 & 0 & 0 & 0 & \ldots & \delta
\end{bmatrix},
\]

so that, identifying basis elements of \( \mathbb{C}^m \) with \( \sqrt{\frac{m}{t}} \chi_{\left[\frac{j-1}{m} t, \frac{j}{m} t\right]}, j = 1, \ldots, m, \)

- the top-left element \( \alpha^m = \left( \cos \lambda \sqrt{\frac{t}{m}} \right)^m \approx \left( 1 - \frac{\lambda^2 t}{2m} \right)^m \to e^{-\frac{\lambda^2}{2} t} \)
\[
\begin{bmatrix}
\alpha^m & \beta & \alpha\beta & \alpha^2\beta & \alpha^3\beta & \ldots & \alpha^{m-1}\beta \\
\gamma^{m-1}\alpha & \delta & \gamma\beta & \gamma\alpha\beta & \gamma\alpha^2\beta & \ldots & \gamma\alpha^{m-2}\beta \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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\end{bmatrix},
\]

so that, identifying basis elements of \( \mathbb{C}^m \) with \( \sqrt{\frac{m}{t}} \chi_{\left[\frac{j-1}{m}, \frac{j}{m}\right]}\), \( j = 1, \ldots, m \),

- the top-left element \( \alpha^m = \left( \cos \lambda \sqrt{\frac{t}{m}} \right)^m \approx \left( 1 - \frac{\lambda^2 t}{2m} \right)^m \to e^{-\frac{\lambda^2}{2} t} \)
- the remainder of the top row converges to \( \langle f_t \rangle \) where \( f_t(x) = e^{-\frac{\lambda^2}{2} x} \chi_{[0,t]}(x) \), and the remainder of the left-most column converges to \( \langle g_t \rangle \) where \( g_t(x) = e^{-\frac{\lambda^2}{2} (t-x)} \chi_{[0,t]}(x) = f_t(t-x) \).
\[
\begin{pmatrix}
\alpha^m & \beta & \alpha \beta & \alpha^2 \beta & \alpha^3 \beta & \cdots & \alpha^{m-1} \beta \\
\gamma^{m-1} \alpha & \delta & \gamma \beta & \gamma \alpha \beta & \gamma \alpha^2 \beta & \cdots & \gamma \alpha^{m-2} \beta \\
\gamma^{m-2} \alpha & 0 & \delta & \gamma \beta & \gamma \alpha \beta & \cdots & \gamma \alpha^{m-3} \beta \\
\gamma^{m-3} \alpha & 0 & 0 & \delta & \gamma \beta & \cdots & \gamma \alpha^{m-4} \beta \\
\gamma^{m-4} \alpha & 0 & 0 & 0 & \delta & \cdots & \gamma \alpha^{m-5} \beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\gamma & 0 & 0 & 0 & 0 & \cdots & \delta \\
\end{pmatrix},
\]

so that, identifying basis elements of \( C^m \) with \( \sqrt{\frac{m}{t}} \chi_{[\frac{j-1}{m} t, \frac{j}{m} t]}, j = 1, \ldots, m, \)

- the top-left element \( \alpha^m = \left( \cos \lambda \sqrt{\frac{t}{m}} \right)^m \approx \left( 1 - \frac{\lambda^2 t}{2m} \right)^m \rightarrow e^{-\frac{\lambda^2}{2} t} \)

- the remainder of the top row converges to \( \langle f_t \rangle \) where \( f_t(x) = e^{-\frac{\lambda^2}{2} x} \chi_{[0, t]}(x) \), and the remainder of the left-most column converges to \( |g_t\rangle \) where \( g_t(x) = e^{-\frac{\lambda^2}{2} (t-x)} \chi_{[0, t]}(x) = f_t(t - x) \).

- The remaining \( m \times m \) matrix converges to \( I + D_t \) where \( D_t \) is the integral operator whose kernel is \( (x, y) \mapsto e^{-\frac{\lambda^2}{2} (y-x)} \chi_{[0, t]}(x, y) = f_t(t - x) \), where \( \chi_{[0, t]}(x, y) = 1 \) if \( 0 \leq x < y < t \) and 0 otherwise.
All this is handwaving heuristics. But it can then be shown rigorously that the resulting operator on $\mathbb{C} \oplus L^2(\mathbb{R}_+)$

$$W_t = \begin{bmatrix} e^{-\frac{t}{2}} & \langle f_t \rangle \\ \langle g_t \rangle & I + D_t \end{bmatrix}$$
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is a cocycle for the semigroup $\left( \begin{bmatrix} 1 & 0 \\ 0 & S_t \end{bmatrix} \right)_{t \geq 0}$ where $(S_t)_{t \geq 0}$ is the shift semigroup on $L^2(\mathbb{R}_+)$,
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and that the second quantization of \( W_t \),

$$\Gamma(W_t) = \prod_{0}^{t} \left( 1 + i\lambda (p \otimes dQ - q \otimes dP) - \frac{\lambda^2}{2} (p^2 + q^2) \otimes dT \right),$$

in so far as it satisfies the stochastic differential equation

$$d\Gamma(W_\bullet) = \Gamma(W_\bullet) \left( i\lambda (p \otimes dQ - q \otimes dP) - \frac{\lambda^2}{2} (p^2 + q^2) \otimes dT \right),$$

with \( \Gamma(W_0) = I \). (See M Evans, Nottingham PhD thesis, 1988).
Let $dr \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathbb{C} \langle dP, dQ, dT \rangle$. For $0 \leq a \leq b$, $0 \leq s \leq t$ we define

$$
\begin{align*}
\frac{b}{a} \prod_{s}^{t} (1 + dr) &= \begin{cases} \\
\text{either} \quad \frac{b}{a} \prod_{s}^{t} \left(1 + \prod_{s}^{t} (1 + dr)\right) \\
\text{or} \quad \prod_{s}^{t} \left(1 + \frac{b}{a} \prod (1 + dr)\right)
\end{cases}.
\end{align*}
$$
Double product integrals.

Let $dr \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathbb{C} \langle dP, dQ, dT \rangle$. For $0 \leq a \leq b$, $0 \leq s \leq t$ we define

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\frac{b}{a} \prod_{s}^{t} (1 + dr) = \left\{ \begin{array}{l}
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\text{or} \quad \prod_{s}^{t} \left( 1 + \frac{b}{a} \prod_{s}^{t} (1 + dr) \right)
\end{array} \right.
$$

Here, for a non-unital system algebra $\mathcal{A}$ (e.g., $\mathbb{C} \langle dP, dQ, dT \rangle$ equipped with the quantum Itô multiplication) and for $d\rho \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathcal{A}$, $\frac{b}{a} \prod_{s}^{t} (1 + d\rho)$ is the solution at $b$ of the qsde

$$
dX = \left( X + I_{\mathcal{F}(L^2(\mathbb{R}^+))} \right) d\rho, \quad X(a) = 0,
$$
Double product integrals.

Let \( dr \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathbb{C} \langle dP, dQ, dT \rangle \). For \( 0 \leq a \leq b \), \( 0 \leq s \leq t \) we define

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\]

Here, for a non-unital system algebra \( \mathcal{A} \) (eg \( \mathbb{C} \langle dP, dQ, dT \rangle \) equipped with the quantum Itô multiplication) and for \( d\rho \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathcal{A} \), \( \frac{b}{a} \prod_{a}^{b} (1 + d\rho) \) is the solution at \( b \) of the qsde

\[
dX = \left( X + I_{\mathcal{F}(L^{2}(\mathbb{R}^{+}))} \right) d\rho, \quad X(a) = 0,
\]

and, for a unital system algebra \( \mathcal{A} \) (eg the algebra of all iterated integrals over \([s, t]\)), \( \frac{b}{a} \prod_{a}^{b} (1 + d\rho) \) is the solution at \( b \) of the qsde

\[
dY = Y d\rho, \quad Y(a) = I.
\]
Double product integrals.

Let \( dr \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathbb{C} \langle dP, dQ, dT \rangle \). For \( 0 \leq a \leq b \), \( 0 \leq s \leq t \) we define

\[
\frac{b}{a} \prod_{s}^{t} (1 + dr) = \begin{cases} \\
\text{either} & \frac{b}{a} \prod_{s}^{t} \left( 1 + \hat{\prod}_{s}^{t} (1 + dr) \right) \\
\text{or} & \prod_{s}^{t} \left( 1 + \frac{b}{a} \hat{\prod}_{s}^{t} (1 + dr) \right) 
\end{cases}
\]

Here, for a non-unital system algebra \( \mathcal{A} \) (eg \( \mathbb{C} \langle dP, dQ, dT \rangle \) equipped with the quantum Itô multiplication) and for \( d\rho \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathcal{A} \), \( \frac{b}{a} \hat{\prod}_{s}^{t} (1 + d\rho) \) is the solution at \( b \) of the qsde

\[
dx = \left( X + I_{\mathcal{F}(L^{2}(\mathbb{R}^{+}))) \right) d\rho, \quad X(a) = 0,
\]

and, for a unital system algebra \( \mathcal{A} \) (eg the algebra of all iterated integrals over \([s, t[)\), \( \frac{b}{a} \hat{\prod}_{s}^{t} (1 + d\rho) \) is the solution at \( b \) of the qsde

\[
dY = Yd\rho, \quad Y(a) = I.
\]

\( \hat{\prod}_{s}^{t} (1 + d\rho) \) and \( \prod_{s}^{t} (1 + d\rho) \) are defined similarly but the system algebra is now on the left.
The two definitions are equivalent. This is not at all obvious, even if you have understood them.
The two definitions are equivalent. This is not at all obvious, even if you have understood them.

It is a continuous analog of the *discrete double product*

\[
m \prod_{j=1}^{m} n \prod_{k=1}^{n} (1 + x_{j,k}) = \begin{cases} 
\text{either} & \prod_{j=1}^{m} \left\{ \prod_{k=1}^{n} (1 + x_{j,k}) \right\} \\
\text{or} & \prod_{k=1}^{n} \left\{ \prod_{j=1}^{m} (1 + x_{j,k}) \right\} 
\end{cases},
\]

where it is quite easy to show that the two definitions agree *if the pairs* \(x_{j,k}, x_{j',k'}\) *commute whenever* \(j \neq j'\) *and* \(k \neq k'\),
The two definitions are equivalent. This is not at all obvious, even if you have understood them.

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\end{array} \right.,
\]

where it is quite easy to show that the two definitions agree if the pairs \(x_{j,k}, x'_{j',k'}\) commute whenever \(j \neq j'\) and \(k \neq k'\),

which they do for example, if, for \(dr \in \mathbb{C} \langle dP, dQ, dT \rangle \otimes \mathbb{C} \langle dP, dQ, dT \rangle\), \(x_{j,k}\) is the discrete double increment \((\delta r)_{j,k}\) over partitions of \([a, b]\) and \([s, t]\), into \(m\) and \(n\) subintervals respectively.
Causal double product integrals.

We also define causal or triangular double product integrals on the single Fock space $\mathcal{F}(L^2(\mathbb{R}_+))$ by

$$\prod_{a \leq x < y < b} (1 + dr(x, y)) = \prod_{a \leq y < b} (1 + \prod_{a \leq x < y} (1 + dr(., dy))).$$

It is related to the previous "rectangular" double product integrals by the coboundary relation

$$\prod_{a \leq x < y < c} (1 + dr(x, y)) = \prod_{a \leq x < y < b} (1 + dr(x, y)) b \prod_{a \leq x < y < c} (1 + dr) \prod_{b \leq x < y < c} (1 + dr(x, y)).$$

Here we regard $b \prod_{a \leq x < y < c} (1 + dr)$ as also living in the single Fock space $\mathcal{F}(L^2(\mathbb{R}_+)) = \mathcal{F}(L^2([0, b])) \otimes \mathcal{F}(L^2([b, \infty[))$ by splitting at $b$ (modulo de-ampliation).
In the rest of this talk we consider the case
\[ dr = i \lambda (dP \otimes dQ - dQ \otimes dP). \]
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\[ dr = i\lambda (dP \otimes dQ - dQ \otimes dP). \]

Imitating the previous procedure, we expect that, for large \( m, n \),

\[
\begin{align*}
\mathcal{R} &= \prod_{j=1}^{b} \prod_{i=1}^{a} \left( 1 + i\lambda (dP \otimes dQ - dQ \otimes dP) \right) \\
\mathcal{R} &\equiv \prod_{j=1}^{m} \prod_{k=1}^{n} \left( 1 + i\lambda \sqrt{\frac{(b-a)(t-s)}{mn}} (p^j \otimes q_k - q^j \otimes p_k) \right) \\
\mathcal{R} &\equiv \Gamma \left( \prod_{j=1}^{m} \prod_{k=1}^{n} R_{(m,n)}^{(i,k)} \right)
\end{align*}
\]

where the \((p^j, q^j)\) and \((p_k, q_k)\) are standard commuting canonical pairs and \(R_{(m,n)}^{(i,k)}\) is the \((m+n) \times (m+n)\) matrix got by embedding the \(2 \times 2\) rotation through the angle

\[
\theta_{m,n} = \lambda \sqrt{\frac{(b-a)(t-s)}{mn}}
\]

at the intersections of the \(j\)th and \((m+k)\)th rows and columns with remaining diagonal and non-diagonal entries 1 and 0 respectively.
I do not have an explicit formula for \( m \prod_{k=1}^{n} R_{(m,n)}^{(i,k)} \). But when \( m = 1 \), from above,

\[
\prod_{k=1}^{n} R_{(1,n)}^{(i,1)} = \begin{bmatrix}
\alpha^n & \beta & \alpha \beta & \alpha^2 \beta & \alpha^3 \beta & \cdots & \alpha^{n-1} \beta \\
\gamma^{n-1} \alpha & \delta & \gamma \beta & \gamma \alpha \beta & \gamma \alpha^2 \beta & \cdots & \gamma \alpha^{n-2} \beta \\
\gamma^{n-2} \alpha & 0 & \delta & \gamma \beta & \gamma \alpha \beta & \cdots & \gamma \alpha^{n-3} \beta \\
\gamma^{n-3} \alpha & 0 & 0 & \delta & \gamma \beta & \cdots & \gamma \alpha^{n-4} \beta \\
\gamma^{n-4} \alpha & 0 & 0 & 0 & \delta & \cdots & \gamma \alpha^{n-5} \beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma & 0 & 0 & 0 & 0 & \cdots & \delta
\end{bmatrix}
\]

where now \( \alpha = \delta = \cos \theta_{1,n} \), \( -\beta = \gamma = \sin \theta_{1,n} \),

\[
\theta_{1,n} = \lambda \sqrt{\frac{(b-a)(t-s)}{n}}.
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I do not have an explicit formula for \( m \prod_{k=1}^{n} R_{(m,n)}^{(i,k)} \). But when \( m = 1 \), from above,

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\alpha^n & \beta & \alpha\beta & \alpha^2\beta & \alpha^3\beta & \cdots & \alpha^{n-1}\beta \\
\gamma^{n-1}\alpha & \delta & \gamma\beta & \gamma\alpha\beta & \gamma\alpha^2\beta & \cdots & \gamma\alpha^{n-2}\beta \\
\gamma^{n-2}\alpha & 0 & \delta & \gamma\beta & \gamma\alpha\beta & \cdots & \gamma\alpha^{n-3}\beta \\
\gamma^{n-3}\alpha & 0 & 0 & \delta & \gamma\beta & \cdots & \gamma\alpha^{n-4}\beta \\
\gamma^{n-4}\alpha & 0 & 0 & 0 & \delta & \cdots & \gamma\alpha^{n-5}\beta \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma & 0 & 0 & 0 & 0 & \cdots & \delta
\end{bmatrix}
\]

where now \( \alpha = \delta = \cos \theta_{1,n} \), \(-\beta = \gamma = \sin \theta_{1,n}\),

\[
\theta_{1,n} = \lambda \sqrt{\frac{(b-a)(t-s)}{n}}.
\]

Notice however that the matrix makes sense as an operator on \( L^2(\mathbb{R}_+) \oplus \mathbb{C}^n \) if \( \alpha \) is an operator on \( L^2(\mathbb{R}_+) \), \( \beta \) and \( \gamma \) are respectively vectors and covectors in \( L^2(\mathbb{R}_+) \) and \( \delta \) is a scalar.
And similarly the product of \((m + 1) \times (m + 1)\) matrices:

\[
\prod_{j=1}^{m} R^{(j,1)}_{(m,1)} = \prod_{j=1}^{m} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha & 0 & \beta \\
0 & 0 & 1 & 0 \\
0 & \gamma & 0 & \delta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-2} \gamma & \beta \delta^{m-1} \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \ldots & \beta \delta^{m-3} \gamma & \beta \delta^{m-2} \\
0 & 0 & \alpha & \beta \gamma & \ldots & \beta \delta^{m-4} \gamma & \beta \delta^{m-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \alpha & \beta \\
\gamma & \delta \gamma & \delta^2 \gamma & \delta^3 \gamma & \ldots & \delta^{m-1} \gamma & \delta^m
\end{bmatrix}
\]

where now \(\alpha = \delta = \cos \theta_{m,1}, \ -\beta = \gamma = \sin \theta_{m,1}, \)

\[
\theta_{m,1} = \lambda \sqrt{\frac{(b-a)(t-s)}{m}}.
\]
And similarly the product of \((m + 1) \times (m + 1)\) matrices:

\[
\prod_{j=1}^{m} R_{(m,1)}^{(j,1)} = \prod_{j=1}^{m} \begin{bmatrix}
[1,j] & [j] & [j,m] & [m+1] \\
1 & 0 & 0 & 0 \\
0 & \alpha & 0 & \beta \\
0 & 0 & 1 & 0 \\
0 & \gamma & 0 & \delta \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\alpha & \beta \gamma & \beta \delta \gamma & \beta \delta^2 \gamma & \ldots & \beta \delta^{m-2} \gamma & \beta \delta^{m-1} \\
0 & \alpha & \beta \gamma & \beta \delta \gamma & \ldots & \beta \delta^{m-3} \gamma & \beta \delta^{m-2} \\
0 & 0 & \alpha & \beta \gamma & \ldots & \beta \delta^{m-4} \gamma & \beta \delta^{m-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \alpha & \beta \\
\gamma & \delta \gamma & \delta^2 \gamma & \delta^3 \gamma & \ldots & \delta^{m-1} \gamma & \delta^m \\
\end{bmatrix}
\]

where now \(\alpha = \delta = \cos \theta_{m,1} \), \(-\beta = \gamma = \sin \theta_{m,1}\),

\[
\theta_{m,1} = \lambda \sqrt{\frac{(b-a)(t-s)}{m}}.
\]

This product formula holds for operators on \(\mathbb{C}^m \oplus L^2 (\mathbb{R}_+)\) if \(\alpha\) is a scalar, \(\beta\) and \(\gamma\) are respectively a covector and a vector in \(L^2 (\mathbb{R}_+)\) and \(\delta\) is an operator on \(L^2 (\mathbb{R}_+)\).
Thus we can take $\alpha$, $\beta$, $\gamma$ and $\delta$ to be the operator, covector, vector and scalar making up the limit

$$\lim_{n \to \infty} \prod_{k=1}^{n} R^{(j,k)}_{(m,n)} = \begin{bmatrix} e^{-\frac{\lambda^2(b-a)(t-s)}{2mn}} & \langle f^k_m \rangle \\ |g^k_m \rangle & D \end{bmatrix}.$$ 

In this way we can form, first

$$\prod_{j=1}^{m} \left( \lim_{n \to \infty} \prod_{k=1}^{n} R^{(j,k)}_{(m,n)} \right) = \lim_{n \to \infty} \left( \prod_{j=1}^{m} \left( \prod_{k=1}^{n} R^{(j,k)}_{(m,n)} \right) \right).$$
Thus we can take $\alpha$, $\beta$, $\gamma$ and $\delta$ to be the operator, covector, vector and scalar making up the limit

$$
\lim_{n \to \infty} \prod_{k=1}^{n} R^{(j,k)}_{(m,n)} = \begin{bmatrix}
    e^{-\frac{\lambda^2 (b-a)(t-s)}{2mn}} \\
    \langle f^k_m | \\
    g^k_m \\
    D
\end{bmatrix}.
$$

- In this way we can form, first

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$$

- and then also

$$
\frac{b}{a} W^t_s = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} \left( \prod_{j=1}^{m} \left( \prod_{k=1}^{n} R^{(j,k)}_{(m,n)} \right) \right) \right\} = \begin{bmatrix}
    I + A & B \\
    C & I + D
\end{bmatrix}
$$

where $A$, $B$, $C$ and $D$ are explicitly determined integral operators on $L^2(\mathbb{R}_+)$. 


Similarly we can find the explicit form for

\[ b \mathcal{W}_s^t = \lim_{n \to \infty} \left\{ \lim_{m \to \infty} \left( \prod_{k=1}^{n} \left\{ \prod_{j=1}^{m} R(j,k) \right\} \right) \right\} = \left[ I + \tilde{A} \quad \tilde{B} \right] \left[ \begin{array}{c} \tilde{C} \\ I + \tilde{D} \end{array} \right]. \]
Similarly we can find the explicit form for

\[
\begin{align*}
\lim_{n \to \infty} \left\{ \lim_{m \to \infty} \left( \prod_{k=1}^{n} \left( \prod_{j=1}^{m} R^{(j,k)}_{(m,n)} \right) \right) \right\} &= \left[ \begin{array}{cc}
I + \tilde{A} & \tilde{B} \\
\tilde{C} & I + \tilde{D}
\end{array} \right].
\end{align*}
\]

Fortunately the two iterated limits are equal; in fact

\[
\begin{align*}
\ker A(x, y) &= \ker \tilde{A}(x, y) = \langle b_{a}^{\top} \rangle_{x, y} \sum_{N=0}^{\infty} \frac{(y - x)^{N} \left( -\frac{\lambda^{2}}{2} (t - s) \right)^{N+1}}{N! (N + 1)!} \\
\ker B(x, y) &= \lambda \chi_{[a,b]}(x) \chi_{[s,t]}(y) \sum_{N=0}^{\infty} \frac{\left( -\frac{\lambda^{2}}{2} (b - x)(y - s) \right)^{N}}{(N!)^{2}} \\
\ker C(x, y) &= -\lambda \chi_{[a,b]}(y) \chi_{[s,t]}(x) \sum_{N=0}^{\infty} \frac{\left( -\frac{\lambda^{2}}{2} (t - x)(y - a) \right)^{N}}{(N!)^{2}} \\
\ker D(x, y) &= \langle t_{s} \rangle_{x, y} \sum_{N=0}^{\infty} \frac{(y - x)^{N} \left( -\frac{\lambda^{2}}{2} (b - a) \right)^{N+1}}{N! (N + 1)!}.
\end{align*}
\]
The construction is heuristic. But it can then be rigorously shown that
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Each $a \overset{b}{\mathcal{W}}_s = a \overset{b}{\tilde{\mathcal{W}}}_s$ is unitary.
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Each \( _a^b W_t = a^b \tilde{W}_t \) is unitary.

For fixed \( s \) and \( t \), \( ( _a^b W_t )_{a \leq b} \) is an evolution in \( a \) and \( b \), and *vice versa*. 
The construction is heuristic. But it can then be rigorously shown that

- Each $b_a W^t_s = b_a \tilde{W}^t_s$ is unitary.
- For fixed $s$ and $t$, $(b_a W^t_s)_{a \leq b}$ is an evolution in $a$ and $b$, and *vice versa*.
- These evolutions are shift-covariant.
The construction is heuristic. But it can then be rigorously shown that

- Each $b^a W^t_s = b^a \tilde{W}^t_s$ is unitary.
- For fixed $s$ and $t$, $(b^a W^t_s)_{a \leq b}$ is an evolution in $a$ and $b$, and vice versa.
- These evolutions are shift-covariant.
- $\Gamma ( (b^a W^t_s) ) = b^a \Pi^t_s (1 + i \lambda (dP \otimes dQ - dQ \otimes dP))$ in so far as it satisfies either set of quantum stochastic differential equations.
The construction is heuristic. But it can then be rigorously shown that 
Each \( {}^b_a W_t^s = {}^b_a \tilde{W}_t^s \) is unitary.
For fixed \( s \) and \( t \), \( ({}^b_a W_t^s)_{a \leq b} \) is an evolution in \( a \) and \( b \), and *vice versa*.
These evolutions are shift-covariant.
\[
\Gamma \left( ({}^b_a W_t^s) \right) = {}^b_a \Pi_t^s \left( 1 + i\lambda \left( dP \otimes dQ - dQ \otimes dP \right) \right)
\]
in so far as it satisfies either set of quantum stochastic differential equations.
The causal product

$$\prod_{a \leq x < y < b} (1 + i\lambda (dP(x)dQ(y) - dQ(x)dP(y)))$$

is of particular interest because of its connection to Lévy’s stochastic area.
A causal product.

- The causal product \( \prod_{a \leq x < y < b} (1 + i\lambda (dP(x)dQ(y) - dQ(x)dP(y))) \) is of particular interest because of its connection to Lévy’s stochastic area.
- We approximate the integral as:
  \[
  \prod_{a \leq x < y < b} (1 + i\lambda (dP(x)dQ(y) - dQ(x)dP(y))) \\
  \approx \prod_{1 \leq j < k \leq n} \left( 1 + i\lambda \frac{b - a}{n} (p_j q_k - q_j p_k) \right)
  \]

where the canonical pairs \((p_j, q_j)\) are the normalised increments of \(P\) and \(Q\).
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\]

where the canonical pairs \((p_j, q_j)\) are the normalised increments of \(P\) and \(Q\).

The method of iterated limits used above does not work in this case. Instead, a different approximation procedure (in fact that used in RLH and Jones, loc. cit.) leads to
\[
\prod (1 + i \lambda (dP(x)dQ(y) - dQ(x)dP(y))) = \Gamma \left( \frac{b}{n} + K^b \right)
\]
where the integral operator \( K^b_a \) has a kernel of the form

\[
\sum_{N=0}^{\infty} \left( \langle x, y \rangle f_N(x - a, y - x, b - x) + \rangle^b_a (x, y) g_N (b - x, x - y, y - \right)
\]

where \( f_N \) and \( g_N \) are homogeneous polynomials of degree \( N \). \( g_N \) is known explicitly. \( f_N \) is not yet known explicitly but its coefficients involve Catalan numbers and their derivatives.

Similar methods can be used to construct causal and rectangular products of form

\[
\prod (1 + dr)
\]

where \( dr \) satisfies the coisometry condition

\[
dr + dr^\dagger + drdr^\dagger = 0
\]

as unitary implementors of Bogolubov transformations constructed as continuum limits of discrete products got by replacing differentials by discrete increments.

where the integral operator $K_a^b$ has a kernel of the form

$$
\sum_{N=0}^{\infty} \left( \langle b_a (x, y) f_N(x - a, y - x, b - x) + \rangle_a^b (x, y) g_N(b - x, x - y, y - \right)
$$

where $f_N$ and $g_N$ are homogeneous polynomials of degree $N$. $g_N$ is known explicitly. $f_N$ is not yet known explicitly but its coefficients involve Catalan numbers and their derivatives.

Similar methods can be used to construct causal and rectangular products of form $\Pi(1 + dr)$ where $dr$ satisfies the coisometry condition

$$
dr + dr^\dagger + drdr^\dagger = 0
$$

as unitary implementors of Bogolubov transformations constructed as continuum limits of discrete products got by replacing differentials by discrete increments.

where the integral operator $K^b_a$ has a kernel of the form

$$
\sum_{N=0}^{\infty} \left( <^b_a (x, y)f_N(x-a, y-x, b-x) + >^b_a (x, y)g_N(b-x, x-y, y-x) \right)
$$

where $f_N$ and $g_N$ are homogeneous polynomials of degree $N$. $g_N$ is known explicitly. $f_N$ is not yet known explicitly but its coefficients involve Catalan numbers and their derivatives.

Similar methods can be used to construct causal and rectangular products of form $\Pi(1 + dr)$ where $dr$ satisfies the coisometry condition

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as unitary implementors of Bogolubov transformations constructed as continuum limits of discrete products got by replacing differentials by discrete increments.

Thankyou for listening.