Walking forward and backward in Euler schemes and random number generators

emmanuel.gobet@polytechnique.edu

In the memory of
Professor Grigori N. Milstein
(1937-2023)

Based on joint works with P. Cohort (Zeliade systems) and M. Mrad (University Paris Nord).

1 Reversing a Random Number Generator

1.1 Application/motivation

Example: Regression Monte-Carlo with low memory constraints

Dynamic programming equation (DPE) with N times:

$$\mathbf{Y_i} = \mathbb{E}\left[\mathbf{g_i}(\mathbf{Y_{i+1}}, \dots, \mathbf{Y_N}, \mathbf{X_i}, \dots, \mathbf{X_N}) \mid \mathbf{X_i}\right], \quad \mathbf{i} = \mathbf{N} - \mathbf{1}, \dots, \mathbf{0},$$

$$\mathbf{Y_N} = \mathbf{g_N}(\mathbf{X_N}),$$

We want to estimate the function y_i such that $Y_i = y_i(X_i)$.

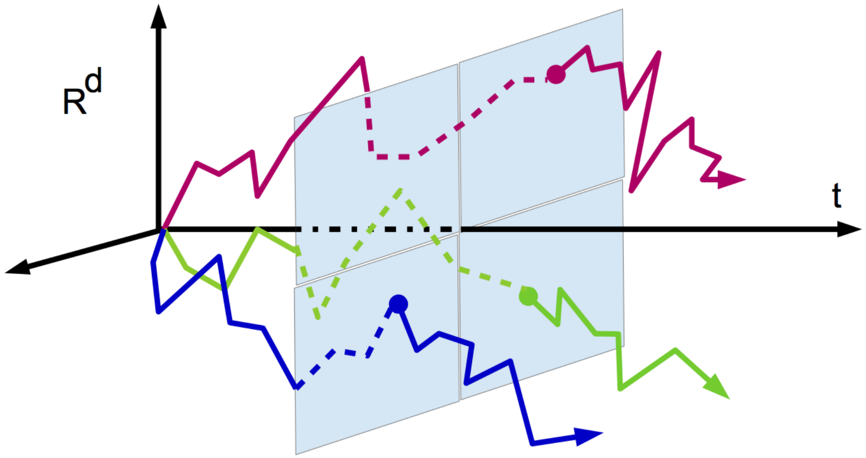
✓ **Optimal Stopping:**

- Value function: $\mathbf{V_i} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{\mathbf{i},\mathbf{N}}} \mathbb{E}\left[\mathbf{g}_\tau(\mathbf{X}_\tau) \mid \mathbf{X_i}\right].$
- Continuation value: $\mathbf{Y_i} = \mathbb{E}\left[\mathbf{V_{i+1}} | \mathbf{X_i}\right]$ solves a DPE.

✓ **Semi-linear equations and Backward Stochastic differential equations also solves approximate DPE:**

- Non-linear PDE: $\partial_\mathbf{t} \mathbf{u} + \mathcal{L}\mathbf{u} + \mathbf{f}(\mathbf{u}, \sigma \nabla \mathbf{u}) = \mathbf{0}$, $u(1, .) = g(.)$.

Regression Monte-Carlo: [Longstaff-Schwartz '01, Belomestny-Milstein-Spokoiny '09, G'-Turkedejiev '16] value functions are computed using M simulations, and K basis functions (or Neural Networks with K parameters ...)



▷ **Convergence of the statistical error:** must have $M \gg K$.
More precisely, $\mathbf{M \sim KN^3}$

▷ **Convergence of the approximation error:** $K \sim N^{\alpha d}$

▷ **Memory constraints:**

✓ M paths of length N: MN

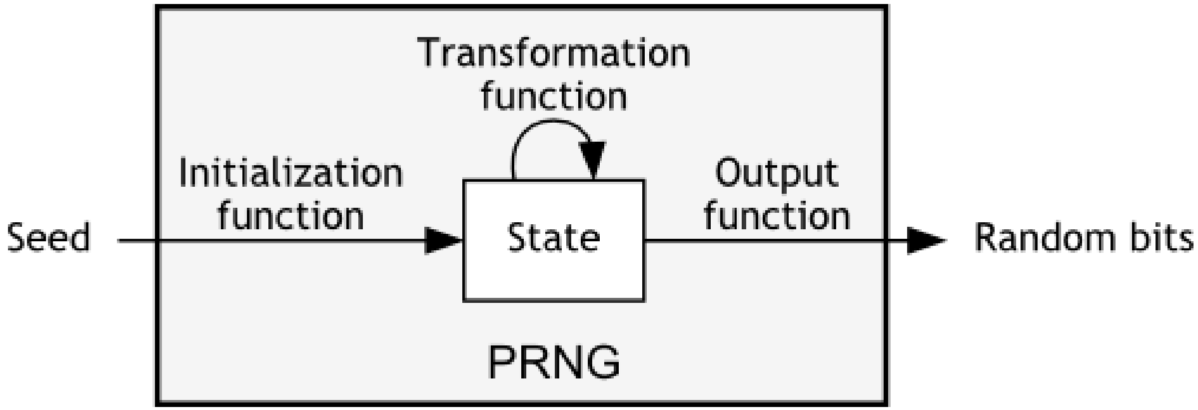
✓ N value functions: KN

In **[Aïd-Campi-Langrené-Pham '14]**, optimal investment in renewable energy.
Time steps: 40 years and 2 decisions per day ➠ $N = 40 \times 2 \times 365 \approx 30000$!!

➠ Impossible to store the simulations

➠ Sequential learning? or resampling data?

1.2 What is a Pseudo-RNG?



Schematic diagram of a pseudo-random number generator

Credits: http://pit-claudel.fr/clement/

Examples.

✓ *Linear congruential generator*: $\mathbf{x_{m+1} = a x_m + b \text{ MOD } L}, \quad \mathbf{u_m = \frac{x_m}{L}}$.
Ex: $a = 7^5$, $b = 0$ et $L = 2^{31} - 1 = 2147483647$.

✓ *Mersenne Twister (1997)*: period $2^{19937} - 1$

✓ *Xorshift (2003)*: period $2^{1024} - 1$ (simple and fast)

1.3 Reversing the RNG

PRNG sequence: $\mathbf{u_i = h(G^i(v_0))}$ for some output function h, transformation function G and initialization state v_0.

Is it possible to invert the sequence

$$u_0 \curvearrowright u_1 \curvearrowright \dots \curvearrowright u_n \qquad \text{to} \qquad u_n \curvearrowright u_1 \curvearrowright \dots \curvearrowright u_0?$$

▷ A classical example : the L'Ecuyer generator

Classical example among the Multiple Recursive family **[L'Ecuyer '88]**

Transformation function G: defined by

$$G(x, y) := \begin{cases} 40014\,x \text{ MOD } 2147483563 \\ 40692\,y \text{ MOD } 2147483399 \end{cases}$$

Generator output: $h(x, y) = (x - y)/2147483563 \text{ MOD } 1.$

$m_1 := 2147483563$ and $m_2 := 2147483399$ are prime numbers ➠ G is invertible and G^{-1} is of the form of G (same computational complexity).

▷ An up-to-date example : the WELL generators

WELL = *Well Equidistributed Long-period Linear*

Belong to family of Linear Recurrence Modulo 2 generators **[Panneton-L'Ecuyer '06]**.

Transformation function: based on linear recursions $x_{n+1} = Ax_n$, for a binary matrix A.

Output function: $h(x_n) := Bx_n$ for a binary matrix B.

Properties.

- ✓ Generator reversible if A is invertible.
- ✓ G and G^{-1} have similar complexity.
- ✓ Among the 17 WELL generators of **[Panneton-L'Ecuyer '06]**, we found 11 invertible generators (e.g. WELL607a, 19937a, 23209b and 44497a)

2 Euler schemes: forward and backward (v1.0)

IDEA: inverting the RNG allows to backward resimulate the Brownian increments

No need anymore to store them

What about the SDE?

2.1 Forward scheme

$\mathbb{R}^d$-valued diffusion on $[0,1]$: $\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t$.

Forward Euler scheme at times $t_i = i/N$:

$$\overrightarrow{X}^{\mathbf{N}}_{\mathbf{0}} = \mathbf{x}, \quad \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}} = \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} + \mu(\mathbf{t_i}, \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}})\frac{\mathbf{1}}{\mathbf{N}} + \sigma(\mathbf{t_i}, \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}})\mathbf{\Delta W_{t_i}}.$$

Theorem. Under Lipschitz assumptions,

$$\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \left\| \sup_{|\mathbf{x}|\leq\lambda} |\mathbf{X_t}(\mathbf{x}) - \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t}}(\mathbf{x})| \right\|_{\mathbb{L}_\mathbf{q}} \leq \frac{\mathbf{C}}{\mathbf{N^{1/2}}}\lambda^{\mathbf{2}}, \quad \forall\lambda \geq \mathbf{1}.$$

2.2 Backward scheme

Techniques of stochastic flows [Kunita '97]

✓ **SDE** $X_{\cdot}(s,x)$ solution from initial condition $x \in \mathbb{R}$ at time $s > 0$.

✓ Inverse flow $\xi_{s,t}(x) := (X_{s,t})^{-1}(x)$.

Theorem ([Kunita '97]). ξ is a semimartingale with **respect to** t for any (s,x):

$$\begin{cases} d\xi_{s,t}(x) &= -\mathrm{Jac}(\xi_{s,t}(x))\big[\mu(t,x) - \sum_j \big(\partial_j\sigma(t,x)\big).\sigma^j(t,x)\big]dt + \sigma(t,x)dW_t \\ &+ \frac{1}{2}\sum_{i,j}\partial^2_{ij}\xi_{s,t}(x)\sigma^i(t,x).\sigma^j(t,x)dt, \\ \xi_{s,s}(x) &= x. \end{cases}$$

ξ is a semimartingale with **respect to** s for any (t,x):

$$\begin{cases} d\xi_{s,t}(x) &= -\big[\mu(s,\xi_{s,t}(x)) - \sum_j \big(\partial_j\sigma(s,\xi_{s,t}(x))\big).\sigma^j(s,\xi_{s,t}(x))\big]ds \\ &-\sigma(s,\xi_{s,t}(x))d\overleftarrow{W}_s, \\ \xi_{t,t}(x) &= x. \end{cases}$$

Definition (Backward Euler scheme (v1.0)).

$$\overleftarrow{X}^{\mathbf{N}}_{\mathbf{1}} = \overrightarrow{X}^{\mathbf{N}}_{\mathbf{1}},$$

$$\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} = \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}} - \mu(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\frac{\mathbf{1}}{\mathbf{N}} - \sigma(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\boldsymbol{\Delta}\mathbf{W}_{\mathbf{t_i}}$$

$$+ \sum_{\mathbf{j}} \left(\partial_{\mathbf{j}}\sigma(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\right).\sigma^{\mathbf{j}}(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\frac{\mathbf{1}}{\mathbf{N}}.$$

- 🙂 Easy to simulate using backward regeneration of Brownian increments
- 🙂 Avoid to invert the forward Euler transform

$$x \mapsto \overrightarrow{X}^{N}_{t_{i+1}} = x + \mu(t_i, x)\frac{1}{N} + \sigma(t_i, x)\Delta W_{t_i}.$$

- ✓ Convergence rate for $\overleftarrow{X}^{N}_{t_i} - X_{t_i}$? or for $\overleftarrow{X}^{N}_{t_i} - \overrightarrow{X}^{N}_{t_i}$?
- ☹ Non standard analysis: composition of dependent stochastic maps

2.3 Approximation of compound maps $F(\Theta)$

- ✓ a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $(\omega, x) \in (\Omega, \mathbb{R}^d) \mapsto F(\omega, x) \in \mathbb{R}^{d'}$, continuous in x for a.e. ω
- ✓ $\Theta : \Omega \mapsto \mathbb{R}^d$ be a $\mathcal{F}$-random variable.

Our aim: control in $\mathbb{L}_q$ the error $\omega \in \Omega \mapsto F^N(\omega, \Theta^N(\omega)) - F(\omega, \Theta(\omega))$

In our case:

- ✓ $\Theta^N = \overrightarrow{X}_1^N$ (forward evolution)
- ✓ $F^N = \overleftarrow{X}_{t_i}^N(y)$ (backward evolution from y at time 1)
- ✓ $F(\Theta) = X_{t_i}$

General result for approximating $F(\Theta)$

(H1) For any $q > 0$, $\exists\ \alpha_q^{(\mathbf{H1})} \geq 0$ s.t.

$$\sup_{\lambda \geq 1} \lambda^{-\alpha_q^{(\mathbf{H1})}} \left\| \sup_{|x| \leq \lambda} |F(\cdot, x)| \right\|_{\mathbb{L}_q} < +\infty.$$

(H2) $\exists\ \kappa \in (0, 1]$ such that $\forall\ q > 0$, $\exists\ \alpha_q^{(\mathbf{H2})} \geq 0$ s.t.

$$\sup_{\lambda \geq 1} \lambda^{-\alpha_q^{(\mathbf{H2})}} \left\| \sup_{x \neq y, |x| \leq \lambda, |y| \leq \lambda} \frac{|F(\cdot, y) - F(\cdot, x)|}{|y - x|^{\kappa}} \right\|_{\mathbb{L}_q} < +\infty.$$

(H3) For any $q > 0$, $\exists\ \alpha_q^{(\mathbf{H3})} \geq 0$ and a non-negative sequence $(\varepsilon_q^{N,(\mathbf{H3})})_{N \geq 1}$ s.t.

$$\sup_{\lambda \geq 1} \lambda^{-\alpha_q^{(\mathbf{H3})}} \left\| \sup_{|x| \leq \lambda} |F^N(\cdot, x) - F(\cdot, x)| \right\|_{\mathbb{L}_q} \leq \varepsilon_{\mathbf{q}}^{\mathbf{N},(\mathbf{H3})}, \quad \forall N \geq 1.$$

(H4) For any $q > 0$, there exists a non-negative sequence $(\varepsilon_q^{N,(\mathbf{H4b})})_{N\geq 1}$ s.t.

$$\sup_{N\geq 1}\left[\|\Theta\|_{\mathbb{L}_q} \vee \|\Theta^N\|_{\mathbb{L}_q}\right] < +\infty,$$

$$\|\Theta^N - \Theta\|_{\mathbb{L}_q} \leq \varepsilon_{\mathbf{q}}^{\mathbf{N},(\mathbf{H4b})}, \quad \forall N \geq 1.$$

Theorem (general result). Assume (**H1-H2-H3-H4**). Then for any $q > 0$ and any $q_2 > q$, there is a constant c independent on N such that

$$\|\mathbf{F}^{\mathbf{N}}(\boldsymbol{\Theta}^{\mathbf{N}}) - \mathbf{F}(\boldsymbol{\Theta})\|_{\mathbb{L}_{\mathbf{q}}} \leq \mathbf{c}\left(\varepsilon_{\mathbf{2q}}^{\mathbf{N},(\mathbf{H3})} + [\varepsilon_{\kappa\mathbf{q_2}}^{\mathbf{N},(\mathbf{H4b})}]^{\kappa}\right).$$

Corollary (rule of thumb). If

- ✓ $F_N - F$ "=" $O(N^{-\gamma_F})$ in any L_q,
- ✓ $\Theta^N - \Theta = O(N^{-\gamma_\theta})$ in any L_q,

the order of L_q-convergence of $F^N(\Theta^N) - F(\Theta)$ is $\gamma_{\mathbf{F}} \wedge (\kappa\gamma_\theta)$.

Application to backward Euler scheme

Corollary (Backward Euler vs forward diffusion). For any $q \geq 1$,

$$\sup_{\mathbf{t_i} \leq \mathbf{1}} \left\| \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} - \mathbf{X}_{\mathbf{t_i}} \right\|_{\mathbb{L}_{\mathbf{q}}} = \mathbf{O}(\mathbf{N}^{-\mathbf{1/2}}).$$

One can not improve the convergence order $1/2$.

Corollary (Backward Euler vs forward Euler). For any $q \geq 1$,

$$\sup_{\mathbf{t_i} \leq \mathbf{1}} \left\| \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} - \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} \right\|_{\mathbb{L}_{\mathbf{q}}} = \mathbf{O}(\mathbf{N}^{-\mathbf{1/2}}).$$

- ✓ Can we improve the accuracy in retrieving $\overrightarrow{X}^{N}_{t_i}$ by backward generation?
- ✓ Rate N^{-1}?
- ☺ It would mean that the backward generation brings a smaller error than the forward generation.

3 Backward Euler scheme (v2.0)

3.1 Definition

$$\overleftarrow{X}^{\mathbf{N}}_{\mathbf{1}} = \overrightarrow{X}^{\mathbf{N}}_{\mathbf{1}},$$

$$\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} = \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}} - \mu(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\frac{\mathbf{1}}{\mathbf{N}} - \sigma(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\mathbf{\Delta W_{t_i}}$$
$$+ \mathrm{Jac}\big(\sigma(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\mathbf{\Delta W_{t_i}}\big)\sigma(\mathbf{t_{i+1}}, \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\mathbf{\Delta W_{t_i}}.$$

In (v1.0), the blue term was $\sum_j \big(\partial_j \sigma(t_{i+1}, \overleftarrow{X}^N_{t_{i+1}})\big).\sigma^j(t_{i+1}, \overleftarrow{X}^N_{t_{i+1}})\frac{1}{N}$.

3.2 Convergence rate and CLT

Theorem (a.s. error). For any $\eta > 0$, $\exists C_\eta : \Omega \to \mathbb{R}^+$ s.t.

$$\sup_{\mathbf{t_i} \le \mathbf{1}} \left| \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} - \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} \right| \le \mathbf{C}_\eta \mathbf{N}^{-\mathbf{1}+\eta}, \qquad \forall \mathbf{N} \ge \mathbf{1}, \mathbf{a.s.}.$$

Theorem (Functional CLT). For some processes a and b and an extra Brownian motion B, the error $\mathcal{E}^N_t = \overleftarrow{X}^N_t - \overrightarrow{X}^N_t$ satisfies

$$\left\{\mathbf{N}\mathcal{E}^{\mathbf{N}}_{\mathbf{t}} : \mathbf{0} \le \mathbf{t} \le \mathbf{1}\right\} \overset{\mathbf{d}}{\Longrightarrow} \int_\cdot^{\mathbf{1}} \mathbf{a_s} \mathrm{d}\mathbf{s} + \int_\cdot^{\mathbf{1}} \mathbf{b_s} \mathbf{dB_s}.$$

Sketch of proof: error $\mathcal{E}_k = \overleftarrow{X}^N_{t_k} - \overrightarrow{X}^N_{t_k}$

Proposition. For any $1 \le i \le d$ and any integer $k \le N-1$, the i-th coordinate of the error approximation satisfies

$$\mathcal{E}^i_k = \mathcal{E}^i_{k+1} + \bar{a}^i_k \mathcal{E}_{k+1} + \mathcal{E}^\top_{k+1} b^i_k \mathcal{E}_{k+1} + \mathcal{E}^\top_{k+1} \Big(\sum_{j=1}^d c^{i,j}_k \mathcal{E}^j_{k+1} \Big) \mathcal{E}_{k+1}$$
$$+ d^i_k (\Delta W^3_k) + e^i_k (\Delta W^1_k) \Delta t + f^i_k (\Delta W^4_k) + g^i_k (\Delta W^2_k) \Delta t + h^i_k \Delta t^2 + \delta^i_k,$$

where

$$\sup_{0 \le k \le N-1} \left\| \delta^i_k \right\|_{\mathbb{L}_q} = O(N^{-5/2}), \qquad \forall q \ge 1,$$

and

$$
\begin{cases}
b_k^i & := \frac{1}{2}\Big[\mathcal{H}\big(\nabla(\sigma^i\Delta W_k)(\sigma\Delta W_k)\big) - \mathcal{H}(\mu^i)\Delta t - \mathcal{H}(\sigma^i\Delta W_k) \\
& \qquad - \sum_{j=1}^{d}\partial_j\mathcal{H}(\sigma^i\Delta W_k)(\sigma^j\Delta W_k)\Big], \\
c_k^{i,j} & := -\frac{1}{6}\partial_j\mathcal{H}(\sigma^i\Delta W_k), \\
\bar{a}_k^i & := \nabla[\nabla(\sigma^i\Delta W_k)(\sigma\Delta W_k)] - \nabla\mu^i\Delta t - \nabla(\sigma^i\Delta W_k) \\
& \quad +(\mu\Delta t+\sigma\Delta W_k)^\top\Big(\mathcal{H}\big(\nabla(\sigma^i\Delta W_k)(\sigma\Delta W_k)\big) - \mathcal{H}(\mu^i)\Delta t - \mathcal{H}(\sigma^i\Delta W_k)\Big) \\
& \quad -\frac{1}{2}(\sigma\Delta W_k)^\top\Big(\sum_{j=1}^d\partial_j\mathcal{H}(\sigma^i\Delta W_k)(\sigma^j\Delta W_k)\Big), \quad \text{for } 1\le i\le d \\
d_k(\Delta W_k^3) & := \Big[\Big(\nabla[\nabla(\sigma^i\Delta W_k)(\sigma\Delta W_k)] - \frac{1}{2}(\sigma\Delta W_k)^\top\mathcal{H}(\sigma^i\Delta W_k)\Big)(\sigma\Delta W_k)\Big]_{1\le i\le d}, \\
e_k(\Delta W_k^1) & := -\nabla\mu(\sigma\Delta W_k) - \nabla(\sigma\Delta W_k)\mu, \\
f_k(\Delta W_k^4) & := \Big[(\sigma\Delta W_k)^\top\Big(\frac{1}{2}\mathcal{H}\big(\nabla(\sigma^i\Delta W_k)(\sigma\Delta W_k)\big) \\
& \qquad -\frac{1}{6}\sum_{j=1}^d\partial_j\mathcal{H}(\sigma^i\Delta W_k)(\sigma^j\Delta W_k)\Big)(\sigma\Delta W_k)\Big]_{1\le i\le d}, \\
g_k(\Delta W_k^2) & := \Big[\Big(\nabla[\nabla(\sigma^i\Delta W_k)(\sigma\Delta W_k)]\mu - \frac{1}{2}(\sigma\Delta W_k)^\top\mathcal{H}(\mu^i)(\sigma\Delta W_k) \\
& \quad -(\sigma\Delta W_k)^\top\mathcal{H}(\sigma^i\Delta W_k)\mu\Big)\Big]_{1\le i\le d}, \\
h_k & := \big[-\nabla(\mu^i)\mu\big]_{1\le i\le d}.
\end{cases}
$$

Propagating errors

$$\mathcal{E}_k^i = \mathcal{E}_{k+1}^i + \bar{a}_k^i \mathcal{E}_{k+1} + \mathcal{E}_{k+1}^\top b_k^i \mathcal{E}_{k+1} + \mathcal{E}_{k+1}^\top \Big(\sum_{j=1}^d c_k^{i,j} \mathcal{E}_{k+1}^j \Big) \mathcal{E}_{k+1}$$
$$+ d_k^i (\Delta W_k^3) + e_k^i (\Delta W_k^1) \Delta t + f_k^i (\Delta W_k^4) + g_k^i (\Delta W_k^2) \Delta t + h_k^i \Delta t^2 + \delta_k^i.$$

- Initialization: $\mathcal{E}_N = 0$
- Local errors: using martingale arguments, we have

$$\sum_k \left(d_k^i (\Delta W_k^3) + e_k^i (\Delta W_k^1) \Delta t + f_k^i (\Delta W_k^4) + g_k^i (\Delta W_k^2) \Delta t + h_k^i \Delta t^2 + \delta_k^i \right)$$
$$= O_{\mathbb{L}_q}(N^{-1})$$

- Error propagation: application of Gronwall lemma does not work because of the quadratic/cubic term in $\mathcal{E}_{k+1}$

Instead,

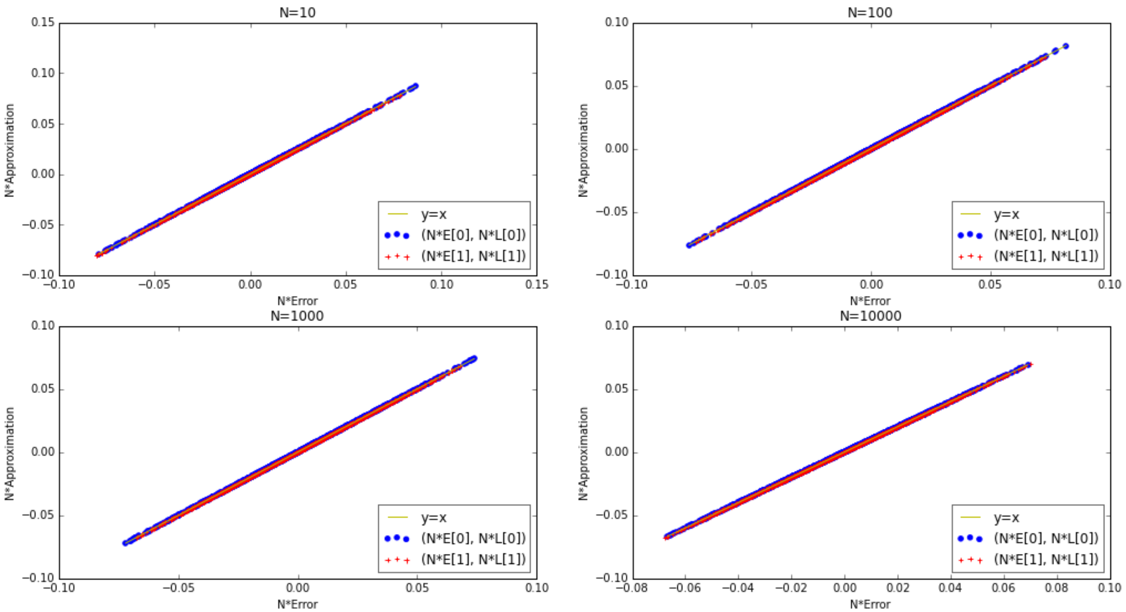
1. Write carefully $\mathcal{E}_k$ with local errors between $k+1$ and N
2. Use that
$$\sup_{t_i \le 1} \|\mathcal{E}_i\|_{\mathbb{L}_q} = O(N^{-1/2}), \qquad \forall q \ge 1,$$
to get (Borel Cantelli)
$$\sup_{t_i \le 1} |\mathcal{E}_i| = O(N^{-1/2+\eta}) \quad a.s. \quad \text{for any given } \eta > 0.$$
3. Plug these a.s. bounds into the one-step error equations to get
$$\sup_{t_i \le 1} |\mathcal{E}_i| = O(N^{-\kappa}) \quad a.s. \quad \text{for a } \kappa \in (1/2, 1).$$
4. Iterate to get $$\sup_{t_i \le 1} |\mathcal{E}_i| = O(N^{-1+\eta}) \quad a.s.$$
5. Finally, prove
$$\sup_{t_i \le 1} |\mathcal{E}_i - L_i| = O(N^{-\kappa'}) \quad a.s.$$
for a $\kappa' \in (1, 3/2)$, where L_i gives the weak-limit at rate N.

3.3 Numerical results

- ✓ 10000 paths
- ✓ empirical joint distribution of $(N\mathcal{E}_0^N, NL_0^N)$
- ✓ examples in dimension 2

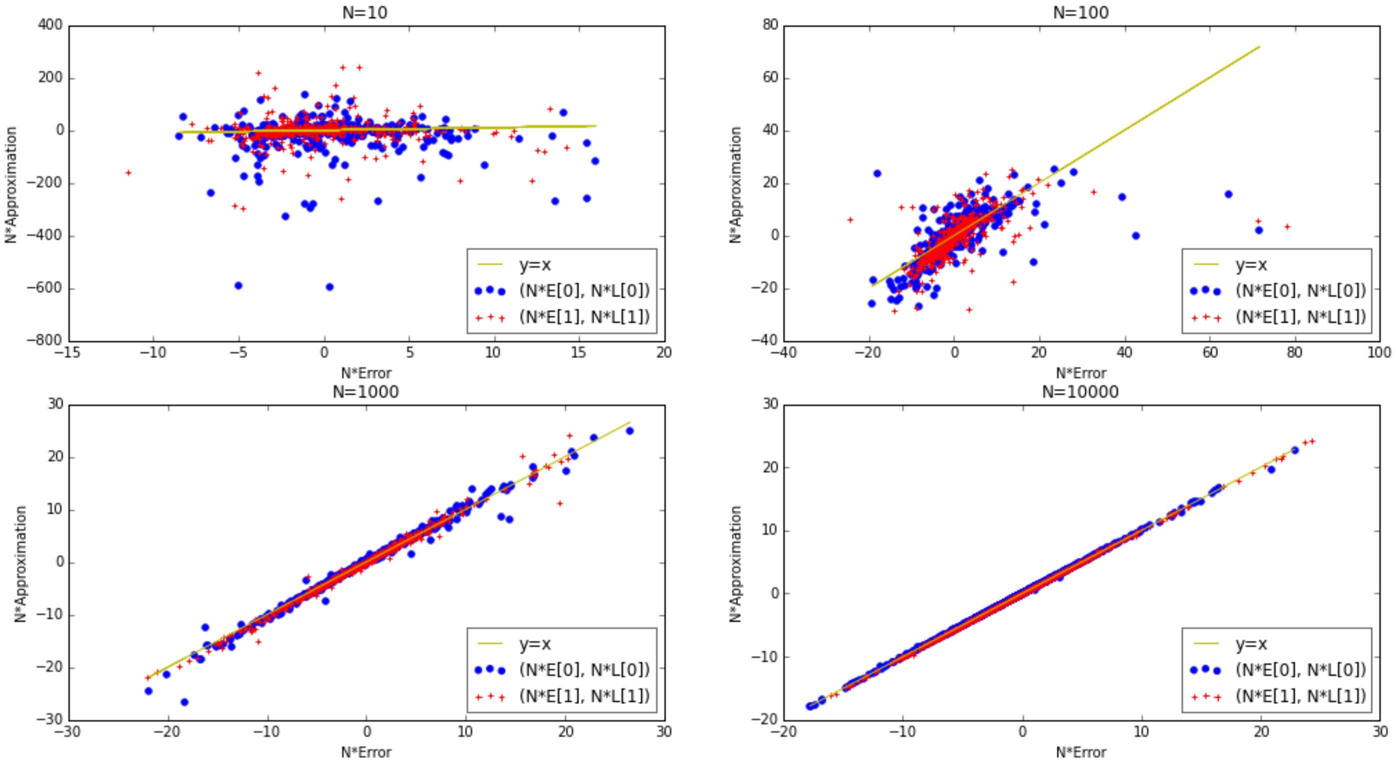
Linear SDE: standard non correlated two-dimensional Black&Scholes case

$$\mu(x) = (0,0), \qquad \sigma(x) = \begin{pmatrix} 0.2\,x_0 & 0 \\ 0 & 0.2\,x_1 \end{pmatrix}, \qquad x_0 = (1,1).$$



Non linear SDE 1:

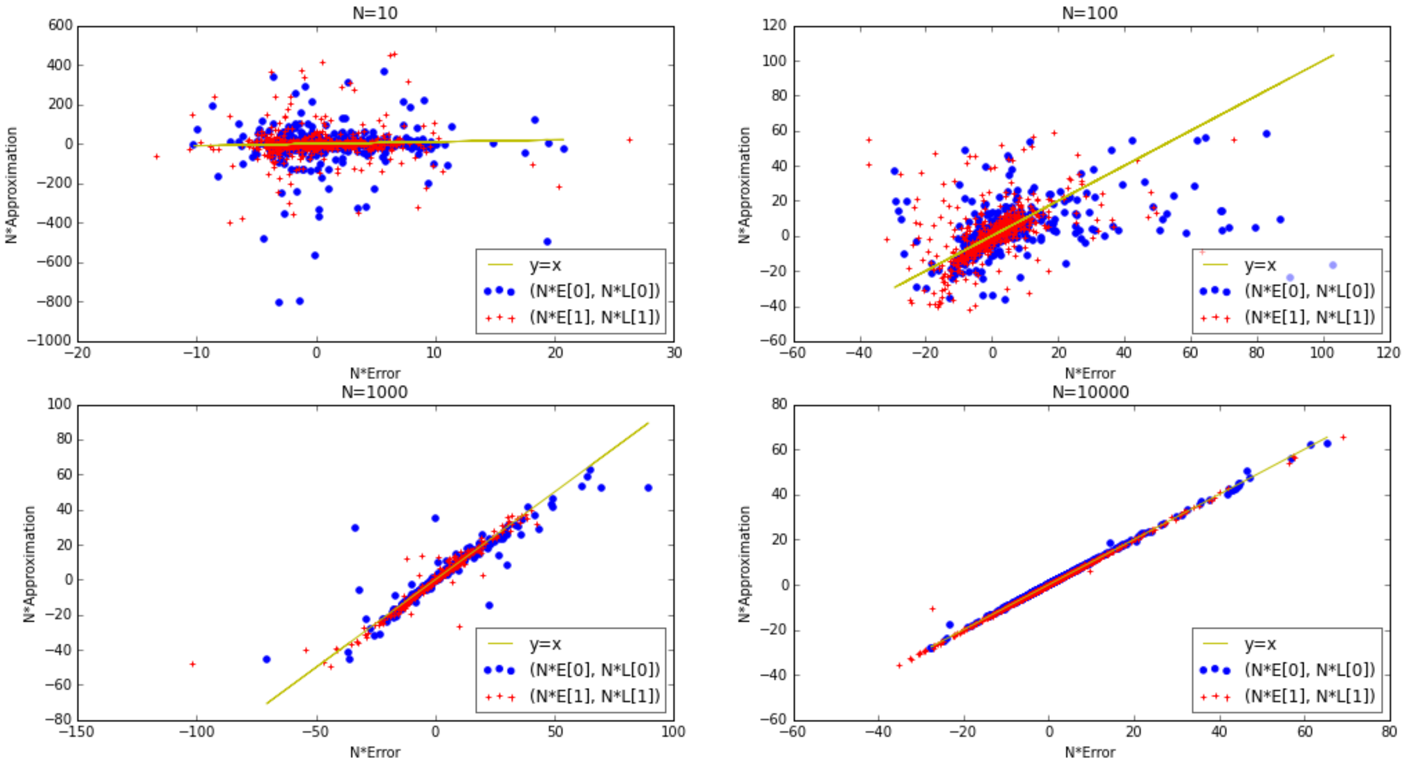
$$\mu(x) = (0,0), \quad \sigma(x) = \begin{pmatrix} 0.2\,x_0(1+\sin(2\pi x_0)) & 0.2\,x_0(1+\cos(2\pi x_1)) \\ 0.2\,x_1(1+\cos(2\pi x_0)) & 0.2\,x_1(1+\sin(2\pi x_1)) \end{pmatrix}, \quad Y_0 = (1,1).$$



Non linear SDE 2:

$$\mu(x) = (0.2\,x_0(1+\sin(2\pi x_0)), 0.2\,x_0(1+\sin(2\pi x_1)))$$

$$\sigma(x) = \begin{pmatrix} 0.2\,x_0(1+\sin(2\pi x_0)) & 0.2\,x_0(1+\cos(2\pi x_1)) \\ 0.2\,x_1(1+\cos(2\pi x_0)) & 0.2\,x_1(1+\sin(2\pi x_1)) \end{pmatrix}, \qquad Y_0 = (1,1).$$



4 Conclusion

✓ Accurate resampling of Euler schemes using backward regeneration of Brownian increments

✓ Convergence rates:

- 🙂 N^{-1^-} in a.s. sense
- 🙂 N^{-1} for weak convergence
- 🙁 $N^{-?}$ for strong convergence

✓ Useful for Regression Monte Carlo algorithms with strong memory constraints

✓ Future work: generalization to Milstein scheme

References

[ACLP14] R. Aïd, L. Campi, N. Langrené, and H. Pham. A probabilistic numerical method for optimal multiple switching problem in high dimension. *SIAM J. on Financial Mathematics*, 5(1):191–231, 2014.

[BMS09] D. Belomestny, G.N. Milstein, and V. Spokoiny. Regression methods in pricing American and Bermudan options using consumption processes. *Quantitative Finance*, 9(3):315–327, 2009.

[CGM25] P. Cohort, E. Gobet, and M. M'Rad. Forward and backward Euler scheme and Random Number Generator inversion. *Preprint*, 2025.

[GM17] E. Gobet and M. M'Rad. Strong approximation of stochastic processes at random times and application to their exact simulation. *Stochastics*, 2017.

[GM18] E. Gobet and M. M'Rad. Convergence rate of strong approximations of compound random maps, applications to SPDEs. *Discrete and Continuous Dynamical Systems Series B*, 23(10):4455–4476, 2018.

[Kun97] H. Kunita. *Stochastic flows and stochastic differential equations.* Cambridge Studies in Advanced Mathematics. 24. Cambridge: Cambridge University Press, 1997.

[L'E88] P. L'Ecuyer. Efficient and portable combined random number generators. *Communications of the ACM*, 31(6):742–774, 1988.

[LS01] F. Longstaff and E.S. Schwartz. Valuing American options by simulation: A simple least squares approach. *The Review of Financial Studies*, 14:113–147, 2001.

[PL06] F. Panneton and P. L'Ecuyer. Improved long-period generators based on linear recurrences modulo 2. *ACM Transactions on Mathematical Software*, 32(1):1–16, 2006.