Quantum Probability, Statistics and Control
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a workshop dedicated to the memory of Slava Belavkin

◎ My memory of Prof. Slava Belavkin …..

He was one of my best friends.

◎ My memory of Prof. Slava Belavkin ……..

He was one of my best teachers.
Memory of Cherry Blossoms
Slava gave me

the deep understandings for the basic notions

on Quantum Probability and Information theory.
Mutual entropy via Belavkin–Ohya entanglement channel

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This talk is based on the following papers:

1) Belavkin, V. P. and Ohya, M.

2) Belavkin, V. P. and Ohya, M.
**Notation & Definitions**

\( \mathcal{H}_1, \mathcal{H}_2: \) separable Hilbert spaces

\( B(\mathcal{H}_i)_{i=1,2}: \) a set of all bounded operators on \( \mathcal{H}_i \)

\( S(\mathcal{H}_i)_{i=1,2}: \) a set of all states (density operators)

**<Def. of separable states and entangled states>**

- \( S(\mathcal{H}_1 \otimes \mathcal{H}_2)_{sep} = \text{conv}(S(\mathcal{H}_1) \otimes S(\mathcal{H}_2)) \)
  \[ \Rightarrow \theta = \sum p_{ij} \rho_i \otimes \sigma_j \]

- \( S(\mathcal{H}_1 \otimes \mathcal{H}_2)_{ent} = S(\mathcal{H}_1 \otimes \mathcal{H}_2) \backslash S(\mathcal{H}_1 \otimes \mathcal{H}_2)_{sep} \)

**<Def. of CP map and CCP map>**

- a linear map \( \alpha : \mathcal{A} \rightarrow \mathcal{B} \) is positive if
  \[ \alpha(\mathcal{A}^+) \subset \mathcal{B}^+ \]

- a linear map \( \tau : \mathcal{A} \rightarrow \mathcal{B} \) is CP iff
  \[ \tau_n : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B}); \begin{bmatrix} a_{ij} \end{bmatrix} \rightarrow \begin{bmatrix} \tau(a_{ij}) \end{bmatrix} \]
  is positive for all \( n \).

\( M_n(\mathcal{A}); \ n \times n \) matrices with entries from \( \mathcal{A} \)

- a linear map \( \tau : \mathcal{A} \rightarrow \mathcal{B} \) is CCP iff
  \[ \tau_n^t : M_n(\mathcal{A}) \rightarrow M_n(\mathcal{B}); \begin{bmatrix} a_{ij} \end{bmatrix} \rightarrow \begin{bmatrix} \tau(a_{ji}) \end{bmatrix} \]
  is positive for all \( n \).

In this talk \( \dim \mathcal{H}_i < \infty \)
Plan of my talk

1. Two methods of constructing compound states and their relations
   (i) Compound state via entanglement mapping
   (ii) Compound state via quantum conditional probability operator
   (iii) Relation between (i) and (ii)

2. Classical correlation and transmitted information on Shannon’s theory

3. Mutual entropy as the quantum correlation on the base of relation between (i) and (ii)

4. Decomposition of mutual entropy
Two methods of constructing compound states

Method 1:
Compound state via entanglement map
(Belavkin, Ohya : 2001, 2002)

$$E_{\theta}(A \otimes B) = \text{Tr}_{12}(A \otimes B) \theta$$

where $\theta \in S(\mathcal{H}_1 \otimes \mathcal{H}_2)$

Entanglement map

$\phi : B(\mathcal{H}_2) \rightarrow B(\mathcal{H}_1^*)$

$$B \mapsto \phi(B) = \text{Tr}_2(I_1 \otimes B) \theta$$

$\phi^* : B(\mathcal{H}_1) \rightarrow B(\mathcal{H}_2^*)$

$$A \mapsto \phi^*(A) = \text{Tr}_1(A \otimes I_2) \theta$$

$$\Rightarrow E_{\theta}(A \otimes B) = \text{Tr}_1(A \phi(B)) = \text{Tr}_2(\phi^*(A)B)$$

Both $\phi$ and $\phi^*$ are always CCP, but not always CP.

Remark : 1. $\phi$ and $\phi^*$ can be given by Hilbert Schmidit operator.
2. B-O scheme can be constructed in the case of general C*-algebraic setting.
The classification by entanglement mapping

In the case of pure states or mixture states on
\[ \mathcal{H}_1 \otimes \mathcal{H}_2 = C^2 \otimes C^2 \text{ or } C^2 \otimes C^3 \text{ or } C^3 \otimes C^2 \]
\( \theta \) is separable iff \( \phi \) is CP.
In the other case:
If \( \theta \) is separable, then \( \phi \) is CP.

<Def.> (q-entanglement)
\( \phi \) is called q-entanglement if \( \phi \) is not CP.
\( \Rightarrow \) If \( \phi \) is q-entanglement, then \( \theta \) is entangled.
\( \phi_q \in \mathcal{E}_q \)

\( \phi \) reconstructs \( \theta \) via Choi-Jamiołkowski’s isomorphism.

\[
\theta = \sum \phi (|e_j\rangle\langle e_i|) \otimes (|e_i\rangle\langle e_j|) = W_j (\phi) \geq 0
\]
\[
\Leftrightarrow \phi \text{ is always complete co-positive.}
\]
where \( \{e_i\} \subset \mathcal{H}_2 \) is an orthogonal basis.
General form of compound state by entanglement map

<Def.>
A CCP map \( \phi : B(\mathcal{H}_2) \to B(\mathcal{H}_1) \) normalized as \( \text{Tr}_1 \phi(I_2) = 1 \) is called the (generalized) entanglement map from the state \( \sigma = \phi^*(I_1) \) to the state \( \rho = \phi(I_2) \).

\[
E_{\theta_\phi} (A \otimes B) = \text{Tr}_{12} (A \otimes B) \theta_\phi
\]

\[
\theta_\phi = \sum_{i,j} \phi\left( |e_j \rangle \langle e_i | \right) \otimes |e_i \rangle \langle e_j |
\]
Method 2:

Compound state via quantum conditional probability operator
(Asorey, Kossakowski, Marmo, Sudarshan : 2005)

For a given $\theta \in S(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\rho = \text{Tr}_2 \theta$, one can define the operator $\pi_\theta$ as

$$
\pi_\theta : \mathcal{H}_1 \otimes \mathcal{H}_2 \mapsto \mathcal{H}_1 \otimes \mathcal{H}_2
$$

$$
\pi_\theta = \left( \rho \frac{1}{2} \otimes I_2 \right) \theta \left( \rho \frac{1}{2} \otimes I_2 \right)
$$

which has a property

$$
\pi_\theta \geq 0, \quad (1)
$$

$$
\text{Tr}_2 \pi_\theta = I_1. \quad (2)
$$

where we assume that $\rho$ is a faithful state.

<Def.> An operator

$$
\pi : \mathcal{H}_1 \otimes \mathcal{H}_2 \mapsto \mathcal{H}_1 \otimes \mathcal{H}_2
$$

is called a quantum conditional probability operator (QCPO) if it satisfies (1) and (2).
For a given $\pi$ and any faithful state $\rho \in S(\mathcal{H}_1)$ we can define a compound state $\theta$

$$\theta = \left( \frac{1}{\rho^2} \otimes I_2 \right) \pi \left( \frac{1}{\rho^2} \otimes I_2 \right)$$

with its marginals as $\rho = \text{Tr}_2 \theta$, $\sigma = \text{Tr}_1 \theta$.

**General form of compound state**

by QCPO given by CP map

For a given CP normalized map $\phi : B(\mathcal{H}_2) \mapsto B(\mathcal{H}_1)$ i.e., $\phi(I_2) = I_1$

$$\pi_\phi = \sum_{i,j} \phi \left( |e_i\rangle\langle e_j| \right) \otimes |e_i\rangle\langle e_j|$$

$$\theta_\phi = \left( \frac{1}{\rho^2} \otimes I_{\mathcal{K}} \right) \pi_\phi \left( \frac{1}{\rho^2} \otimes I_{\mathcal{K}} \right) = \sum_{i,j} \rho^2 \phi \left( |e_i\rangle\langle e_j| \right) \rho^2 \otimes |e_i\rangle\langle e_j|$$

<Th.> (Kossakowski et al)

In the case of pure state or mixture states on $\mathcal{H}_1 \otimes \mathcal{H}_2 = C^2 \otimes C^2$ or $C^2 \otimes C^3$ or $C^3 \otimes C^2$

$\theta$ is separable iff $\phi$ is CCP.

In the other case:

If $\theta$ is separable, then $\phi$ is CCP.
Relation between Entanglement Map & QCPO
(Chruscinski, Kossakowski, T.M., Ohya :2011)

\[ \theta_\phi = \left( \frac{1}{\rho^2} \otimes I_2 \right) \pi_\phi \left( \frac{1}{\rho^2} \otimes I_2 \right) \]

<Th.> (Belavkin, Dai :2008)

Every entanglement \( \phi \) with \( \phi(I_2) = \rho \) has a decomposition

\[ \phi(\bullet) = \rho^{\frac{1}{2}} \left( \varphi \circ \tau(\bullet) \right) \rho^{\frac{1}{2}} \]

where \( \tau \) is a transpose operation and \( \varphi \) is a CP normalized map to be found as a unique solution as

\[ \varphi(\bullet) = \rho^{\frac{1}{2}} \left( \phi \circ \tau(\bullet) \right) \rho^{\frac{1}{2}} \]

On classical probability scheme:

\[ p(i, j) = p(i) p(j | i) \]

<Question>

On the base of above quantum relation how can we recognized the quantum correlation by using the mutual entropy?
Classical correlation and transmitted information on Shannon’s theory

In classical system

System \( (X, P(X)) \): \( X = \{x_i\}_{i=1, \ldots, n}, P(X) = (p(x_i)) \)

System \( (Y, P(Y)) \): \( Y = \{y_j\}_{j=1, \ldots, n}, P(Y) = (p(y_j)) \)

Correlation \( \Leftrightarrow \) Joint prob. or Conditional prob.

\[
p(x_i, y_j) = p(y_j) p(x_i | y_j) = p(x_i) p(y_j | x_i)
\]

\(<\text{Mutual entropy}>\)

\[
I(X, Y) \equiv \sum p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i) p(y_j)} = S(X) + S(Y) - S(X \otimes Y)
\]

\[
S(X \otimes Y)
\]

\[
S(X)
\]

\[
I(X,Y)
\]

\[
S(Y)
\]
Mutual entropy on the communication scheme

Input system: \( X = \{x_i\}_{i=1,\ldots,n}, P(X) = (p(x_i)) \)

Output system: \( Y = \{y_j\}_{j=1,\ldots,n}, P(Y) = (p(y_j)) \)

Channel: \( \Lambda^* = (\Lambda^*_{ij}) \Leftrightarrow (p(y_j | x_i)) \)
we call it a “transition prob. matrix”

\[
P(Y) = \Lambda^*(P(X)) \Leftrightarrow p(y_j) = \sum_{i=1}^{n} \Lambda^*_{ij} p(x_i)
\]

\[
I(X,Y) \equiv \sum p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i) p(y_j)}
\]

\[
= \sum p(x_i) p(y_j | x_i) \log \frac{p(y_j | x_i)}{p(y_j)}
\]

\[
= \sum \Lambda^*_{ij} p(x_i) \log \frac{\Lambda^*_{ij}}{\sum_k \Lambda^*_k p(x_k)} \equiv I(P(X); \Lambda^*)
\]

The mutual entropy \( I(P(X); \Lambda^*) \) can be recognized as a proper measure of the amount of information transmitted from input to output through the channel \( \Lambda^* \).
Crucial points in classical information theory

Correlation on probabilities

\[ p(x_i, y_j) = p(y_j) p(x_i | y_j) = p(x_i) p(y_j | x_i) \]

Mutual entropy as a measure of correlation

\[ I(X,Y) \]

Mutual entropy on the communication scheme

\[ P(Y) = \Lambda^*(P(X)) \iff p(y_j) = \sum_{i=1}^{n} \Lambda^*_{ij} p(x_i) \]

\[ I(X,Y) = I(P(X); \Lambda^*) \]

The classical correlation can be equivalent to the amount of transmitted information from a marginal state to another marginal state.

**<The fundamental inequality>**

\[ 0 \leq I(p(X); \Lambda^*) \leq \min\{S(p(X)), S(\Lambda^*(p(X)))\} \]

How can we extend the Shannon’s scheme to a quantum system?
Mutual entropy on quantum correlation

Quantum mutual entropy  (Belavkin and Ohya : 2001, 2002)

\[ I_\phi (\rho, \sigma) \equiv S(\theta_\phi, \rho \otimes \sigma) \]
\[ = \text{Tr} \theta_\phi (\log \theta_\phi - \log \rho \otimes \sigma) \]
\[ = S(\rho) + S(\sigma) - S(\theta_\phi) \]

where \( S(\bullet, \bullet) \) is the Umegaki relative entropy

and \( \rho = \phi(I_2), \sigma = \phi^*(I_1) \).

Remark: The above entropic measure are discussed also by Cref and Adami, Groisman et all, Henderson and Vedral, Horodeckies, etc.
Example 1 (a product state):

\[
\phi_{\text{product}}(b) = \text{Tr}_1 \sigma b \cdot \rho, \quad \phi^*_{\text{product}}(a) = \text{Tr}_2 \rho a \cdot \sigma, \\
\theta_{\phi_{\text{product}}} = \rho \otimes \sigma
\]

\[
I_{\phi_{\text{product}}} (\rho, \sigma) = 0 
\]

⇔ No correlation


A state \( \theta \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2) \) is called a product state if it can be represented in the form

\[
\theta = \rho \otimes \sigma.
\]

A correlated state is a state not belonged to the set of all product states, so a correlated state is simply considered as a non product state.

\[
\downarrow
\]

In quantum scheme we have two types of correlated states.

\[
\mathcal{S}_c = \mathcal{S}_{sc} \cup \mathcal{S}_{ec}
\]

\( \mathcal{S}_c \): set of all correlated states

\( \mathcal{S}_{sc} \): set of all separable correlated states

\( \mathcal{S}_{ec} \): set of all entangled correlated states
Example 2 (a separable correlated state):
(Belavkin, Ohya : 2001, 2002)

\[
\phi_{\text{sep}}(b) = \sum p_i \rho_i \text{Tr}_2 \sigma_i b, \quad \phi^*_{\text{sep}}(a) = \sum p_i \sigma_i \text{Tr}_1 \rho_i a,
\]

\[
\theta_{\phi_{\text{sep}}} = \sum p_i \rho_i \otimes \sigma_i.
\]

\[
0 \leq I_{\phi_{\text{sep}}} (\rho, \sigma) \leq \min \{S(\rho), S(\sigma)\}
\]

\[\iff\] Corresponding to classical property

Example 3 (a pure entangle correlated state):

For \( \lambda_i \in \mathbb{C}, \sum |\lambda_i|^2 = 1, \{f_j\} \subset \mathcal{H}_1, \{e_i\} \subset \mathcal{H}_2 \)

\[
\phi_{\text{pure ent}}(b) = \sum \lambda_i \overline{\lambda}_j |f_i\rangle\langle f_j| \text{Tr}_2 \langle e_j, be_i |, \quad \phi^*_{\text{pure ent}}(a) = \sum \lambda_i \overline{\lambda}_j |e_i\rangle\langle e_j| \text{Tr}_1 \langle f_j, af_i |,
\]

\[
\theta_{\phi_{\text{pure ent}}} = \sum \lambda_i \overline{\lambda}_j |f_i\rangle\langle f_j| \otimes |e_i\rangle\langle e_j|.
\]

\[
I_{\phi_{\text{pure ent}}} (\rho, \sigma) = 2S(\rho) > S(\rho) = S(\sigma)
\]

\[\iff\] Non-classical property

\[I_{\phi_{\text{pure ent}}} (\rho, \sigma) \] does not satisfy the fundamental inequality.
Mutual entropy on quantum communication scheme

Channel representation of the mutual entropy by QCPO

For a given entanglement map $\phi$

$$\sigma = \Lambda^*_{\phi}(\rho) = \text{Tr}_1\left(\frac{1}{\rho^2} \otimes I_2\right)\pi_{\phi}\left(\frac{1}{\rho^2} \otimes I_2\right)$$

We call $\pi_{\phi}$ the quantum channel density.

Notice: this naming was given by Slava

$$I_{\phi}(\rho, \sigma) = \text{Tr}\left(\frac{1}{\rho^2} \otimes I_2\right)\pi_{\phi}\left(\frac{1}{\rho^2} \otimes I_2\right)$$

$$= (\log\left(\frac{1}{\rho^2} \otimes I_2\right))\pi_{\phi}\left(\frac{1}{\rho^2} \otimes I_2\right) - \log \rho \otimes \Lambda^*_{\phi}(\rho))$$

$$\equiv I_{\phi}(\rho; \Lambda^*_{\phi})$$

$I_{\phi}(\rho; \Lambda^*_{\phi})$ does not always satisfy the fundamental inequality. $\iff$ a pure entangled correlated state

$I_{\phi}(\rho, \sigma)$ cannot be recognized as a proper measure of transmitted information through the channel $\Lambda^*_{\phi}$

**Question:**

How can we connect $I_{\phi}(\rho, \sigma)$ to the transmitted information through the channel $\Lambda^*_{\phi}$?
**Ohya mutual entropy via the channel** $\Lambda^*_\phi$

We can measure the transmitted information through the entanglement channel $\Lambda^*_\phi$ by using O-mutual entropy.

**Step1:** **Schatten decomposition of input state** $\rho_{in}$

$$\rho_{in} = \sum \lambda_k E_k$$

where $E_k$ is the one dimensional projection associated with eigenvalue $\lambda_k$ and the degenerated eigenvalue $\lambda_k$ are repeated it’s the degeneracy, for instance, if the eigenvalue $\lambda_1$ has the degeneracy 3, then $\lambda_1 = \lambda_2 = \lambda_3 > \lambda_4$.

This **Schatten decomposition is not unique unless every eigenvalue is non-degenerated.**

**Step2:** **Diagonal compound state**

$$\theta_E = \sum \lambda_k E_k \otimes \Lambda^* (E_k)$$

with its marginal $\rho_{in}$ and $\rho_{out} = \Lambda^* (\rho_{in})$.

**Step3:** **Transmitted information through the channel**

$$I_O (\rho_{in} : \Lambda^*) \equiv \sup_{E = \{E_i\}} \left\{ S (\theta_E, \rho_{in} \otimes \Lambda^* (\rho_{in})) \right\}$$

**<Theorem>**

$$0 \leq I_O (\rho_{in} ; \Lambda^*) \leq \min \left\{ S (\rho_{in} ) , S (\Lambda^* (\rho_{in} )) \right\}$$
We can apply for \( I_O(\rho_{in};\Lambda^*) \) to \( \Lambda^*_\phi \).

Input state and output state:
\[
\rho_{in} = \rho = \phi(I_2), \quad \rho_{out} = \Lambda^*_\phi(\rho) = \sigma = \phi^*(I_1)
\]

O-compound state:
\[
\theta_E = \theta^*_E = \lambda_i E_i \otimes \Lambda^*_\phi(E_i), \quad \rho = \sum \lambda_k E_k
\]

\[
I^T_\phi(\rho,\sigma) \equiv I_O(\rho;\Lambda^*) = \sup \left\{ S(\theta^*_E, \rho \otimes \sigma) : \rho = \sum \lambda_k E_k \right\}
\]

\( I^T_\phi(\rho,\sigma) \) means the transmitted the information through \( \Lambda^*_\phi \) which can be transmitted by using the quantum correlation between two marginal stats.
In this sense we call it the transmitted correlation \( \text{(T-correlation)} \).

**Question:**
\( I^T_\phi(\rho,\sigma) \) satisfies the Shannon's fundamental inequality. So that \( I^T_\phi(\rho,\sigma) \) and \( I_\phi(\rho,\sigma) \) do not always coincide. How can we understand the difference between both mutual entropies?
Decomposition of mutual entropy $I_\phi(\rho,\sigma)$

We introduce the criterion by using relative entropy as

$$I^Q_\phi(\rho,\sigma) \equiv \inf_{\{E_k\}} \left\{ S(\theta_\phi, \theta^\phi_E); \rho = \sum \lambda_k E_k \right\}.$$  

**<Theorem>**

$$I_\phi(\rho,\sigma) = I^Q_\phi(\rho,\sigma) + I^T_\phi(\rho,\sigma)$$  

In the classical system (or classical information theory),

$I(X,Y) = I(P(X);\Lambda^*) \iff I^Q_\phi(\rho,\sigma) = 0$

$$\Downarrow$$

$$I^Q_\phi(\rho,\sigma) > 0 \iff \text{non-classical property}$$

$I^Q_\phi(\rho,\sigma)$ represents the correlation which corresponding to the distance from the separable state (O-compound state) $\theta^\phi_E$ in the sense of relative entropy.  
In this sense we may call it the quantum correlation (Q-correlation).

**Remark:** $I^T_\phi(\rho,\sigma)$ also represents some quantum correlation.

$$\Downarrow$$

O-compound state $\theta^\phi_E$ can separate the total correlation (or whole correlation) $I_\phi(\rho,\sigma)$ into two parts, Q-correlation $I^Q_\phi(\rho,\sigma)$ and T-correlation $I^T_\phi(\rho,\sigma)$.  

$$\Downarrow$$
Necessary and sufficient condition of $I^Q_\phi(\rho, \sigma) = 0$

**<Lemma>**

$$S(\theta_\phi, \theta^\phi_E) = 0 \text{ iff } \theta_\phi = \theta^\phi_E$$

(proof)
For each $\rho, \sigma \in \mathcal{S}$,

$$\|\rho - \sigma\| \leq \sqrt{2S(\rho, \sigma)}$$

(Hiai, Ohya and Tukada: 1983)

This inequality means

$$S(\rho, \sigma) = 0 \text{ iff } \rho = \sigma$$

**<Corollary>** Necessary condition of entanglement:
If $\phi = \phi_{\text{ent}}$, then $I^Q_{\phi_{\text{ent}}}(\rho, \sigma) > 0$

**<Theorem>**
For a separable state $\theta^\text{sep}_{\phi} = \sum p_k \rho_k \otimes \sigma_k$ we have

$$I^Q_{\text{sep}}(\rho, \sigma) = 0 \text{ iff } \rho_k \rho_l = \rho_l \rho_k \text{ for any } k, l.$$  

$\Leftrightarrow$  $I_\phi(\rho, \sigma) = I^T_\phi(\rho, \sigma)$
Summary

1) \[ p(i, j) = p(i) p(j | i) \]
   \[ \iff (q_j) = \Lambda^*(P) = \left( \sum \Lambda_i^* p_i \right) = \left( \sum p(j | i) p_i \right) \]

\[ \theta_\phi = \left( \rho^2 \otimes I_2 \right) \pi_\phi \left( \rho^2 \otimes I_2 \right) \]
   \[ \iff \Lambda^*_\phi(\rho) \equiv \text{Tr}_1 \left( \rho^2 \otimes I_2 \right) \pi_\phi \left( \rho^2 \otimes I_2 \right) \]

2) \[ I(X, Y) = I(P(X); \Lambda^*) \] satisfies the Shannon’s inequality.

\[ I_\phi(\rho, \sigma) = I_\phi(\rho; \Lambda^*_\phi) \] does not satisfy the Shannon’s inequality.

3) \[ I^T_\phi(\rho, \sigma) \equiv I_o(\rho; \Lambda^*_\phi) \] satisfies the Shannon’s inequality.

4) \[ I_\phi(\rho, \sigma) = I^Q_\phi(\rho, \sigma) + I^T_\phi(\rho, \sigma) \]

5) For a separable state \[ \theta_{\phi_{sep}} = \sum p_k \rho_k \otimes \sigma_k \],
   \[ I_{\phi_{sep}}(\rho, \sigma) = I^T_{\phi_{sep}}(\rho, \sigma) \iff \rho_k \rho_l = \rho_l \rho_k \]

We have to check how useful this decomposition for the classification of quantum composite systems.
\[ \theta = p |\Psi_1\rangle\langle \Psi_1| + (1-p)|\Psi_2\rangle\langle \Psi_2| \quad (0 \leq p \leq 1) \]

where
\[ \Psi_1 = \alpha e_0 \otimes e_0 + \beta e_1 \otimes e_1, \quad \Psi_1 = \alpha e_1 \otimes e_0 + \beta e_0 \otimes e_1, \]
\[ \alpha > 0, \beta > 0, \alpha^2 + \beta^2 = 1 \]

It’s marginal:
\[ \rho = \text{Tr}_2 \theta = (p\alpha^2 + (1-p)\beta^2)|e_0\rangle\langle e_0| + ((1-p)\alpha^2 + p\beta^2)|e_1\rangle\langle e_1|, \]
\[ \sigma = \text{Tr}_1 \theta = \alpha^2 |e_0\rangle\langle e_0| + \beta^2 |e_1\rangle\langle e_1| \]

\[ \boxed{\text{By using PPT criterion}} \]
\[ \theta \text{ is separable} \iff p = \frac{1}{2} \quad \theta \text{ is entangled} \iff p \neq \frac{1}{2} \]

\[ p \neq \frac{1}{2}; \quad \implies \quad I_\phi(\rho, \sigma) = S(\sigma) + S(\rho) - S(\theta), \]
\[ I_\phi^Q(\rho, \sigma) = S(\sigma), \quad I_\phi^T(\rho, \sigma) = S(\rho) - S(\theta) \]

\[ p = \frac{1}{2}; \quad \implies \quad I_\phi(\rho, \sigma) = I_\phi^T(\rho, \sigma) = S(\sigma), \quad I_\phi^Q(\rho, \sigma) = 0 \]
\[ \rho = \frac{1}{2}(|e_0\rangle\langle e_0| + |e_1\rangle\langle e_1|) \implies I_\phi^Q(\rho, \sigma) = S(\sigma) \]
\[ \rho = \frac{1}{2}(|\varphi_0\rangle\langle \varphi_0| + |\varphi_1\rangle\langle \varphi_1|) \implies I_\phi^Q(\rho, \sigma) = 0 \]

where \( \varphi_0 = 1/\sqrt{2}(e_0 + e_1), \varphi_1 = 1/\sqrt{2}(e_0 - e_1). \)

\[ \theta_\phi = \theta_\phi^E = 1/2(|\varphi_0\rangle\langle \varphi_0| \otimes |\psi_0\rangle\langle \psi_0| + |\varphi_1\rangle\langle \varphi_1| \otimes |\psi_1\rangle\langle \psi_1|) \]

where \( \psi_0 = \alpha e_0 + \beta e_1, \psi_1 = \alpha e_0 - \beta e_1. \)
This talk is dedicated to the memory of Slava

Thank you for your attention!!