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Numerical solution of the Dirichlet problem for nonlinear parabolic equations by a probabilistic approach

G. N. MILSTEIN†

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany

AND

M. V. TRETYAKOV‡

Institute of Mathematics and Mechanics, Russian Academy of Sciences, S. Kovalevskaya Str. 16, 620219 Ekaterinburg, Russia

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A number of new layer methods for solving the Dirichlet problem for semilinear parabolic equations are constructed by using probabilistic representations of their solutions. The methods exploit the ideas of weak sense numerical integration of stochastic differential equations in a bounded domain. Despite their probabilistic nature these methods are nevertheless deterministic. Some convergence theorems are proved. Numerical tests are presented.

Keywords: weak approximation of stochastic differential equations; probabilistic representations for parabolic equations.

1. Introduction

Numerical analysis of nonlinear partial differential equations (nonlinear PDEs) is generally based on deterministic approaches (see, for example, Strikwerda, 1989; Vreugdenhil & Koren, 1993; Quarteroni & Valli, 1994 and Samarskii, 1977). A probabilistic approach to constructing new layer methods for solving nonlinear PDEs of parabolic type is proposed in Milstein (1997) (see also Milstein & Tretyakov, 2000). It is based on making use of the well known probabilistic representations of solutions to linear parabolic equations (see, for example, Dynkin, 1965 and Freidlin, 1985) and the ideas of weak sense numerical integration of stochastic differential equations (SDEs) (Milstein, 1995a; Kloeden & Platen, 1992; Pardoux & Talay, 1985). Despite their probabilistic nature these methods are nevertheless deterministic. The probabilistic approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic manner. In particular, the layer methods allow us to avoid difficulties stemming from essentially changing coefficients and strong advection. Other probabilistic

†Also at Department of Mathematics, Ural State University, Lenin Street 51, 620083 Ekaterinburg, Russia. Email: milstein@wias-berlin.de

‡Also at Department of Mathematics, University of Wales Swansea, Swansea SA2 8PP UK. Email: Michael.Tretyakov@usu.ru

applications to numerically solving nonlinear PDEs are available, for example, in Kushner (1977) and Talay & Tubaro (1996).

The papers of Milstein (1997) and Milstein & Tretyakov (2000) are devoted to layer methods for the nonlinear Cauchy problem. The aim of this paper is to develop such methods for nonlinear problems with Dirichlet boundary conditions. Some probability methods for solving boundary value problems for linear parabolic equations are proposed in Milstein (1995b,c, 1996) and Milstein & Tretyakov (2001).

Let G be a bounded domain in $\mathbb{R}^d$, $Q = [t_0, T) \times G$ be a cylinder in $\mathbb{R}^{d+1}$, $\Gamma = \overline{Q} \setminus Q$. The set Γ is a part of the boundary of the cylinder Q consisting of the upper base and the lateral surface.

Consider the Dirichlet problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x,u)\frac{\partial u}{\partial x^i} + g(t,x,u) = 0, \ \ (t,x) \in Q, \tag{1.1}$$

$$u(t,x)|_{\Gamma} = \varphi(t,x). \tag{1.2}$$

The form of (1.1) is relevant to a probabilistic approach, i.e. the equation is considered under $t < T$, and the 'initial' conditions are prescribed at $t = T$. Using the well known probabilistic representation of the solution to (1.1), (1.2) (see Dynkin (1965) and Freidlin (1985)), we get

$$u(t,x) = \mathbf{E}(\varphi(\tau, X_{t,x}(\tau)) + Z_{t,x,0}(\tau)). \tag{1.3}$$

In (1.3) $X_{t,x}(s)$, $Z_{t,x,z}(s)$, $(t,x) \in Q$, $s \geqslant t$, is the solution of the Cauchy problem for the Ito system of stochastic differential equations

$$\begin{gathered} \mathrm{d}X = b(s, X, u(s,X))\,\mathrm{d}s + \sigma(s, X, u(s,X))\,\mathrm{d}w(s), \ \ X(t) = x, \\ \mathrm{d}Z = g(s, X, u(s,X))\,\mathrm{d}s, \ \ Z(t) = z, \end{gathered} \tag{1.4}$$

where $w(s) = (w^1(s), \ldots, w^d(s))^{\top}$ is a standard Wiener process, $b(s,x,u) = (b^1(s,x,u), \ldots, b^d(s,x,u))^{\top}$ is a column vector, the matrix $\sigma = \sigma(s,x,u)$ is obtained from the equation

$$\sigma\sigma^{\top} = a, \ \ \sigma = \{\sigma^{ij}(s,x,u)\}, \ \ a = \{a^{ij}(s,x,u)\},$$

and $\tau = \tau_{t,x}$ is the first exit time of the trajectory $(s, X_{t,x}(s))$ from the domain Q.

If (1.1) is linear, the system (1.4) does not contain the unknown function $u(s,x)$ and therefore one can use weak approximation schemes for solving (1.4) with the Monte Carlo realization of representation (1.3). The representation involves the point $(\tau, X_{t,x}(\tau))$. To get a sufficiently effective approximation of this point is rather a hard problem. Some constructive schemes solving this problem in the case of linear parabolic equations are presented in Milstein (1995b,c). The procedures of Milstein (1995b,c) together with the Monte Carlo approach allow us to find a value $u(t,x)$ at a single point even when the domain G has high dimension.

Of course, the nonlinear case is much more complicated. But we aim to construct layer methods and due to this fact it becomes possible to use a one-step (local) version

of the representation (1.3) (see (2.3) below). We will introduce a time discretization, for definiteness the equidistant one:

$$T = t_N > t_{N-1} > \cdots > t_0, \ h := \frac{T - t_0}{N}.$$

The methods proposed here give an approximation $\bar{u}(t_k, x)$ of the solution $u(t_k, x)$, $k = N, \ldots, 0$, $x \in \overline{G}$, i.e. step by step everywhere in the domain $\overline{G}$. This is feasible if the dimension of the domain G is comparatively small ($d \leqslant 3$). To construct the layer methods, we exploit the ideas of weak sense numerical integration of SDE in a bounded domain and obtain some approximate relations on the basis of (2.3), (1.4). The relations allow us to express $\bar{u}(t_k, x)$, $k = N-1, \ldots, 0$, recurrently in terms of $\bar{u}(t_{k+1}, x)$. Despite their probabilistic nature these methods turn out to be deterministic as in Milstein (1997) and Milstein & Tretyakov (2000).

In Section 2, we derive a layer method for nonlinear parabolic equations relying on the numerical integration of ordinary SDEs. In Section 3, we prove a convergence theorem for this method using deterministic-type arguments. We propose a second layer method in Section 4. A convergence theorem for it is proved by probabilistic-type arguments. To realize layer methods in practice, we need a discretization in the variable x with some kind of interpolation at every step to turn an applied method into an algorithm. Such numerical algorithms are constructed in Section 5. A majority of ideas can be demonstrated for $d = 1$, and we restrict ourselves to this case in Sections 2–5. The case $d \geqslant 2$ is briefly discussed in Section 6. Numerical tests are presented in the last section.

2. Construction of a layer method with one-step error $O(h^2)$

The boundary value problem (1.1), (1.2) in the one-dimensional case has the following form:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t, x, u)\frac{\partial^2 u}{\partial x^2} + b(t, x, u)\frac{\partial u}{\partial x} + g(t, x, u) = 0, \quad (t, x) \in Q, \tag{2.1}$$

$$u(t, x)|_{\Gamma} = \varphi(t, x). \tag{2.2}$$

In this case Q is the partly open rectangle: $Q = [t_0, T) \times (\alpha, \beta)$, and Γ consists of the upper base $\{T\} \times [\alpha, \beta]$ and two vertical intervals: $[t_0, T) \times \{\alpha\}$ and $[t_0, T) \times \{\beta\}$. We assume that $\sigma(t, x, u) \geqslant \sigma_* > 0$ for $(t, x) \in \overline{Q}$, $-\infty < u < \infty$.

Let $u = u(t, x)$ be the solution to problem (2.1), (2.2), which is supposed to exist, to be unique, and to be sufficiently smooth. One can find many theoretical results on this topic in Ladyzhenskaya (1988); Smoller (1983); Samarskii *et al.* (1995); Grindrod (1996) and Taylor (1996) (see also references therein).

Analogously to (1.3), we have

$$u(t_k, x) = \mathbf{E}(u(\vartheta_{t_k,x}, X_{t_k,x}(\vartheta_{t_k,x})) + Z_{t_k,x,0}(\vartheta_{t_k,x})), \tag{2.3}$$

where $\vartheta_{t_k,x} = \vartheta_{t_k,x}(t_{k+1}) := \tau_{t_k,x} \wedge t_{k+1}$, and X, Z satisfy system (1.4).

Let us suppose for a while that it is possible to extend the coefficients of (2.1) so that the new equation has a solution $u(t, x)$ on $[t_0, T) \times \mathbf{R}$ which is an extension of the solution

to the boundary value problem (2.1), (2.2). Then, instead of (2.3), we obtain (we suppose the layer $u(t_{k+1}, x)$ to be known)

$$u(t_k, x) = \mathbf{E}(u(t_{k+1}, X_{t_k,x}(t_{k+1})) + Z_{t_k,x,0}(t_{k+1})). \tag{2.4}$$

Applying the explicit weak Euler scheme with the simplest simulation of noise to system (1.4), we get

$$\bar{X}_{t_k,x}(t_{k+1}) = x + b(t_k, x, u(t_k, x))h + \sigma(t_k, x, u(t_k, x))\sqrt{h}\xi, \tag{2.5}$$

$$\bar{Z}_{t_k,x,0}(t_{k+1}) = g(t_k, x, u(t_k, x))h, \tag{2.6}$$

where the ξ is distributed by the law: $P(\xi = \pm 1) = \frac{1}{2}$.

Using (2.4), we get to within $O(h^2)$:

$$\begin{aligned} u(t_k, x) &\simeq \mathbf{E}(u(t_{k+1}, \bar{X}_{t_k,x}(t_{k+1})) + \bar{Z}_{t_k,x,0}(t_{k+1})) \\ &= \tfrac{1}{2}u\big(t_{k+1}, x + b(t_k, x, u(t_k, x))h - \sigma(t_k, x, u(t_k, x))\sqrt{h}\big) \\ &+\tfrac{1}{2}u\big(t_{k+1}, x + b(t_k, x, u(t_k, x))h + \sigma(t_k, x, u(t_k, x))\sqrt{h}\big) + g(t_k, x, u(t_k, x))h\cdot \end{aligned} \tag{2.7}$$

Now we can obtain an implicit relation for an approximation of $u(t_k, x)$. Applying the method of simple iteration to the implicit relation and taking $u(t_{k+1}, x)$ as a null iteration, we get the following explicit one-step approximation $v(t_k, x)$ of $u(t_k, x)$:

$$v(t_k, x) = \tfrac{1}{2}u\big(t_{k+1}, x + b_k \cdot h - \sigma_k \cdot \sqrt{h}\big) + \tfrac{1}{2}u\big(t_{k+1}, x + b_k \cdot h + \sigma_k \cdot \sqrt{h}\big) + g_k \cdot h, \tag{2.8}$$

where b_k, σ_k, g_k are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ calculated at the point $(t_k, x, u(t_{k+1}, x))$.

But in reality we know the layer $u(t_{k+1}, x)$ for $\alpha \leqslant x \leqslant \beta$ only. At the same time the argument $x+b_k h-\sigma_k\sqrt{h}$ for x close to α is less than α and the argument $x+b_k h+\sigma_k\sqrt{h}$ for x close to β is more than β. Thus, we need to extend the layer $u(t_{k+1}, x)$ in a constructive manner.

Using the explicit weak Euler scheme for the initial point (t, α) with $t_k \leqslant t \leqslant t_{k+1}$, we put (cf. (2.5), (2.6))

$$\begin{aligned} \bar{X}_{t,\alpha}(t_{k+1}) &= x + b(t, \alpha, u(t, \alpha)) \cdot (t_{k+1} - t) + \sigma(t, \alpha, u(t, \alpha)) \cdot \sqrt{t_{k+1} - t} \cdot \xi, \\ \bar{Z}_{t,\alpha,0}(t_{k+1}) &= g(t, \alpha, u(t, \alpha)) \cdot (t_{k+1} - t). \end{aligned} \tag{2.9}$$

Analogously, we define $\bar{X}_{t,\beta}(t_{k+1})$, $\bar{Z}_{t,\beta,0}(t_{k+1})$.

We have (see (2.7) and (2.9)) for $t_k \leqslant t \leqslant t_{k+1}$

$$\begin{aligned} u(t, \alpha) &\simeq \mathbf{E}(u(t_{k+1}, \bar{X}_{t,\alpha}(t_{k+1})) + \bar{Z}_{t,\alpha,0}(t_{k+1})) \\ &= \tfrac{1}{2}u\big(t_{k+1}, \alpha + b(t, \alpha, u(t, \alpha)) \cdot (t_{k+1} - t) - \sigma(t, \alpha, u(t, \alpha)) \cdot \sqrt{t_{k+1} - t}\big) \\ &+\tfrac{1}{2}u\big(t_{k+1}, \alpha + b(t, \alpha, u(t, \alpha)) \cdot (t_{k+1} - t) + \sigma(t, \alpha, u(t, \alpha)) \cdot \sqrt{t_{k+1} - t}\big) \\ &+g(t, \alpha, u(t, \alpha)) \cdot (t_{k+1} - t). \end{aligned} \tag{2.10}$$

If we replace (remember, $u(t,\alpha)=\varphi(t,\alpha)$ due to (2.2)) the argument $(t,\alpha,u(t,\alpha))=(t,\alpha,\varphi(t,\alpha))$ by $(t_k,\alpha,\varphi(t_{k+1},\alpha))$, the right-hand side of (2.10) is changed by a quantity of order $O(h^2)$. Since the approximation in (2.10) is also $O(h^2)$, we get

$$\begin{aligned}\tfrac{1}{2}u\big(t_{k+1},\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{t_{k+1}-t}\big)\\ =\varphi(t,\alpha)-\tfrac{1}{2}u\big(t_{k+1},\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)\\ +\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{t_{k+1}-t}\big)\\ -g(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)+O(h^2).\end{aligned}\tag{2.11}$$

Introduce

$$\alpha_0:=\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot h-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{h}.$$

Clearly $\alpha_0<\alpha$ and $\alpha_0\leqslant\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{t_{k+1}-t}\leqslant\alpha$ for $t_k\leqslant t\leqslant t_{k+1}$ under a sufficiently small h.

Analogously

$$\begin{aligned}\tfrac{1}{2}u\big(t_{k+1},\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot(t_{k+1}-t)+\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{t_{k+1}-t}\big)\\ =\varphi(t,\beta)-\tfrac{1}{2}u\big(t_{k+1},\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot(t_{k+1}-t)\\ -\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{t_{k+1}-t}\big)\\ -g(t_k,\beta,\varphi(t_{k+1},\beta))\cdot(t_{k+1}-t)+O(h^2),\\ \beta_0:=\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot h+\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{h}.\end{aligned}\tag{2.12}$$

The relations (2.11), (2.12) give the desired extension of the function $u(t_{k+1},x)$ on the interval $[\alpha_0,\beta_0]$.

Let us return to (2.8) now. The arguments $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}$ and $x+b_k\cdot h+\sigma_k\cdot\sqrt{h}$ are monotone increasing functions in $x\in[\alpha,\beta]$ for a sufficiently small h. Their values belong to $[\alpha_0,\beta_0]$, and $x+b_k\cdot h+\sigma_k\cdot\sqrt{h}$ is always (for $x\in[\alpha,\beta]$) more than α while $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}$ is less than β. Let $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}<\alpha$ (clearly it is possible for x close to α). Due to the above, there exists a unique root $\gamma_k(x),\ 0<\gamma_k(x)\leqslant 1$, of the quadratic equation

$$\begin{aligned}\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{\gamma_k h}\\ =x+b_k\cdot h-\sigma_k\cdot\sqrt{h}.\end{aligned}\tag{2.13}$$

Analogously, if $x+b_k\cdot h+\sigma_k\cdot\sqrt{h}>\beta$, then there exists a unique root $\delta_k(x),\ 0<\delta_k(x)\leqslant 1$, of the quadratic equation

$$\begin{aligned}\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\delta_k h+\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{\delta_k h}\\ =x+b_k\cdot h+\sigma_k\cdot\sqrt{h}.\end{aligned}\tag{2.14}$$

If, for instance, $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}<\alpha$, then one can replace the value $u(t_{k+1},x+$

$b_k \cdot h - \sigma_k \cdot \sqrt{h})/2$ in (2.8) by the value due to (2.13) and (2.11):

$$\begin{aligned}
&\tfrac{1}{2}u\big(t_{k+1}, x + b_k \cdot h - \sigma_k \cdot \sqrt{h}\big) \\
&= \tfrac{1}{2}u\big(t_{k+1}, \alpha + b(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \gamma_k h - \sigma(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \sqrt{\gamma_k h}\big) \\
&\approx \varphi(t_{k+1-\gamma_k}, \alpha) - \tfrac{1}{2}u\big(t_{k+1}, \alpha + b(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \gamma_k h + \sigma(t_k, \alpha, \varphi(t_{k+1}, \alpha))\sqrt{\gamma_k h}\big) \\
&\quad - g(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \gamma_k h,
\end{aligned}$$

where $t_{k+1-\gamma_k} = t_k + (1 - \gamma_k) \cdot h$.

As a result, we obtain the following one-step approximation $v(t_k, x)$ for $u(t_k, x)$:

$$\begin{aligned}
v(t_k, x) &= \tfrac{1}{2}u\big(t_{k+1}, x + b_k \cdot h - \sigma_k \cdot \sqrt{h}\big) + \tfrac{1}{2}u\big(t_{k+1}, x + b_k \cdot h + \sigma_k \cdot \sqrt{h}\big) \\
&\quad + g_k \cdot h, \qquad \text{if } x + b_k \cdot h \pm \sigma_k \cdot \sqrt{h} \in [\alpha, \beta]; \\
v(t_k, x) &= \varphi(t_{k+1-\gamma_k}, \alpha) - g(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \gamma_k h \\
&\quad - \tfrac{1}{2}u\big(t_{k+1}, \alpha + b(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \gamma_k h + \sigma(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \sqrt{\gamma_k h}\big) \\
&\quad + \tfrac{1}{2}u(t_{k+1}, x + b_k \cdot h + \sigma_k \cdot \sqrt{h}) + g_k \cdot h, \qquad \text{if } x + b_k \cdot h - \sigma_k \cdot \sqrt{h} < \alpha; \\
v(t_k, x) &= \tfrac{1}{2}u\big(t_{k+1}, x + b_k \cdot h - \sigma_k \cdot \sqrt{h}\big) + \varphi(t_{k+1-\delta_k}, \beta) - g(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \delta_k h \\
&\quad - \tfrac{1}{2}u\big(t_{k+1}, \beta + b(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \delta_k h - \sigma(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \sqrt{\delta_k h}\big) \\
&\quad + g_k \cdot h, \qquad \text{if } x + b_k \cdot h + \sigma_k \cdot \sqrt{h} > \beta, \ k = N - 1, \ldots, 1, 0,
\end{aligned} \tag{2.15}$$

where (let us recall) b_k, σ_k, g_k are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ calculated at the point $(t_k, x, u(t_{k+1}, x))$ and γ_k, δ_k are the corresponding roots of (2.13) and (2.14).

Thus the layer method acquires the form

$$\bar{u}(t_N, x) = \varphi(t_N, x), \ x \in [\alpha, \beta], \tag{2.16}$$

$$\begin{aligned}
\bar{u}(t_k, x) &= \tfrac{1}{2}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) + \tfrac{1}{2}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) \\
&\quad + \bar{g}_k \cdot h, \qquad \text{if } x + \bar{b}_k \cdot h \pm \bar{\sigma}_k \cdot \sqrt{h} \in [\alpha, \beta]; \\
\bar{u}(t_k, x) &= \varphi(t_{k+1-\bar{\gamma}_k}, \alpha) - g(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \bar{\gamma}_k h \\
&\quad - \tfrac{1}{2}\bar{u}\big(t_{k+1}, \alpha + b(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \bar{\gamma}_k h + \sigma(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \sqrt{\bar{\gamma}_k h}\big) \\
&\quad + \tfrac{1}{2}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) + \bar{g}_k \cdot h, \qquad \text{if } x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h} < \alpha; \\
\bar{u}(t_k, x) &= \tfrac{1}{2}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) + \varphi(t_{k+1-\bar{\delta}_k}, \beta) - g(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \bar{\delta}_k h \\
&\quad - \tfrac{1}{2}\bar{u}\Big(t_{k+1}, \beta + b(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \bar{\delta}_k h - \sigma(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \sqrt{\bar{\delta}_k h}\Big) \\
&\quad + \bar{g}_k \cdot h, \qquad \text{if } x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h} > \beta; \\
&\quad k = N - 1, \ldots, 1, 0,
\end{aligned}$$

where $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$ are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$ and $\bar{\gamma}_k$, $\bar{\delta}_k$ are the corresponding roots of (2.13) and (2.14) with the right sides $x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}$ and $x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}$.

The method (2.16) is an explicit layer method for solving the Dirichlet problem (2.1), (2.2). This method is deterministic, even though it is constructed by a probabilistic

approach. The method is of the first order of convergence with respect to h (see Theorem 3.1).

REMARK 2.1 Let us briefly discuss some differences between the layer methods obtained here and the well known finite-difference methods (see a more detailed discussion in Milstein (1997)). Finite-difference methods also allow us to express an approximate solution on the layer $t = t_k$ recurrently in terms of the solution on the layer $t = t_{k+1}$. For their construction, both the time step Δt and the space step Δx are used. Moreover, the knots of the layer $t = t_{k+1}$ used to evaluate $\bar{u}(t_k, x_j)$ are definitely prescribed. In our methods we use the time step h only, and the points from the layer $t = t_{k+1}$ to evaluate $\bar{u}(t_k, x)$ arise automatically. A location of these points depends on the coefficients of the problem considered and on the weak scheme chosen. As a result, the probabilistic approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic manner. In particular, the layer methods allow us to avoid difficulties stemming from essentially changing coefficients and strong advection. We should also note that the probabilistic approach gives a natural way to derive many new methods.

3. Convergence theorem

We shall make the following assumptions.

(i) There exists a unique solution $u(t, x)$ to problem (2.1), (2.2) such that

$$u_\circ < u_* \leqslant u(t, x) \leqslant u^* < u^\circ, \ t_0 \leqslant t \leqslant T, \ x \in [\alpha, \beta], \tag{3.1}$$

where $u_\circ$, u_*, u^*, u° are some constants, and there exist uniformly bounded derivatives:

$$\left| \frac{\partial^{i+j} u}{\partial t^i \partial x^j} \right| \leqslant K, \ i = 0, \ j = 1, 2, 3, 4; \ i = 1, \ j = 0, 1, 2; \ i = 2, \ j = 0;$$
$$t_0 \leqslant t \leqslant T, \ x \in [\alpha, \beta]. \tag{3.2}$$

(ii) The coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ and their first and second derivatives in x and u are uniformly bounded:

$$\left| \frac{\partial^{i+j} b}{\partial x^i \partial u^j} \right| \leqslant K, \left| \frac{\partial^{i+j} \sigma}{\partial x^i \partial u^j} \right| \leqslant K, \left| \frac{\partial^{i+j} g}{\partial x^i \partial u^j} \right| \leqslant K, \ 0 \leqslant i + j \leqslant 2,$$
$$t_0 \leqslant t \leqslant T, \ x \in [\alpha, \beta], \ u_\circ < u < u^\circ. \tag{3.3}$$

Below we use the letters K and C without any index for various constants which do not depend on h, k, x.

First of all let us evaluate the one-step error $\rho(t_k, x)$ of method (2.16).

LEMMA 3.1 Under assumptions (i) and (ii), the one-step error $\rho(t_k, x)$ of method (2.16) has the second order of smallness with respect to h, i.e.

$$|\rho(t_k, x)| = |v(t_k, x) - u(t_k, x)| \leqslant Ch^2,$$

where $v(t_k, x)$ is defined by (2.16), C does not depend on h, k, x.

Proof. If both points $x + b_k \cdot h \pm \sigma_k \cdot \sqrt{h}$ belong to $[\alpha, \beta]$, the statement of this lemma follows directly from Lemma 4.1 of Milstein (1997).

Let us consider the case when the point $x + b_k \cdot h - \sigma_k \cdot \sqrt{h} < \alpha$. Introduce the notation b_α, σ_α, g_α for the coefficients b, σ, g calculated at the point $(t_k, \alpha, \varphi(t_{k+1}, \alpha))$. We get from (2.13) that

$$\alpha - x = b_k h - \sigma_k \sqrt{h} - b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h} = \big(\sigma_\alpha \sqrt{\gamma_k} - \sigma_k\big)\sqrt{h} + O(h) = O(\sqrt{h}). \tag{3.4}$$

Expand the terms of (2.16) at the point (t_k, x):

$$\begin{aligned}
\varphi(t_{k+1-\gamma_k}, \alpha) &= u(t_k + (1-\gamma_k)h, x + (\alpha - x)) = u + \frac{\partial u}{\partial t} \cdot (1-\gamma_k)h \\
&\quad + \frac{\partial u}{\partial x} \cdot (\alpha - x) + \frac{\partial^2 u}{\partial t \partial x} \cdot (1-\gamma_k)(\alpha - x)h \\
&\quad + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \cdot (\alpha - x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3} \cdot (\alpha - x)^3 + O(h^2),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&u\big(t_{k+1}, \alpha + b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h}\big) = u\big(t_k + h, x + \big(\alpha - x + b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h}\big)\big) \\
&= u + \frac{\partial u}{\partial t} h + \frac{\partial u}{\partial x} \cdot \big(\alpha - x + b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h}\big) + \frac{\partial^2 u}{\partial t \partial x} \cdot \big(\alpha - x + \sigma_\alpha \sqrt{\gamma_k h}\big)h \\
&\quad + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \cdot \Big(\big(\alpha - x + \sigma_\alpha \sqrt{\gamma_k h}\big)^2 + 2(\alpha - x + \sigma_\alpha \sqrt{\gamma_k h}) b_\alpha \gamma_k h\Big) \\
&\quad + \frac{1}{6}\frac{\partial^3 u}{\partial x^3} \cdot \big(\alpha - x + \sigma_\alpha \sqrt{\gamma_k h}\big)^3 + O(h^2),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
u\big(t_{k+1}, x + b_k h + \sigma_k \sqrt{h}\big) &= u + \frac{\partial u}{\partial t} h + \frac{\partial u}{\partial x} \cdot \big(b_k h + \sigma_k \sqrt{h}\big) + \frac{\partial^2 u}{\partial t \partial x} \cdot \sigma_k h^{3/2} \\
&\quad + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \cdot (\sigma_k^2 h + 2 b_k \sigma_k h^{3/2}) + \frac{1}{6}\frac{\partial^3 u}{\partial x^3} \cdot \sigma_k^3 h^{3/2} + O(h^2).
\end{aligned} \tag{3.7}$$

Here the function u and its derivatives are calculated at the point (t_k, x).

Substituting (3.5)–(3.7) in the corresponding expression for $v(t_k, x)$ of (2.16) and using (3.4), we obtain

$$\begin{aligned}
v(t_k, x) &= u + h(1-\gamma_k) \cdot \left(\frac{\partial u}{\partial t} + \frac{\sigma_k^2}{2}\frac{\partial^2 u}{\partial x^2} + b_k \frac{\partial u}{\partial x} + g_k\right) + \frac{\partial u}{\partial x} \cdot (b_k - b_\alpha)\gamma_k h \\
&\quad + \frac{\partial^2 u}{\partial t \partial x} \cdot \big(\sigma_k - \sigma_\alpha \sqrt{\gamma_k}\big)\gamma_k h^{3/2} + \frac{\partial^2 u}{\partial x^2} \cdot \big(\tfrac{1}{2}(\sigma_k^2 - \sigma_\alpha^2)\gamma_k h + b_\alpha(\sigma_k - \sigma_\alpha \sqrt{\gamma_k})\gamma_k h^{3/2}\big) \\
&\quad + \tfrac{1}{2}\frac{\partial^3 u}{\partial x^3} \cdot \sigma_\alpha^2\big(\sigma_k - \sigma_\alpha \sqrt{\gamma_k}\big)\gamma_k h^{3/2} + (g_k - g_\alpha)\gamma_k h + O(h^2).
\end{aligned} \tag{3.8}$$

Due to assumptions (i) and (ii) and relation (3.4), we get

$$\begin{aligned}
\sigma_\alpha = \sigma(t_k, \alpha, u(t_{k+1}, \alpha)) &= \sigma(t_k, x, u(t_k, x)) + \frac{\partial \sigma}{\partial x} \cdot (\alpha - x) \\
&\quad + \frac{\partial \sigma}{\partial u} \cdot (u(t_{k+1}, \alpha) - u(t_k, x)) + O(h) \\
&= \sigma + \frac{\partial \sigma}{\partial x} \cdot \big(\sigma_\alpha \sqrt{\gamma_k} - \sigma_k\big)\sqrt{h} + \frac{\partial \sigma}{\partial u}\frac{\partial u}{\partial x} \cdot \big(\sigma_\alpha \sqrt{\gamma_k} - \sigma_k\big)\sqrt{h} + O(h) \\
&= \sigma + \left(\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial u}\frac{\partial u}{\partial x}\right) \cdot \sigma_k\big(\sqrt{\gamma_k} - 1\big)\sqrt{h} + O(h), \\
b_\alpha = b(t_k, \alpha, u(t_{k+1}, \alpha)) &= b + \left(\frac{\partial b}{\partial x} + \frac{\partial b}{\partial u}\frac{\partial u}{\partial x}\right) \cdot \sigma_k\big(\sqrt{\gamma_k} - 1\big)\sqrt{h} + O(h), \\
g_\alpha = g(t_k, \alpha, u(t_{k+1}, \alpha)) &= g + \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial x}\right) \cdot \sigma_k\big(\sqrt{\gamma_k} - 1\big)\sqrt{h} + O(h), \\
b_k = b(t_k, x, u(t_{k+1}, x)) &= b + O(h),\ \sigma_k = \sigma + O(h),\ g_k = g + O(h),
\end{aligned} \tag{3.9}$$

where b, σ, g (without any indices) and their derivatives are calculated at the point $(t_k, x, u(t_k, x))$.

Using (3.9), we obtain from (3.8):

$$\begin{aligned}
v(t_k, x) &= u(t_k, x) + h(1 - \gamma_k) \cdot \left[\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + g\right] \\
&\quad + h^{3/2}\gamma_k \sigma\big(1 - \sqrt{\gamma_k}\big) \cdot \left[\frac{\partial^2 u}{\partial t \partial x} + \frac{\sigma^2}{2}\frac{\partial^3 u}{\partial x^3} + \sigma \cdot \left(\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial u}\frac{\partial u}{\partial x}\right)\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x^2}\right. \\
&\quad \left. + \left(\frac{\partial b}{\partial x} + \frac{\partial b}{\partial u}\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x} + \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial x}\right] + O(h^2) \\
&= u(t_k, x) + \left(h(1 - \gamma_k) + h^{3/2}\gamma_k \sigma\big(1 - \sqrt{\gamma_k}\big)\frac{\partial}{\partial x}\right)\left[\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + g\right] + O(h^2).
\end{aligned} \tag{3.10}$$

Taking into account that $u(t, x)$ is the solution to problem (2.1), (2.2), the relation (3.10) implies

$$v(t_k, x) = u(t_k, x) + O(h^2).$$

The case $x + b_k \cdot h + \sigma_k \cdot \sqrt{h} > \beta$ can be considered analogously. □

Let us prove the following theorem on global convergence.

THEOREM 3.1 Under assumptions (i) and (ii), the method (2·16) has the first order of convergence with respect to h i.e.

$$|\bar{u}(t_k, x) - u(t_k, x)| \leqslant Kh,$$

where K does not depend on h, k, x.

Proof. Denote the error of method (2.16) on the kth layer ($(N-k)$th step) as

$$R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x). \tag{3.11}$$

If $x + \bar{b}_k \cdot h \pm \bar{\sigma}_k \cdot \sqrt{h} \in [\alpha, \beta]$, we have (see (2.16) and (3.11)):

$$u(t_k, x) + R(t_k, x) = \tfrac{1}{2}u\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) + \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big)$$
$$+\tfrac{1}{2}u\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) + \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) + \bar{g}_k \cdot h. \tag{3.12}$$

Expanding the functions $u(t_{k+1}, x + \bar{b}_k \cdot h \pm \bar{\sigma}_k \cdot \sqrt{h})$ at the point (t_k, x), we get

$$u\big(t_{k+1}, x + \bar{b}_k \cdot h \pm \bar{\sigma}_k \cdot \sqrt{h}\big) = u(t_k, x) + \frac{\partial u}{\partial t}h + \big(\bar{b}_k \cdot h \pm \bar{\sigma}_k \cdot \sqrt{h}\big)\frac{\partial u}{\partial x}$$
$$+\frac{\bar{\sigma}_k^2}{2}\frac{\partial^2 u}{\partial x^2} \cdot h \pm \bar{b}_k\bar{\sigma}_k\frac{\partial^2 u}{\partial x^2} \cdot h^{3/2} \pm \bar{\sigma}_k\frac{\partial^2 u}{\partial t\,\partial x} \cdot h^{3/2} \pm \frac{\bar{\sigma}_k^3}{6}\frac{\partial^3 u}{\partial x^3} \cdot h^{3/2} + O(h^2), \tag{3.13}$$

where the derivatives are calculated at the point (t_k, x).

Here we have to assume for a while that the value $u(t_{k+1}, x) + R(t_{k+1}, x)$ remains in the interval $(u_\circ, u^\circ)$ for a sufficiently small h (see condition (ii)). Clearly, $R(t_N, x) = 0$, and below we prove recurrently that $R(t_k, x)$ is sufficiently small for a sufficiently small h. Thereupon, thanks to (3.1) this assumption will be justified for such h.

Due to assumptions (i) and (ii) and notation (3.11), we obtain

$$\bar{b}_k = b(t_k, x, \bar{u}(t_{k+1}, x)) = b(t_k, x, u(t_{k+1}, x) + R(t_{k+1}, x))$$
$$= b(t_k, x, u(t_{k+1}, x)) + \Delta b = b(t_k, x, u(t_k, x)) + \Delta b + O(h),$$
$$\bar{\sigma}_k = \sigma(t_k, x, u(t_k, x)) + \Delta\sigma + O(h),\ \bar{\sigma}_k^2 = \sigma^2(t_k, x, u(t_k, x)) + \Delta\sigma^2 + O(h),$$
$$\bar{g}_k = g(t_k, x, u(t_k, x)) + \Delta g + O(h), \tag{3.14}$$

where

$$|\Delta b|,\ |\Delta\sigma|,\ |\Delta\sigma^2|,\ |\Delta g| \leqslant K \cdot |R(t_{k+1}, x)|.$$

Substituting (3.13) in (3.12) and taking into account (3.14), we come to the relation

$$u(t_k, x) + R(t_k, x) = u(t_k, x) + h \cdot \left(\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + g\right) + r(t_k, x) + O(h^2)$$
$$+\tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) + \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big), \tag{3.15}$$

where the derivatives are calculated at (t_k, x), b, σ, g are calculated at $(t_k, x, u(t_k, x))$, and

$$|r(t_k, x)| \leqslant Kh|R(t_{k+1}, x)|.$$

Since $u(t, x)$ is the solution to (2.1), (2.2), the relation (3.15) implies

$$R(t_k, x) = \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) + \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big)$$
$$+r(t_k, x) + O(h^2). \tag{3.16}$$

For x such that $x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h} < \alpha$, we get (see (3.11) and (2.16))

$$\begin{aligned} u(t_k, x) + R(t_k, x) = \bar{u}(t_k, x) = \varphi(t_{k+1-\bar{\gamma}_k}, \alpha) \\ -\tfrac{1}{2}u\big(t_{k+1}, \alpha + b_\alpha \bar{\gamma}_k h + \sigma_\alpha \sqrt{\bar{\gamma}_k h}\big) + \tfrac{1}{2}u\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) \\ -\tfrac{1}{2}R\big(t_{k+1}, \alpha + b_\alpha \bar{\gamma}_k h + \sigma_\alpha \sqrt{\bar{\gamma}_k h}\big) + \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) \\ -g(t_k, \alpha, u(t_{k+1}, \alpha)) \cdot \bar{\gamma}_k h + \bar{g}_k \cdot h, \end{aligned} \tag{3.17}$$

where b_α, σ_α, g_α are the corresponding coefficients calculated at $(t_k, \alpha, \varphi(t_{k+1}, \alpha))$.

In accordance with (2.16) and (2.13), we have (cf. (3.4))

$$\alpha - x = \bar{b}_k h - \bar{\sigma}_k \sqrt{h} - b_\alpha \bar{\gamma}_k h + \sigma_\alpha \sqrt{\bar{\gamma}_k h} = \big(\sigma_\alpha \sqrt{\bar{\gamma}_k} - \bar{\sigma}_k\big)\sqrt{h} + O(h) = O(\sqrt{h}).$$

Recall that $\bar{\gamma}_k$ is a root of (2.13) with the right side $x + \bar{b}_k h - \bar{\sigma}_k \sqrt{h}$.

Now we expand the first three terms in the right side of (3.17) in powers of h at the point (t_k, x) as was done in the proof of Lemma 3.1 (see (3.5)–(3.8)). The obtained new relation contains $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$, $\bar{\gamma}_k$ (instead of b_k, σ_k, g_k, γ_k in (3.8)) and b_α, σ_α, g_α. We present $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$ due to (3.14) and b_α, σ_α, g_α due to (3.9). As a result, we get (cf. (3.10))

$$\begin{aligned} u(t_k, x) + R(t_k, x) = u(t_k, x) \\ +\left(h(1 - \bar{\gamma}_k) + h^{3/2}\bar{\gamma}_k \sigma\big(1 - \sqrt{\bar{\gamma}_k}\big)\frac{\partial}{\partial x}\right)\left[\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + g\right] \\ +r_1(t_k, x) + O(h^2) - \tfrac{1}{2}R\big(t_{k+1}, \alpha + b_\alpha \bar{\gamma}_k h + \sigma_\alpha \sqrt{\bar{\gamma}_k h}\big) + \tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big), \end{aligned}$$

where the derivatives of u are calculated at the point (t_k, x), the coefficients b, σ, g and their derivatives are calculated at the point $(t_k, x, u(t_k, x))$, and

$$|r_1(t_k, x)| \leqslant Kh|R(t_{k+1}, x)|.$$

Since $u(t, x)$ is the solution to (2.1), (2.2), finally we arrive at

$$\begin{aligned} R(t_k, x) = -\tfrac{1}{2}R\big(t_{k+1}, \alpha + b_\alpha \bar{\gamma}_k h + \sigma_\alpha \sqrt{\bar{\gamma}_k h}\big) \\ +\tfrac{1}{2}R\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) + r_1(t_k, x) + O(h^2). \end{aligned} \tag{3.18}$$

Clearly, for x such that $x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h} > \beta$, we can obtain the relation similar to (3.18):

$$\begin{aligned} R(t_k, x) = -\tfrac{1}{2}R\left(t_{k+1}, \beta + b_\beta \bar{\delta}_k h - \sigma_\beta \sqrt{\bar{\delta}_k h}\right) \\ +\tfrac{1}{2}R(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}) + r_2(t_k, x) + O(h^2) \end{aligned} \tag{3.19}$$

with

$$|r_2(t_k, x)| \leqslant Kh|R(t_{k+1}, x)|.$$

Now we introduce

$$R_k := \max_{x \in [\alpha, \beta]} |R(t_k, x)|.$$

The relations (3.16), (3.18), and (3.19) imply (recall that $R(t_N, x) = 0$)

$$R_N = 0,\ R_k \leqslant R_{k+1} + K R_{k+1} h + Ch^2,\ k = N-1, \dots, 1, 0.$$

Therefore

$$R_k \leqslant \frac{C}{K}(\mathrm{e}^{K(T-t_0)} - 1) \cdot h,\ k = N, \dots, 0.$$

□

4. Layer method with one-step boundary error $O(h^{3/2})$

Without exploiting the idea used above of involving the points outside the interval $[\alpha, \beta]$ while constructing a layer method, it is possible to get a layer method that is simpler but with a larger one-step error near the boundary than (2.16) (see Lemma 4.1 below). Let us note that in spite of the greater one-step boundary error the global error of this method will be $O(h)$ again (see Theorem 4.1). Here we approximate the solution $u(t_k, x)$, when the point x is close to α (or β), using values of the solution at a point $(t_{k+\lambda_k}, \alpha)$ with some $\lambda_k \in (0, 1)$ (or at a point $(t_{k+\mu_k}, \beta)$ with some $\mu_k \in (0, 1)$) and at the point $(t_{k+1}, x + b_k \cdot h + \sigma_k \cdot \sqrt{h})$ (or $(t_{k+1}, x + b_k \cdot h - \sigma_k \cdot \sqrt{h})$) with some (positive) weights. These two weights may be interpreted as probabilities of reaching and not reaching α (or β). The method obtained in this way has the form

$$\bar{u}(t_N, x) = \varphi(t_N, x),\ x \in [\alpha, \beta], \tag{4.1}$$

$$\begin{aligned}
\bar{u}(t_k, x) &= \tfrac{1}{2}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) + \tfrac{1}{2}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) \\
&\quad + \bar{g}_k \cdot h, \qquad \text{if } x + \bar{b}_k \cdot h \pm \bar{\sigma}_k \cdot \sqrt{h} \in [\alpha, \beta]; \\
\bar{u}(t_k, x) &= \frac{1}{1 + \sqrt{\bar{\lambda}_k}}\varphi(t_{k+\bar{\lambda}_k}, \alpha) + \frac{\sqrt{\bar{\lambda}_k}}{1 + \sqrt{\bar{\lambda}_k}}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}\big) \\
&\quad + \bar{g}_k \cdot \sqrt{\bar{\lambda}_k} h, \qquad \text{if } x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h} < \alpha; \\
\bar{u}(t_k, x) &= \frac{1}{1 + \sqrt{\bar{\mu}_k}}\varphi(t_{k+\bar{\mu}_k}, \beta) + \frac{\sqrt{\bar{\mu}_k}}{1 + \sqrt{\bar{\mu}_k}}\bar{u}\big(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}\big) \\
&\quad + \bar{g}_k \cdot \sqrt{\bar{\mu}_k} h, \qquad \text{if } x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h} > \beta; \\
&\qquad k = N-1, \dots, 1, 0,
\end{aligned}$$

where $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$ are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$ and $0 < \bar{\lambda}_k, \bar{\mu}_k < 1$ are roots of the quadratic equations (it is not difficult to verify that the roots exist and are unique)

$$\alpha = x + \bar{b}_k \cdot \bar{\lambda}_k h - \bar{\sigma}_k \cdot \sqrt{\bar{\lambda}_k h},\ \beta = x + \bar{b}_k \cdot \bar{\mu}_k h + \bar{\sigma}_k \cdot \sqrt{\bar{\mu}_k h}.$$

This method involves one value of the function $\varphi(t, x)$ and one value of the approximate solution $\bar{u}(t_{k+1}, y)$ on the previous layer in contrast to the method (2.16) which requires evaluating one value of the function $\varphi(t, x)$ and two values of the approximate solution $\bar{u}(t_{k+1}, y)$ on the previous layer.

LEMMA 4.1 Under assumptions (i) and (ii), the one-step error $\rho(t_k, x)$ of (4.1) is estimated as

$$|\rho(t_k, x)| \leqslant Ch^2 \qquad \text{if } x + b_k \cdot h \pm \sigma_k \cdot \sqrt{h} \in [\alpha, \beta];$$
$$|\rho(t_k, x)| \leqslant Ch^{3/2} \qquad \text{if } x + b_k \cdot h - \sigma_k \cdot \sqrt{h} < \alpha \text{ or } x + b_k \cdot h + \sigma_k \cdot \sqrt{h} > \beta.$$

The proof is very similar (even simpler) to that of Lemma 3.1 and we do not give it here. The following convergence theorem for (4.1) takes place.

THEOREM 4.1 Under assumptions (i) and (ii), the method (4.1) has the global error estimated as

$$|\bar{u}(t_k, x) - u(t_k, x)| \leqslant Kh, \tag{4.2}$$

where K does not depend on h, k, x.

Proof. If we followed the way of proving Theorem 3.1, we would get that the global error of method (4.1) is $O(\sqrt{h})$. To prove the estimate (4.2), we exploit ideas of proving convergence theorems for probabilistic methods solving linear boundary value problems (Milstein, 1995b,c).

To this end, in connection with the layer method (4.1), we introduce the Markov chain (ϑ_i, X_i), $i \geqslant k$, $(\vartheta_k, X_k) = (t_k, x) \in \overline{Q}$, which stops on Γ at a random moment $\varkappa \leqslant N$. For $(\vartheta_i, X_i) \notin \Gamma$, we define

$$X_{i+1}^{\pm} := X_i + h\bar{b}_i \pm h^{1/2}\bar{\sigma}_i,$$

and if $X_{i+1}^{\pm} \in [\alpha, \beta]$, then $\vartheta_{i+1} = \vartheta_i + h$, X_{i+1} takes values X_{i+1}^- or X_{i+1}^+ with

$$P\{(\vartheta_{i+1}, X_{i+1}) = (\vartheta_i + h, X_{i+1}^-)\} = P\{(\vartheta_{i+1}, X_{i+1}) = (\vartheta_i + h, X_{i+1}^+)\} = \tfrac{1}{2};$$

if $X_{i+1}^- < \alpha$, then

$$P\{(\vartheta_{i+1}, X_{i+1}) = (\vartheta_i + \bar{\lambda}_i h, \alpha)\} = \frac{1}{1 + \sqrt{\bar{\lambda}_i}},$$
$$P\{(\vartheta_{i+1}, X_{i+1}) = (\vartheta_i + h, X_{i+1}^+)\} = \frac{\sqrt{\bar{\lambda}_i}}{1 + \sqrt{\bar{\lambda}_i}};$$

if $X_{i+1}^+ > \beta$, then

$$P\{(\vartheta_{i+1}, X_{i+1}) = (\vartheta_i + h, X_{i+1}^-)\} = \frac{\sqrt{\bar{\mu}_i}}{1 + \sqrt{\bar{\mu}_i}},$$
$$P\{(\vartheta_{i+1}, X_{i+1}) = (\vartheta_i + \bar{\mu}_i h, \beta)\} = \frac{1}{1 + \sqrt{\bar{\mu}_i}}.$$

Here $\bar{b}_i = b(\vartheta_i, X_i, \bar{u}(\vartheta_i + h, X_i))$, $\bar{\sigma}_i = \sigma(\vartheta_i, X_i, \bar{u}(\vartheta_i + h, X_i))$, $\bar{u}(t_k, x)$ is considered to be known from (4.1), and $0 < \bar{\lambda}_i, \bar{\mu}_i < 1$ are roots of the quadratic equations

$$\alpha = X_i + \bar{b}_i \cdot \bar{\lambda}_i h - \bar{\sigma}_i\sqrt{\bar{\lambda}_i h}, \quad \beta = X_i + \bar{b}_i \cdot \bar{\mu}_i h + \bar{\sigma}_i\sqrt{\bar{\mu}_i h}. \tag{4.3}$$

If $(\vartheta_i, X_i) \in \Gamma$, the Markov chain stops and $\varkappa = i$.

Let us note that ϑ_i coincides with t_i except, maybe, the last moment $\varkappa$.

Now introduce the random sequence Z_i, $Z_k = 0$:

$$\text{if } X_{i+1}^{\pm} \in [\alpha, \beta], \text{ then } Z_{i+1} = Z_i + h\bar{g}_i; \quad \text{if } X_{i+1}^{-} < \alpha, \quad \text{then } Z_{i+1} = Z_i + \sqrt{\bar{\lambda}_i} h \bar{g}_i;$$
$$\text{if } X_{i+1}^{+} > \beta, \quad \text{then } Z_{i+1} = Z_i + \sqrt{\bar{\mu}_i} h \bar{g}_i; \quad i = k, \ldots, \varkappa - 1,$$

where $\bar{g}_i = g(\vartheta_i, X_i, \bar{u}(\vartheta_i + h, X_i))$.

Define the boundary layer $\partial\Gamma \in \overline{Q}$. For all the points $(t_k, x) \in \overline{Q} \setminus \partial\Gamma$ both points $x + h\bar{b}_k \pm h^{1/2}\bar{\sigma}_k$ belong to $[\alpha, \beta]$. Clearly, for the points $(t_k, x) \in \partial\Gamma$ either $x + h\bar{b}_k - h^{1/2}\bar{\sigma}_k \notin [\alpha, \beta]$ or $x + h\bar{b}_k + h^{1/2}\bar{\sigma}_k \notin [\alpha, \beta]$.

It is not difficult to show that the mean of the number of steps $\nu(t_k, x)$, which the Markov chain (ϑ_i, X_i), $i = k, \ldots, \varkappa$, $\vartheta_k = t_k$, $X_k = x$, spends in the layer $\partial\Gamma$ is estimated as

$$E\nu(t_k, x) \leqslant H, \tag{4.4}$$

where H does not depend on h, k, x (see Milstein (1995b,c)).

One can see that

$$\bar{u}(t_k, x) = E[\bar{u}(\vartheta_\varkappa, X_\varkappa) + Z_\varkappa] = E[\varphi(\vartheta_\varkappa, X_\varkappa) + Z_\varkappa] = E[u(\vartheta_\varkappa, X_\varkappa) + Z_\varkappa].$$

We have $R(t_N, x) := \bar{u}(t_N, x) - u(t_N, x) = 0$ and for $k = N - 1, \ldots, 0$:

$$R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x) = E\sum_{i=k}^{\varkappa-1}[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i]$$
$$= E\sum_{i=k}^{\varkappa-1} I_{\overline{Q}\setminus\partial\Gamma}(\vartheta_i, X_i)[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i]$$
$$+E\sum_{i=k}^{\varkappa-1} I_{\partial\Gamma}(\vartheta_i, X_i)[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i]$$
$$= \sum_{i=k}^{N-1} EE(\chi_{\varkappa>i} I_{\overline{Q}\setminus\partial\Gamma}(\vartheta_i, X_i)[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i]/X_i, Z_i)$$
$$+\sum_{i=k}^{N-1} EE(\chi_{\varkappa>i} I_{\partial\Gamma}(\vartheta_i, X_i)[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i]/X_i, Z_i)$$
$$= \sum_{i=k}^{N-1} E(\chi_{\varkappa>i} I_{\overline{Q}\setminus\partial\Gamma}(\vartheta_i, X_i) E[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i/X_i, Z_i])$$

$$+\sum_{i=k}^{N-1} E(\chi_{\varkappa>i} I_{\partial\Gamma}(\vartheta_i, X_i) E[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i/X_i, Z_i]). \tag{4.5}$$

In (4.5) and below we use the ordinary properties of conditional mathematical expectations taking into account that the indicator functions $\chi_{\varkappa>i}$, $I_{\overline{Q}\setminus\partial\Gamma}(\vartheta_i, X_i)$, and $I_{\partial\Gamma}(\vartheta_i, X_i)$ are measurable with respect to X_i.

To calculate the conditional expectations in (4.5), we exploit a lemma from Gichman & Skorochod (1968, Section 10). In our case the lemma allows us to evaluate a conditional expectation as the ordinary mathemetical expectation under fixed values of the random variables X_i, Z_i.

We get for $(\vartheta_i, X_i) \in \overline{Q} \setminus \partial\Gamma$ that

$$A_i := E[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i / X_i, Z_i]$$
$$= \tfrac{1}{2}u(t_{i+1}, X_i + h\bar{b}_i - h^{1/2}\bar{\sigma}_i) + \tfrac{1}{2}u(t_{i+1}, X_i + h\bar{b}_i + h^{1/2}\bar{\sigma}_i) - u(t_i, X_i) + h\bar{g}_i. \quad (4.6)$$

We expand the terms of (4.6) at the point (t_i, X_i) (cf.(3.13)). Then, attracting relations like (3.14) and taking into account that $u(t, x)$ is the solution to the problem (2.1), (2.2), we obtain

$$A_i = r_i + O(h^2), \quad (4.7)$$

where

$$|r_i| \leqslant Kh|R(\vartheta_i + h, X_i)|.$$

Now let $(\vartheta_i, X_i) \in \partial\Gamma$ be such that X_i is close to α. We have

$$B_i := E[u(\vartheta_{i+1}, X_{i+1}) - u(\vartheta_i, X_i) + Z_{i+1} - Z_i / X_i, Z_i]$$
$$= \frac{1}{1+\sqrt{\bar{\lambda}_i}}\varphi(t_{i+\bar{\lambda}_i}, \alpha) + \frac{\sqrt{\bar{\lambda}_i}}{1+\sqrt{\bar{\lambda}_i}}u(t_{i+1}, X_i + h\bar{b}_i + h^{1/2}\bar{\sigma}_i) - u(t_i, X_i) + \sqrt{\bar{\lambda}_i}h\bar{g}_i$$
$$= \frac{1}{1+\sqrt{\bar{\lambda}_i}}u(t_{i+\bar{\lambda}_i}, \alpha) + \frac{\sqrt{\bar{\lambda}_i}}{1+\sqrt{\bar{\lambda}_i}}u(t_{i+1}, X_i + h\bar{b}_i + h^{1/2}\bar{\sigma}_i) - u(t_i, X_i) + \sqrt{\bar{\lambda}_i}h\bar{g}_i. \quad (4.8)$$

We expand the terms of (4.8) at the point (t_i, X_i). Then, using (4.3), attracting relations like (3.14), and taking into account that $u(t, x)$ is the solution to the problem (2.1), (2.2), we get

$$B_i = \bar{r}_i + O(h^{3/2}), \quad (4.9)$$

where

$$|\bar{r}_i| \leqslant Kh|R(\vartheta_i + h, X_i)|.$$

An analogous relation can be obtained for $(\vartheta_i, X_i) \in \partial\Gamma$ with X_i being close to β.

Substituting (4.7) and (4.9) in (4.5), we obtain

$$R(t_k, x) = E\sum_{i=k}^{\varkappa-1} I_{\overline{Q}\setminus\partial\Gamma}(\vartheta_i, X_i)[r_i + O(h^2)] + E\sum_{i=k}^{\varkappa-1} I_{\partial\Gamma}(\vartheta_i, X_i)[\bar{r}_i + O(h^{3/2})]. \quad (4.10)$$

Due to (4.4),

$$\left| E \sum_{i=k}^{\varkappa-1} I_{\partial\Gamma}(\vartheta_i, X_i) O(h^{3/2}) \right| \leqslant CHh^{3/2}.$$

Then, from (4.10), we obtain for $R_k := \max\limits_{x\in[\alpha,\beta]} |R(t_k, x)|$ that

$$R_k \leqslant Kh \sum_{i=k}^{N-1} R_{i+1} + Ch. \tag{4.11}$$

Introduce $\varepsilon_k := Kh \sum_{i=k}^{N-1} R_{i+1} + Ch$, $k = N-1, \ldots, 0$. Due to (4.11), $R_k \leqslant \varepsilon_k$. Consequently, $\varepsilon_k = KhR_{k+1} + \varepsilon_{k+1} \leqslant (1 + Kh)\varepsilon_{k+1}$, $k = N-2, \ldots, 0$. Then (since $\varepsilon_{N-1} = Ch$)

$$R_k \leqslant \varepsilon_k \leqslant C\mathrm{e}^{K(T-t_0)} \cdot h, \ k = N, \ldots, 0.$$

□

REMARK 4.1 The assertions of Lemma 4.1 and Theorem 4.1 are also valid if we take weaker assumptions on the coefficients than (ii), namely

$$|b| \leqslant K, \ |\sigma| \leqslant K, \ |g| \leqslant K,$$
$$|b(t, x_2, u_2) - b(t, x_1, u_1)| + |\sigma(t, x_2, u_2) - \sigma(t, x_1, u_1)| + |g(t, x_2, u_2) - g(t, x_1, u_1)|$$
$$\leqslant K(|x_2 - x_1| + |u_2 - u_1|), \ t_0 \leqslant t \leqslant T, \ x \in [\alpha, \beta], \ u_\circ < u < u^\circ.$$

REMARK 4.2 It follows from the proof of Theorem 4.1 that to construct a first-order method we can use an approximation of $u(t_k, x)$ for which the one-step error near the boundary (i.e. when $(t_k, x) \in \partial\Gamma$) is estimated as $O(h)$ only (cf. Lemma 4.1). For instance, we can approximate the solution $u(t_k, x)$, when x is close to α, by values of the solution at the point (t_{k+1}, α) and at a point $(t_{k+1}, \hat{x}_k) \in \overline{Q} \setminus \partial\Gamma$ (for example, one can take $\hat{x}_k = \alpha + h^{1/2} \max \sigma + h \max |b|$, where the maxima are taken over $(t, x) \in \overline{Q}, u \in [u_\circ, u^\circ]$) with the weights $p = \frac{\hat{x}_k - x}{\hat{x}_k - \alpha}$ and $q = 1 - p$ respectively. Analogously, we can approximate $u(t_k, x)$ when x is close to β. Making use of this approximation for $(t_k, x) \in \partial\Gamma$ and the Euler approximation for $(t_k, x) \in \overline{Q} \setminus \partial\Gamma$, we get a new layer method with the first order of convergence (see also Milstein & Tretyakov (2001), where such a construction is used for solving linear Dirichlet problems by the Monte Carlo approach). This layer method has practically the same computational complexity as (4.1). But a generalization of this method to the multi-dimensional case can be easier for realization than a generalization of (4.1) (see also Section 6).

REMARK 4.3 We can also conclude from the proof of Theorem 4.1 that if we use an approximation of $u(t_k, x)$ for which the one-step error is $O(h^{3/2})$ for $(t_k, x) \in \partial\Gamma$ (as it is for (4.1)) and is at least $O(h^{5/2})$ for $(t_k, x) \in \overline{Q} \setminus \partial\Gamma$, we will obtain the new layer method with the global error $O(h^{3/2})$. For instance, in the case of constant σ it is possible to get such a method by attracting the second-order weak Runge–Kutta scheme from Milstein (1995a) instead of the weak Euler scheme (2.5) used for construction of (4.1) (see also Milstein (1997)).

REMARK 4.4 The layer methods of Sections 2 and 4 can be applied to solving the Dirichlet problem for linear parabolic equations. But if the dimension d of the linear problem is high ($d \geqslant 3$ in practice) and it is enough to find the solution in a few points only, the Monte Carlo approach is preferable (Milstein, 1995b,c; Milstein & Tretyakov, 2001).

REMARK 4.5 Using other weak approximations for SDEs, some new layer methods can be constructed (cf. Milstein (1997) and Milstein & Tretyakov (2000)). In particular, there are special methods of numerical integration in the weak sense for SDEs with small noise which are more effective than general ones (Milstein & Tretyakov, 1997). In Milstein & Tretyakov (2000) they are used for constructing special layer methods for the Cauchy problem for semilinear parabolic equations with small parameter at higher derivatives. It is also possible to get some special layer methods in the case of the Dirichlet problem for semilinear parabolic equations with small parameter.

5. Numerical algorithms

To become a numerical algorithm, the method (2.16) (just as other layer methods) needs a discretization in the variable x. Consider an equidistant space discretization with a space step h_x (recall that the notation for time step is h): $x_j = \alpha + jh_x$, $j = 0, 1, 2, \ldots, M$, $h_x = (\beta - \alpha)/M$. Using, for example, linear interpolation, we construct the following algorithm (we denote it as $\bar{u}(t_k, x)$ again, since this does not cause any confusion):

$$\bar{u}(t_N, x) = \varphi(t_N, x), \ x \in [\alpha, \beta], \tag{5.1}$$

$$\begin{aligned}
\bar{u}(t_k, x_j) = {} & \tfrac{1}{2}\bar{u}\big(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}\big) + \tfrac{1}{2}\bar{u}\big(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h}\big) \\
& + \bar{g}_{k,j} \cdot h, \qquad \text{if } x_j + \bar{b}_{k,j} \cdot h \pm \bar{\sigma}_{k,j} \cdot \sqrt{h} \in [\alpha, \beta]; \\
\bar{u}(t_k, x_j) = {} & \varphi(t_{k+1-\bar{\gamma}_{k,j}}, \alpha) - g(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \bar{\gamma}_{k,j} h \\
& - \tfrac{1}{2}\bar{u}\big(t_{k+1}, \alpha + b(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \bar{\gamma}_{k,j} h + \sigma(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \sqrt{\bar{\gamma}_{k,j} h}\big) \\
& + \tfrac{1}{2}\bar{u}\big(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h}\big) + \bar{g}_{k,j} \cdot h, \qquad \text{if } x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h} < \alpha; \\
\bar{u}(t_k, x_j) = {} & \tfrac{1}{2}\bar{u}\big(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}\big) + \varphi(t_{k+1-\bar{\delta}_{k,j}}, \beta) \\
& - g(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \bar{\delta}_{k,j} h \\
& - \tfrac{1}{2}\bar{u}\left(t_{k+1}, \beta + b(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \bar{\delta}_{k,j} h - \sigma(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \sqrt{\bar{\delta}_{k,j} h}\right) \\
& + \bar{g}_{k,j} \cdot h, \qquad \text{if } x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h} > \beta; \\
& j = 1, 2, \ldots, M-1,
\end{aligned}$$

$$\bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x}\, \bar{u}(t_k, x_j) + \frac{x - x_j}{h_x}\, \bar{u}(t_k, x_{j+1}), \ x_j < x < x_{j+1}, \tag{5.2}$$
$$j = 0, 1, 2, \ldots, M-1, \ k = N-1, \ldots, 1, 0,$$

where $\bar{b}_{k,j}$, $\bar{\sigma}_{k,j}$, $\bar{g}_{k,j}$ are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$, $g(t, x, u)$ calculated at the point $(t_k, x_j, \bar{u}(t_{k+1}, x_j))$ and $0 < \bar{\gamma}_{k,j}, \bar{\delta}_{k,j} \leqslant 1$ are roots of (2.13) and (2.14) with the right sides $x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}$ and $x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h}$ respectively.

THEOREM 5.1 If the value of h_x is taken equal to $\varkappa h$, $\varkappa$ is a positive constant, then under assumptions (i) and (ii) the algorithm (5.1), (5.2) has the first order of convergence, i.e. the approximation $\bar{u}(t_k, x)$ from (5.1), (5.2) satisfies the relation

$$|\bar{u}(t_k, x) - u(t_k, x)| \leqslant Kh, \tag{5.3}$$

where K does not depend on x, h, k.

The proof of Theorem 5.1 differs only little from the proof of the corresponding theorem in Milstein (1997) and is therefore omitted.

REMARK 5.1 Using probabilistic arguments, it is possible to prove that the algorithm based on the method (4.1) and linear interpolation has the global error $O(h)$ for $h_x = \varkappa h$.

REMARK 5.2 It is natural to consider cubic interpolation instead of the linear one for constructing numerical algorithms. The use of cubic interpolation allows us to take the space step $h_x = \varkappa\sqrt{h}$ (in contrast to $h_x = \varkappa h$ for linear interpolation) and, thus, to reduce the volume of computations. Moreover, if we use cubic interpolation, we can avoid special formulas near the boundary choosing some appropriate $\varkappa$ (indeed, we can take, for example, $\varkappa = 2 \max_{t\in[t_0,T],\, x\in\overline{G},\, u\in[u_\circ,\, u^\circ]} \sigma(t, x, u)$, then for a sufficiently small h the points $x_j + \bar{b}_{k,j}\cdot h \pm \bar{\sigma}_{k,j}\cdot\sqrt{h}$, $j = 1, 2, \ldots, M-1$, always belong to $[\alpha, \beta]$). Unfortunately, we have not succeeded in proving a convergence theorem in the case of cubic interpolation. The way of proving Theorem 5.1 gives us some restriction on the type of interpolation procedure which can be used for constructing the numerical algorithm. The restriction is such that the sum of the absolute values of the coefficients staying at $\bar{u}(t_k, \cdot)$ in the interpolation procedure must not be greater than 1. Linear interpolation and B-splines of $O(h_x^2)$ satisfy the restriction. However, cubic interpolation of $O(h_x^4)$ does not satisfy it. In Sections 7.1 and 7.2 we test an algorithm based on cubic interpolation. The tests give fairly good results. See also some theoretical explanations and numerical tests in Milstein (1997) and Milstein & Tretyakov (2000).

REMARK 5.3 Clearly, the algorithms can be considered with variable time steps and space steps. An algorithm with variable space steps is used in our numerical tests (see Section 7.1).

6. Extension to the multi-dimensional Dirichlet problem

In this section we generalize the layer method (4.1) to the multi-dimensional case ($d > 1$). A generalization of the layer method (2.16) to the multi-dimensional case is complicated and is not considered here.

As has been mentioned in the Introduction, layer methods are feasible if the dimension d of the domain G is not more than 3. This is why we restrict ourselves here to the cases $d = 2$ and $d = 3$. We remark only that it is not difficult to generalize the layer method (4.1) for an arbitrary d.

Consider the case $d = 2$. Introduce the notation ${}_iX_{k+1} := \left({}_iX^1_{k+1},\ {}_iX^2_{k+1}\right)$,

$$
\begin{gathered}
{}_iX^1_{k+1} = x^1 + \bar b^1_k h + \bar\sigma^{11}_k \sqrt{h}\cdot {}_i\xi^1 + \bar\sigma^{12}_k \sqrt{h}\cdot {}_i\xi^2,\\
{}_iX^2_{k+1} = x^2 + \bar b^2_k h + \bar\sigma^{21}_k \sqrt{h}\cdot {}_i\xi^1 + \bar\sigma^{22}_k \sqrt{h}\cdot {}_i\xi^2,\\
i = 1, 2, 3, 4,\ \ x = (x^1, x^2) \in G \subset R^2,
\end{gathered}
$$

where ${}_1\xi = (-1, -1)$, ${}_2\xi = (-1, 1)$, ${}_3\xi = -{}_1\xi$, ${}_4\xi = -{}_2\xi$ and $\bar b_k = (\bar b^1_k, \bar b^2_k)$, $\bar\sigma_k = \{\bar\sigma^{jl}_k\}$ are the coefficients $b(t, x, u)$, $\sigma(t, x, u)$ calculated at the point $(t_k, x, \bar u(t_{k+1}, x))$.

If the point $x = (x^1, x^2) \in G$ is sufficiently far from the boundary ∂G (more precisely, if the points ${}_iX_{k+1}$, $i = 1, 2, 3, 4$, belong to $\overline{G}$), the layer method has the form (cf. Milstein (1997)):

$$
\bar u(t_k, x^1, x^2) = \sum_{i=1}^{4} \tfrac{1}{4} \bar u(t_{k+1},\ {}_iX^1_{k+1},\ {}_iX^2_{k+1}) + \bar g_k \cdot h, \tag{6.1}
$$

where $\bar g_k$ is the coefficient $g(t, x, u)$ calculated at the point $(t_k, x, \bar u(t_{k+1}, x))$.

If the point $x = (x^1, x^2) \in G$ is close to the boundary ∂G, then some of the points ${}_iX_{k+1} = ({}_iX^1_{k+1},\ {}_iX^2_{k+1})$, $i = 1, 2, 3, 4$, may be outside of the domain $\overline{G}$. Let us connect the point x with the points ${}_{i^*}X_{k+1}$, which are outside of $\overline{G}$, by the curves $\psi_{i^*}(\lambda) = (\psi^1_{i^*}(\lambda), \psi^2_{i^*}(\lambda))$:

$$
\begin{gathered}
\psi^1_{i^*}(\lambda) = x^1 + \bar b^1_k\, \lambda h + \bar\sigma^{11}_k \sqrt{\lambda h}\cdot {}_{i^*}\xi^1 + \bar\sigma^{12}_k \sqrt{\lambda h}\cdot {}_{i^*}\xi^2,\\
\psi^2_{i^*}(\lambda) = x^2 + \bar b^2_k \lambda h + \bar\sigma^{21}_k \sqrt{\lambda h}\cdot {}_{i^*}\xi^1 + \bar\sigma^{22}_k \sqrt{\lambda h}\cdot {}_{i^*}\xi^2,\ 0 \leqslant \lambda \leqslant 1.
\end{gathered}
$$

The boundary ∂G is assumed to be sufficiently smooth. For a sufficiently small h there is a unique value of $\lambda = {}_{i^*}\bar\lambda_k$, $0 < {}_{i^*}\bar\lambda_k < 1$, such that the point ${}_{i^*}\eta_k = ({}_{i^*}\eta^1_k,\ {}_{i^*}\eta^2_k)$, where

$$
\begin{gathered}
{}_{i^*}\eta^1_k = x^1 + \bar b^1_k \cdot {}_{i^*}\bar\lambda_k h + \bar\sigma^{11}_k \sqrt{{}_{i^*}\bar\lambda_k h}\cdot {}_{i^*}\xi^1 + \bar\sigma^{12}_k \sqrt{{}_{i^*}\bar\lambda_k h}\cdot {}_{i^*}\xi^2,\\
{}_{i^*}\eta^2_k = x^2 + \bar b^2_k \cdot {}_{i^*}\bar\lambda_k h + \bar\sigma^{21}_k \sqrt{{}_{i^*}\bar\lambda_k h}\cdot {}_{i^*}\xi^1 + \bar\sigma^{22}_k \sqrt{{}_{i^*}\bar\lambda_k h}\cdot {}_{i^*}\xi^2,
\end{gathered}
$$

belongs to the boundary ∂G.

Put ${}_j\bar\lambda_k = 1$ and ${}_j\eta_k = {}_jX_{k+1}$ for the points ${}_jX_{k+1}$ belonging to $\overline{G}$. Then the layer

method takes the form

$$\bar u(t_k,x^1,x^2)=\frac{\sqrt{{}_2\bar\lambda_k\cdot{}_3\bar\lambda_k\cdot{}_4\bar\lambda_k}}{\bigl(\sqrt{{}_1\bar\lambda_k}+\sqrt{{}_3\bar\lambda_k}\bigr)\bigl(\sqrt{{}_1\bar\lambda_k\cdot{}_3\bar\lambda_k}+\sqrt{{}_2\bar\lambda_k\cdot{}_4\bar\lambda_k}\bigr)}\bar u(t_{k+{}_1\bar\lambda_k},\,{}_1\eta_k^1,\,{}_1\eta_k^2)$$
$$+\frac{\sqrt{{}_1\bar\lambda_k\cdot{}_3\bar\lambda_k\cdot{}_4\bar\lambda_k}}{\bigl(\sqrt{{}_2\bar\lambda_k}+\sqrt{{}_4\bar\lambda_k}\bigr)\bigl(\sqrt{{}_1\bar\lambda_k\cdot{}_3\bar\lambda_k}+\sqrt{{}_2\bar\lambda_k\cdot{}_4\bar\lambda_k}\bigr)}\bar u\bigl(t_{k+{}_2\bar\lambda_k},\,{}_2\eta_k^1,\,{}_2\eta_k^2\bigr)$$
$$+\frac{\sqrt{{}_1\bar\lambda_k\cdot{}_2\bar\lambda_k\cdot{}_4\bar\lambda_k}}{\bigl(\sqrt{{}_1\bar\lambda_k}+\sqrt{{}_3\bar\lambda_k}\bigr)\bigl(\sqrt{{}_1\bar\lambda_k\cdot{}_3\bar\lambda_k}+\sqrt{{}_2\bar\lambda_k\cdot{}_4\bar\lambda_k}\bigr)}\bar u(t_{k+{}_3\bar\lambda_k},\,{}_3\eta_k^1,\,{}_3\eta_k^2)$$
$$+\frac{\sqrt{{}_1\bar\lambda_k\cdot{}_2\bar\lambda_k\cdot{}_3\bar\lambda_k}}{\bigl(\sqrt{{}_2\bar\lambda_k}+\sqrt{{}_4\bar\lambda_k}\bigr)\bigl(\sqrt{{}_1\bar\lambda_k\cdot{}_3\bar\lambda_k}+\sqrt{{}_2\bar\lambda_k\cdot{}_4\bar\lambda_k}\bigr)}\bar u(t_{k+{}_4\bar\lambda_k},\,{}_4\eta_k^1,\,{}_4\eta_k^2)$$
$$+\bar g_k\cdot\frac{2\sqrt{{}_1\bar\lambda_k\cdot{}_2\bar\lambda_k\cdot{}_3\bar\lambda_k\cdot{}_4\bar\lambda_k}}{\sqrt{{}_1\bar\lambda_k\cdot{}_3\bar\lambda_k}+\sqrt{{}_2\bar\lambda_k\cdot{}_4\bar\lambda_k}}h. \tag{6.2}$$

Recall that if ${}_i\eta_k=({}_i\eta_k^1,\,{}_i\eta_k^2)\in\partial G$ then $\bar u(t_{k+{}_i\bar\lambda_k},\,{}_i\eta_k^1,\,{}_i\eta_k^2)=\varphi(t_{k+{}_i\bar\lambda_k},\,{}_i\eta_k^1,\,{}_i\eta_k^2)$ (see (1.2)).

The errors of the one-step approximations corresponding to (6.1) and (6.2) are $O(h^2)$ and $O(h^{3/2})$ respectively. By probabilistic arguments (see the proof of Theorem 4.1), the global error of the layer method (6.1), (6.2) is estimated by $O(h)$.

Now consider the case $d=3$. Introduce the notation ${}_iX_{k+1}=({}_iX_{k+1}^1,\,{}_iX_{k+1}^2,\,{}_iX_{k+1}^3)$, $i=1,2,\dots,8$, where

$${}_iX_{k+1}^j:=x^j+\bar b_k^j h+\bar\sigma_k^{j1}\sqrt{h}\cdot{}_i\xi^1+\bar\sigma_k^{j2}\sqrt{h}\cdot{}_i\xi^2+\bar\sigma_k^{j3}\sqrt{h}\cdot{}_i\xi^3,\ \ j=1,2,3,$$
$$x=(x^1,x^2,x^3)\in G\subset\mathbb{R}^3.$$

Here $\bar b_k=\{\bar b_k^j\}$ and $\bar\sigma_k=\{\bar\sigma_k^{jl}\}$ are the coefficients $b(t,x,u)$ and $\sigma(t,x,u)$ calculated at the point $(t_k,x,\bar u(t_k,x))$ and ${}_i\xi=({}_i\xi^1,\,{}_i\xi^2,\,{}_i\xi^3)$, $i=1,\dots,8$, are the following vectors:

$${}_1\xi=(-1,-1,-1),\ {}_2\xi=(-1,-1,1),\ {}_3\xi=(-1,1,-1),\ {}_4\xi=(1,-1,-1),$$
$${}_{i+4}\xi=-{}_i\xi,\ \ i=1,2,3,4.$$

If the points ${}_iX_{k+1}$, $i=1,2,\dots,8$, belong to $\overline{G}$, the layer method has the form

$$\bar u(t_k,x)=\sum_{i=1}^{8}\tfrac{1}{8}\bar u(t_{k+1},\,{}_iX_{k+1})+\bar g_k\cdot h, \tag{6.3}$$

where $\bar g_k$ is the coefficient $g(t,x,u)$ calculated at the point $(t_k,x,\bar u(t_{k+1},x))$.

If some points ${}_{i^*}X_{k+1}\notin\overline{G}$, we connect the point x with the points ${}_{i^*}X_{k+1}$ by the curves $\psi_{i^*}(\lambda)=(\psi_{i^*}^1(\lambda),\psi_{i^*}^2(\lambda),\psi_{i^*}^3(\lambda))$,

$$\psi_{i^*}^j(\lambda)=x^j+\bar b_k^j\lambda h+\bar\sigma_k^{j1}\sqrt{\lambda h}\cdot{}_{i^*}\xi^1+\bar\sigma_k^{j2}\sqrt{\lambda h}\cdot{}_{i^*}\xi^2+\bar\sigma_k^{j3}\sqrt{\lambda h}\cdot{}_{i^*}\xi^3,$$
$$j=1,2,3,\ \ 0\leqslant\lambda\leqslant 1.$$

Due to the smoothness of the boundary ∂G, for a sufficiently small h there is a unique value of $\lambda = {}_{i^*}\bar{\lambda}_k$, $0 < {}_{i^*}\bar{\lambda}_k < 1$, such that the point ${}_{i^*}\eta_k = ({}_{i^*}\eta_k^1, {}_{i^*}\eta_k^2, {}_{i^*}\eta_k^3)$, where

$$ {}_{i^*}\eta_k^j = x^j + \bar{b}_k^j \cdot {}_{i^*}\bar{\lambda}_k h + \bar{\sigma}_k^{j1}\sqrt{{}_{i^*}\bar{\lambda}_k h}\cdot {}_{i^*}\xi^1 + \bar{\sigma}_k^{j2}\sqrt{{}_{i^*}\bar{\lambda}_k h}\cdot {}_{i^*}\xi^2 + \bar{\sigma}_k^{j3}\sqrt{{}_{i^*}\bar{\lambda}_k h}\cdot {}_{i^*}\xi^3, \quad j = 1, 2, 3, $$

belongs to the boundary ∂G.

Put ${}_j\bar{\lambda}_k = 1$ and ${}_j\eta_k = {}_jX_{k+1}$ for the points ${}_jX_{k+1}$ belonging to $\overline{G}$. Then the layer method takes the form

$$ \bar{u}(t_k, x) = \sum_{i=1}^{4} \frac{\gamma_k}{\sqrt{{}_i\bar{\lambda}_k} + \sqrt{{}_{i+4}\bar{\lambda}_k}} \left(\frac{1}{\sqrt{{}_i\bar{\lambda}_k}} \bar{u}(t_{k+{}_i\bar{\lambda}_k}, {}_i\eta_k) + \frac{1}{\sqrt{{}_{i+4}\bar{\lambda}_k}} \bar{u}(t_{k+{}_{i+4}\bar{\lambda}_k}, {}_{i+4}\eta_k) \right) + \bar{g}_k \cdot 4\gamma_k h, \tag{6.4} $$

where

$$ \gamma_k = \left(\sum_{i=1}^{4} \frac{1}{\sqrt{{}_i\bar{\lambda}_k \cdot {}_{i+4}\bar{\lambda}_k}} \right)^{-1}. $$

To construct the corresponding numerical algorithms, we attract linear interpolation as in the previous section. For example, consider the case $d = 2$. To this end, put the domain $\overline{G}$ into a rectangle Π with corners (x_0^1, x_0^2), $(x_0^1, x_{M_2}^2)$, $(x_{M_1}^1, x_0^2)$, $(x_{M_1}^1, x_{M_2}^2)$ and introduce the equidistant space discretization of the rectangle Π:

$$ \Delta_{M_1,M_2} := \{(x_j^1, x_l^2) : x_j^1 = x_0^1 + jh_{x^1}, x_l^2 = x_0^2 + lh_{x^2}, j = 0, \ldots, M_1, l = 0, \ldots, M_2\}, $$

$$ h_{x^1} = \frac{x_{M_1}^1 - x_0^1}{M_1}, \; h_{x^2} = \frac{x_{M_1}^2 - x_0^2}{M_2}. $$

The values of $\bar{u}(t_k, x_j^1, x_l^2)$ at the nodes of $\Delta_{M_1,M_2} \cap \overline{G}$ are found in accordance with (6.1), (6.2). Let $(x^1, x^2) \in \overline{G}$ and $x_j^1 \leqslant x^1 \leqslant x_{j+1}^1$, $x_l^2 \leqslant x^2 \leqslant x_{l+1}^2$. If all the nodes (x_j^1, x_l^2), (x_j^1, x_{l+1}^2), (x_{j+1}^1, x_l^2), $(x_{j+1}^1, x_{l+1}^2) \in \overline{G}$, the value of $\bar{u}(t_k, x^1, x^2)$ is evaluated as

$$ \bar{u}(t_k, x^1, x^2) = \frac{x_{j+1}^1 - x^1}{h_{x^1}} \cdot \frac{x_{l+1}^2 - x^2}{h_{x^2}} \bar{u}(t_k, x_j^1, x_l^2) + \frac{x_{j+1}^1 - x^1}{h_{x^1}} \cdot \frac{x^2 - x_l^2}{h_{x^2}} \bar{u}(t_k, x_j^1, x_{l+1}^2) + \frac{x^1 - x_j^1}{h_{x^1}} \cdot \frac{x_{l+1}^2 - x^2}{h_{x^2}} \bar{u}(t_k, x_{j+1}^1, x_l^2) + \frac{x^1 - x_j^1}{h_{x^1}} \cdot \frac{x^2 - x_l^2}{h_{x^2}} \bar{u}(t_k, x_{j+1}^1, x_{l+1}^2). \tag{6.5} $$

If the point $x = (x^1, x^2) : x_j^1 \leqslant x^1 \leqslant x_{j+1}^1$, $x_l^2 \leqslant x^2 \leqslant x_{l+1}^2$ is such that some of the nodes (x_j^1, x_l^2), (x_j^1, x_{l+1}^2), (x_{j+1}^1, x_l^2), (x_{j+1}^1, x_{l+1}^2) do not belong to $\overline{G}$, then we use some points on the boundary ∂G (due to (1.2) we know values of $u(t, x)$ for $x \in \partial G$) to find $\bar{u}(t_k, x^1, x^2)$ by linear interpolation.

If we take $h_{x^i} = \varkappa^i h$, $i = 1, 2$, $\varkappa^1, \varkappa^2 > 0$ are positive constants, the error of the proposed algorithm is estimated as $O(h)$.

7. Numerical tests

In the previous sections we dealt with semilinear parabolic equations with negative direction of time t: the equations are considered under $t < T$ and the 'initial' conditions are given at $t = T$. This form of equation is suitable for the probabilistic approach which we use to construct numerical methods. Of course, the proposed methods are adaptable to semilinear parabolic equations with positive direction of time, and this adaptation is particularly easy in the autonomous case. In our numerical tests we use algorithms with positive direction of time (see, for example, (7.15), (7.16)).

7.1 *The Burgers equation*

Consider the Dirichlet problem for the one-dimensional Burgers equation:

$$\frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}, \quad t > 0,\ x \in (-1, 1), \tag{7.1}$$

$$u(0, x) = -A \sin \pi x,\ x \in [-1, 1], \tag{7.2}$$

$$u(t, \pm 1) = 0,\ t > 0. \tag{7.3}$$

This problem was used for testing various numerical methods in, for example, Anderson *et al.* (1984), Fletcher (1984) and Basdevant *et al.* (1986) (see also references therein). By means of the Cole–Hopf transformation, one can find the explicit solution of problem (7.1)–(7.3) in the form

$$u(t, x) = -A\frac{\int_{-\infty}^{\infty} \sin \pi(x - y)\, \exp(-\frac{A}{\pi\varepsilon^2}\cos \pi(x - y) - \frac{y^2}{2\varepsilon^2 t})\, \mathrm{d}y}{\int_{-\infty}^{\infty} \exp(-\frac{A}{\pi\varepsilon^2}\cos \pi(x - y) - \frac{y^2}{2\varepsilon^2 t})\, \mathrm{d}y} \tag{7.4}$$

or

$$u(t, x) = \frac{\pi\varepsilon^2}{2}\frac{\sum_{n=1}^{\infty} n a_n \exp(-\frac{1}{8}\varepsilon^2\pi^2 n^2 t) \sin \frac{1}{2}\pi n(x + 1)}{\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \exp(-\frac{1}{8}\varepsilon^2\pi^2 n^2 t) \cos \frac{1}{2}\pi n(x + 1)} \tag{7.5}$$

with

$$a_n = \int_{-1}^{1} \exp\left(-\frac{A}{\pi\varepsilon^2}\cos \pi x\right) \cos \frac{1}{2}\pi n(x + 1)\, \mathrm{d}x.$$

We shall simulate the problem (7.1)–(7.3) on relatively small time intervals $[0, T]$, where the formula (7.4) is more convenient. For a small ε, there is a thin internal layer with width $\sim\varepsilon^2$, where the solution to (7.1)–(7.3) has singular behaviour (see, for example, Il'in (1992) and references therein). Derivatives of the solution go to infinity as $\varepsilon \to 0$. A typical behaviour of the solution is demonstrated on Fig. 1.

Here we test three algorithms. The first one is algorithm (5.1), (5.2). The second one is the algorithm based on layer method (4.1) and linear interpolation. In these two algorithms (they both use linear interpolation), we take the space step h_x being equal to the time step h. The third algorithm is based on cubic interpolation (see Remark 5.2). In the case of the

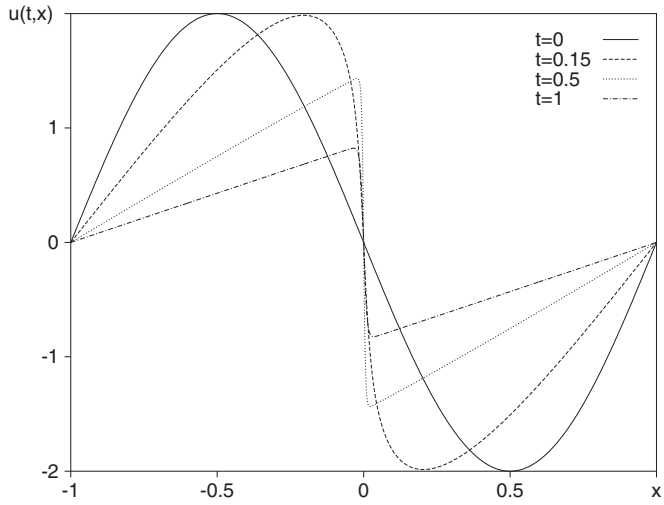

FIG. 1. A typical solution $u(t, x)$ of problem (7.1)–(7.3) for $\varepsilon = 0{\cdot}1$, $A = 2$ and various time moments.

problem (7.1)–(7.3) it has the form (cf. Milstein & Tretyakov (2000))

$$\bar{u}(0, x) = -A \sin \pi x, \ \ x \in [-1, 1], \tag{7.6}$$
$$\bar{u}(t_{k+1}, x_0) = \bar{u}(t_{k+1}, -1) = 0,$$
$$\bar{u}(t_{k+1}, x_M) = \bar{u}(t_{k+1}, 1) = 0,$$
$$\bar{u}(t_{k+1}, x_j) = \tfrac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j) - \varepsilon h^{1/2}) + \tfrac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j) + \varepsilon h^{1/2}),$$
$$j = 1, \ldots, M-1,$$
$$\bar{u}(t_k, x) = \sum_{i=0}^{3} \Phi_{j,i}(x)\bar{u}(t_k, x_{j+i}), \quad x_j < x < x_{j+3},$$
$$\Phi_{j,i}(x) = \prod_{m=0, m\neq i}^{3} \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}},$$
$$k = 0, \ldots, N-1.$$

Here we use a nonequidistant discretization of the interval $[-1, 1]$. We take $h_x := x_{j+1} - x_j = \varepsilon\sqrt{h}$ in $[-0{\cdot}1, 0{\cdot}1]$ and $h_x = \sqrt{h}$ outside $[-0{\cdot}1, 0{\cdot}1]$. Such a choice of h_x is dictated by the fact that if h is comparatively large (for example, $h_x = \varepsilon^2$), the equidistant discretization with $h_x = \sqrt{h}$ has not more than one node in the thin internal layer. For a sufficiently small h, it is possible to use the equidistant discretization for cubic interpolation as well.

Since $\varepsilon \ll 1$, the points $x_j - h\bar{u}(t_k, x_j) \pm \varepsilon h^{1/2}$, $j = 1, \ldots, M-1$, belong to the interval $(-1, 1)$. Thus, we avoid using special formulas near the boundary in (7.6) (see Remark 5.2 as well).

Table 1 gives numerical results obtained by using the algorithms (5.1), (5.2) and (7.6). The algorithm based on the layer method (4.1) and linear interpolation gives results being

TABLE 1 *The Burgers equation. Dependence of the errors* $err^c(t,h)$ *and* $err^l(t,h)$ *in* h *for algorithms* (5.1), (5.2) *and* (7.6) *under* $t = 0{\cdot}5$, $\varepsilon = 0{\cdot}1$ *and* $A = 2$

h	Algorithm (5.1), (5.2)		Algorithm (7.6)	
	$\mathrm{err}^c(t,h)$	$\mathrm{err}^l(t,h)$	$\mathrm{err}^c(t,h)$	$\mathrm{err}^l(t,h)$
0·01	$1{\cdot}239 \cdot 10^{-1}$	$3{\cdot}035 \cdot 10^{-2}$	$1{\cdot}854 \cdot 10^{-1}$	$3{\cdot}081 \cdot 10^{-2}$
0·001 6	$4{\cdot}574 \cdot 10^{-2}$	$5{\cdot}311 \cdot 10^{-3}$	$5{\cdot}855 \cdot 10^{-2}$	$5{\cdot}481 \cdot 10^{-3}$
0·000 1	$2{\cdot}673 \cdot 10^{-3}$	$3{\cdot}288 \cdot 10^{-4}$	$3{\cdot}737 \cdot 10^{-3}$	$3{\cdot}466 \cdot 10^{-4}$
0·000 016	$4{\cdot}261 \cdot 10^{-4}$	$5{\cdot}259 \cdot 10^{-5}$	$5{\cdot}919 \cdot 10^{-4}$	$5{\cdot}527 \cdot 10^{-5}$

practically identical to the ones for (5.1), (5.2). We present the errors of the approximate solutions $\bar{u}$ in the discrete Chebyshov norm and in l^1-norm:

$$\mathrm{err}^c(t,h) = \max_{x_i} |\bar{u}(t,x_i) - u(t,x_i)|,$$
$$\mathrm{err}^l(t,h) = \sum_i |\bar{u}(t,x_i) - u(t,x_i)| \cdot h_x.$$

The algorithms based on linear interpolation require both larger volume of computations per time layer and larger amount of memory than the algorithm (7.6) based on cubic interpolation. For instance, in the considered case the algorithm (5.1), (5.2) with $h = 0{\cdot}0001$ needs $2 \cdot 10^4$ computations of $\bar{u}(t_k, x)$ per layer $t = t_k$ and to store an array of $2 \cdot 10^4$ elements, and the algorithm (7.6) with the same step $h = 0{\cdot}0001$ requires only 380 computations of $\bar{u}(t_k, x)$ per layer and an array of 380 elements (see also Remark 5.2).

7.2 *Comparison analysis*

In this section we give some comparison analysis of the layer methods proposed in the paper and the well known finite-difference schemes (see also Remark 2.1). We use (7.1)–(7.3) as a test problem again. Here we compare the algorithm (7.6) with two explicit finite-difference schemes (7.7) and (7.8) of order $O(\Delta t, \Delta x^2)$, where Δt is a time step and Δx is a space step. These finite-difference schemes are used for simulation of the Burgers equation in Anderson *et al.* (1984) and Fletcher (1991).

The method of differences forward in time and central differences in space applied to the divergent form of the Burgers equation is written as

$$\bar{u}(0,x) = -A \sin \pi x, \quad x \in [-1,1], \tag{7.7}$$
$$\bar{u}(t_{k+1}, x_0) = \bar{u}(t_{k+1}, -1) = 0, \quad \bar{u}(t_{k+1}, x_M) = \bar{u}(t_{k+1}, 1) = 0,$$
$$\bar{u}(t_{k+1}, x_j) = \bar{u}(t_k, x_j) - \frac{\Delta t}{4\Delta x}(\bar{u}^2(t_k, x_{j+1}) - \bar{u}^2(t_k, x_{j-1}))$$
$$+ \frac{\varepsilon^2}{2}\frac{\Delta t}{\Delta x^2}(\bar{u}(t_k, x_{j+1}) - 2\bar{u}(t_k, x_j) + \bar{u}(t_k, x_{j-1})),$$
$$j = 1, \ldots, M-1, \quad k = 0, \ldots, N-1,$$

where the step of a time discretization $\Delta t := T/N$ and $t_k = k \cdot \Delta t$ and the step of space discretization $\Delta x := 2/M$ and $x_j = -1 + j \cdot \Delta x$.

TABLE 2 *The Burgers equation. The relative errors* $\delta^l(t,h)$ *(top position) and* $\delta^c(t,h)$ *(lower position) of algorithm* (7.6) *and finite-difference schemes* (7.7) *and* (7.8) *are given for* $t = 0{\cdot}08$, $\varepsilon = 0{\cdot}1$, $h = \Delta t = 0{\cdot}0016$ *and various A*

A	Algorithm (7.6)	Scheme (7.7)	Scheme (7.8)
5	$7{\cdot}79 \cdot 10^{-3}$	$2{\cdot}22 \cdot 10^{-2}$	$2{\cdot}01 \cdot 10^{-2}$
	$5{\cdot}28 \cdot 10^{-2}$	$2{\cdot}05 \cdot 10^{-1}$	$1{\cdot}66 \cdot 10^{-1}$
		oscillations	oscillations
10	$1{\cdot}87 \cdot 10^{-2}$	$\gg 100$	$2{\cdot}35 \cdot 10^{-2}$
	$9{\cdot}96 \cdot 10^{-1}$	$\gg 100$	$6{\cdot}94 \cdot 10^{-2}$
			oscillations
15	$2{\cdot}70 \cdot 10^{-2}$	overflow	$3{\cdot}64 \cdot 10^{-1}$
	$9{\cdot}84 \cdot 10^{-1}$		$3{\cdot}42 \cdot 10^{0}$
			big oscillations

In the case of the problem (7.1)–(7.3) the Brailovskaya scheme has the form (see Anderson *et al.* (1984, p. 161))

$$\bar{u}(0,x) = -A \sin \pi x, \quad x \in [-1,1], \tag{7.8}$$

$$\bar{u}(t_{k+1},x_0) = \hat{u}(t_{k+1},x_0) = 0, \quad \bar{u}(t_{k+1},x_M) = \hat{u}(t_{k+1},x_M) = 0,$$

$$\hat{u}(t_{k+1},x_j) = \bar{u}(t_k,x_j) - \frac{\Delta t}{4\Delta x}(\bar{u}^2(t_k,x_{j+1}) - \bar{u}^2(t_k,x_{j-1})) + \frac{\varepsilon^2}{2}\frac{\Delta t}{\Delta x^2}(\bar{u}(t_k,x_{j+1}) - 2\bar{u}(t_k,x_j) + \bar{u}(t_k,x_{j-1})),$$

$$\bar{u}(t_{k+1},x_j) = \bar{u}(t_k,x_j) - \frac{\Delta t}{4\Delta x}(\hat{u}^2(t_k,x_{j+1}) - \hat{u}^2(t_k,x_{j-1})) + \frac{\varepsilon^2}{2}\frac{\Delta t}{\Delta x^2}(\bar{u}(t_k,x_{j+1}) - 2\bar{u}(t_k,x_j) + \bar{u}(t_k,x_{j-1})),$$

$$j = 1,\ldots,M-1, \quad k = 0,\ldots,N-1.$$

The space step Δx in the finite-difference schemes (7.7) and (7.8) is selected as $\Delta x = \varkappa \cdot \varepsilon\sqrt{\Delta t}$. The results of Tables 2 and 3 correspond to $\varkappa = 4$.

As in Section 7.1, we realize the algorithm (7.6) using a nonequidistant discretization of the interval $[-1, 1]$. For the time step $h = 0{\cdot}0016$ (Table 2), we take $h_x := x_{j+1} - x_j = \varepsilon\sqrt{h}$ in $[-0{\cdot}1, 0{\cdot}1]$ and $h_x = 2\sqrt{h}$ outside $[-0{\cdot}1, 0{\cdot}1]$. And for $h = 0{\cdot}0001$ (Table 3) we choose $h_x = \varepsilon\sqrt{h}$ in $[-0{\cdot}02, 0{\cdot}02]$ and $h_x = 2\sqrt{h}$ outside $[-0{\cdot}02, 0{\cdot}02]$ (see also the explanations in Section 7.1).

Tables 2 and 3 present the relative errors $\delta^c(t,h)$ and $\delta^l(t,h)$. The error $\delta^c(t,h)$ is equal to

$$\delta^c(t,h) = \frac{\max_{x_i} |\bar{u}(t,x_i) - u(t,x_i)|}{\max_{x_i} |u(t,x_i)|}$$

TABLE 3 *The Burgers equation. The relative errors* $\delta^l(t,h)$ (*top position*) *and* $\delta^c(t,h)$ (*lower position*) *of algorithm* (7.6) *and finite-difference schemes* (7.7) *and* (7.8) *are given for* $h = \Delta t = 0{\cdot}0001$*; the other parameters are as in Table* 2

A	Algorithm (7.6)	Scheme (7.7)	Scheme (7.8)
5	$1{\cdot}26 \cdot 10^{-3}$	$8{\cdot}88 \cdot 10^{-4}$	$7{\cdot}81 \cdot 10^{-4}$
	$4{\cdot}68 \cdot 10^{-2}$	$1{\cdot}55 \cdot 10^{-2}$	$1{\cdot}10 \cdot 10^{-2}$
10	$1{\cdot}24 \cdot 10^{-3}$	$3{\cdot}56 \cdot 10^{-3}$	$3{\cdot}58 \cdot 10^{-3}$
	$9{\cdot}25 \cdot 10^{-2}$	$1{\cdot}54 \cdot 10^{-1}$	$1{\cdot}54 \cdot 10^{-1}$
		oscillations	oscillations
15	$1{\cdot}91 \cdot 10^{-3}$	$5{\cdot}07 \cdot 10^{-3}$	$5{\cdot}11 \cdot 10^{-3}$
	$1{\cdot}99 \cdot 10^{-1}$	$1{\cdot}81 \cdot 10^{-1}$	$1{\cdot}84 \cdot 10^{-1}$
		oscillations	oscillations

for all three methods. The error $\delta^l(t,h)$ is equal to

$$\delta^l(t,h) = \frac{1}{\max_{x_i} |u(t,x_i)|} \sum_i |\bar{u}(t,x_i) - u(t,x_i)| \cdot h_x$$

for algorithm (7.6) while for the schemes (7.7) and (7.8) it is given by

$$\delta^l(t,h) = \frac{1}{\max_{x_i} |u(t,x_i)|} \sum_i |\bar{u}(t,x_i) - u(t,x_i)| \cdot \Delta x.$$

The comment 'oscillations' means that the numerical solution has oscillations in a neighbourhood of $x = 0$. An illustration of such oscillations is given in Fig. 2. The comment 'overflow' indicates that overflow error occurs during simulation.

Let us observe that if we take $\varkappa = 2$ in order to improve accuracy of the results obtained by the Brailovskaya scheme (7.8), for example, for $\Delta t = 0{\cdot}0016$, $A = 10$ (see Table 2 and Fig. 2), the numerical solution becomes more unstable and overflow error occurs. If we take $\varkappa = 8$ in this case, the errors and amplitude of oscillations become greater than for $\varkappa = 4$.

We can conclude from the results presented in Tables 2 and 3 and in Fig. 2 that the algorithm (7.6) based on the layer method demonstrates a more stable behavior than the finite-difference schemes when the parameter A is sufficiently large. In the considered problem (7.1)–(7.3) large values of A lead, in particular, to large advection in a neighborhood of $x = 0$. Our experiments confirm that the layer methods allow us to avoid difficulties stemming from strong advection (see Introduction and Remark 2.1). It should also be mentioned that the algorithms based on layer methods require more CPU time than finite-difference schemes. For example, in the case of parameters as in Table 3 to solve (7.1)–(7.3) by the algorithm (7.6) we need $\approx$2 s while the schemes (7.7) and (7.8) require $\approx 0{\cdot}3$ and $\approx 0{\cdot}4$ s correspondingly. But the algorithm (7.6) gives us quite appropriate results with the greater step $h = 0{\cdot}0016$ (see Table 2 and Fig. 2) and in this case it requires $\approx 0{\cdot}06$ s. Simulations were made on PC with Intel Pentium 233 MHz processor using Borland C compiler. A more detailed comparison analysis requires a special consideration and will be done in a later publication.

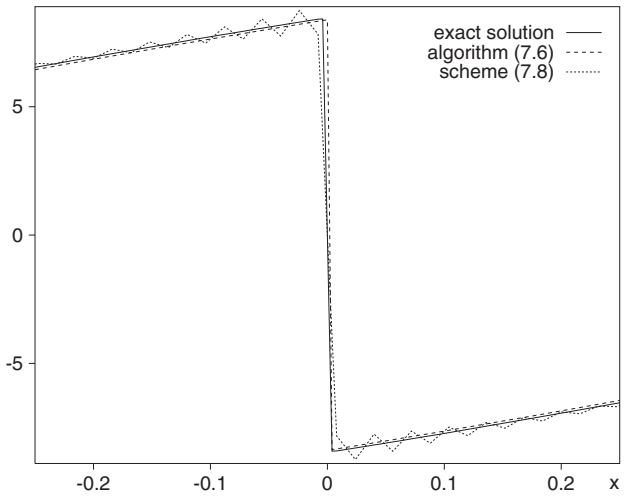

FIG. 2. Solution of problem (7.1)–(7.3) for $A = 10$, other parameters are as in Table 2.

7.3 *Quasilinear equation with power law nonlinearities*

Consider the Dirichlet problem for quasilinear parabolic equation with power law nonlinearities (Samarskii, 1977; Samarskii *et al.*, 1995):

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}\left(u^q \frac{\partial u}{\partial x}\right),\ t \in (0,1),\ x > 0,\ q > 0, \tag{7.9}$$

with the initial condition

$$\begin{aligned} u(0,x) &= (1 - x/L)^{2/q},\ x \in [0, L], \\ u(0,x) &= 0,\ x > L, \end{aligned} \tag{7.10}$$

and the boundary regime

$$u(t,0) = (1-t)^{-1/q},\ t \in (0,1), \tag{7.11}$$

where $L = \sqrt{(q+2)/q}$.

The exact solution to this problem has the form (Samarskii, 1977; Samarskii *et al.*, 1995)

$$u(t,x) = \left(\frac{1 - x/L}{\sqrt{1-t}}\right)^{2/q} \qquad \text{for } x \in [0, L]$$

and

$$u(t,x) = 0 \qquad \text{for } x > L.$$

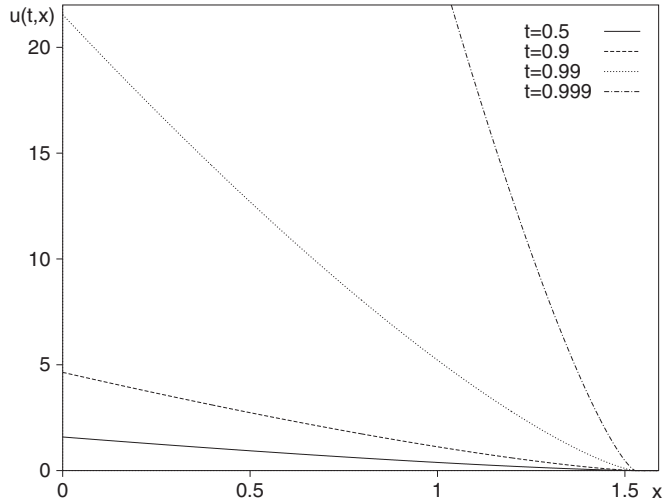

FIG. 3. A typical solution $u(t, x)$ of problem (7.9)–(7.11) for $q = 1{\cdot}5$ and various time moments.

The temperature $u(t, x)$ grows infinitely as $t \to 1$. At the same time the heat remains localized in the interval $[0, L)$. Figure 3 presents a typical behavior of the solution to (7.9)–(7.11).

Equation (7.9) is not of the form (2.1). The function

$$v = u^{q+1}$$

satisfies the problem

$$\frac{\partial v}{\partial t} = \frac{1}{2} v^{q/(q+1)} \frac{\partial^2 v}{\partial x^2}, \ t \in (0, 1), \ x > 0, \tag{7.12}$$

$$v(0, x) = (1 - x/L)^{2(q+1)/q}, \ x \in [0, L], \tag{7.13}$$

$$v(0, x) = 0, \ x > L,$$

$$v(t, 0) = (1 - t)^{-(q+1)/q}, \ t \in (0, 1). \tag{7.14}$$

Equation (7.12) has the form (2.1).

We simulate the solution to (7.12)–(7.14) by two algorithms: algorithm (5.1), (5.2) and the algorithm based on the layer method (4.1) and linear interpolation. The last one in the case of the problem (7.12)–(7.14) has the form

$$\bar{v}(0, x) = \begin{cases} (1 - x/L)^{2(q+1)/q}, & x \in [0, L], \\ 0, & x \in (L, \infty), \end{cases} \tag{7.15}$$

TABLE 4 *Quasilinear equation with power law nonlinearities. Dependence of errors* $\mathrm{err}_{\bar v}(t,h)$ *(top position) and* $\mathrm{err}_{\bar u}(t,h)$ *(lower position) in h and t for algorithm* (7.15), (7.16) *under* $q = 1{\cdot}5$

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$
$t = 0{\cdot}5$	$0{\cdot}8664 \cdot 10^{-1}$	$0{\cdot}8786 \cdot 10^{-2}$	$0{\cdot}9705 \cdot 10^{-3}$	$1{\cdot}018 \cdot 10^{-4}$
	$0{\cdot}3542 \cdot 10^{-1}$	$0{\cdot}7693 \cdot 10^{-2}$	$1{\cdot}685 \cdot 10^{-3}$	$3{\cdot}622 \cdot 10^{-4}$
$t = 0{\cdot}9$	>5	$8{\cdot}094 \cdot 10^{-1}$	$8{\cdot}265 \cdot 10^{-2}$	$8{\cdot}817 \cdot 10^{-3}$
	$5{\cdot}910 \cdot 10^{-1}$	$8{\cdot}109 \cdot 10^{-2}$	$8{\cdot}656 \cdot 10^{-3}$	$8{\cdot}918 \cdot 10^{-4}$

$$\bar v(t_{k+1},x_j) = \tfrac12 \bar v\big(t_k, x_j - (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}\big) + \tfrac12 \bar v\big(t_k, x_j + (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}\big),$$
$$\text{if } x_j - (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h} \geqslant 0;$$
$$\bar v(t_{k+1},x_j) = \frac{1}{1+\sqrt{\bar\lambda_k}} (1 - t_{k+1-\bar\lambda_k})^{-(q+1)/q}$$
$$+ \frac{\sqrt{\bar\lambda_k}}{1+\sqrt{\bar\lambda_k}} \bar v\big(t_k, x_j + (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}\big),$$
$$\bar\lambda_k = \left(\frac{x_j}{(\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}}\right)^2, \qquad \text{if } x_j - (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h} < 0;$$

$$\bar v(t_{k+1},x) = \frac{x_{j+1}-x}{h_x}\, \bar v(t_{k+1},x_j) + \frac{x-x_j}{h_x}\, \bar v(t_{k+1},x_{j+1}),\ x_j \leqslant x \leqslant x_{j+1}, \tag{7.16}$$
$$j = 0,1,2,\ldots,\ k = 1,\ldots,N,$$

where $x_j = j \cdot h_x$, $t_k = k \cdot h$.

In our tests we take $h_x = h$. Tables 4 and 5 give numerical results obtained by using algorithm (7.15), (7.16). The algorithm (5.1), (5.2) gives similar results and they are omitted here. Table 4 presents the errors

$$\mathrm{err}_{\bar v}(t,h) := \max_j |\bar v(t,x_j) - v(t,x_j)|,$$
$$\mathrm{err}_{\bar u}(t,h) := \max_j |\bar u(t,x_j) - u(t,x_j)|,\ \bar u(t,x_j) = (\bar v(t,x_j))^{1/(q+1)}.$$

For times t which are close to the explosion time $t = 1$, the functions $u(t,x)$ and $v(t,x)$ take big values and the absolute errors become fairly large. In Table 5 we present the relative error

$$\delta(t,h) := \frac{\mathrm{err}_{\bar u}(t,h)}{u(t,0)}$$

at times close to the explosion.

In the experiments, the tested algorithms converge as $O(h)$ which is in complete agreement with our theoretical results.

TABLE 5 *Quasilinear equation with power law nonlinearities. Dependence of the relative error* $\delta(t, h)$ *in* h *and* t *for algorithm* (7.15), (7.16) *under* $q = 1{\cdot}5$

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$
$t = 0{\cdot}9$	$1{\cdot}273 \cdot 10^{-1}$	$1{\cdot}747 \cdot 10^{-2}$	$1{\cdot}865 \cdot 10^{-3}$	$1{\cdot}921 \cdot 10^{-4}$
$t = 0{\cdot}99$	—	$1{\cdot}392 \cdot 10^{-1}$	$1{\cdot}789 \cdot 10^{-2}$	$1{\cdot}913 \cdot 10^{-3}$
$t = 0{\cdot}999$	—	—	$1{\cdot}398 \cdot 10^{-1}$	$1{\cdot}801 \cdot 10^{-2}$
$t = 0{\cdot}9999$	—	—	—	$1{\cdot}400 \cdot 10^{-1}$

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