A novel methodology to test infinite expectations :

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Outline

A long time ago

ANALYSE NUMERIQUE
DES
EQUATIONS DIFFERENTIELLES STOCHASTIQUES
DE ITÔ

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Pour ces deux types, Milshtein propose deux algorithmes numériquement exploitables donnant des précisions en $O(\Delta t)^2$; ses démonstrations de ces estimations font appel à un résultat portant sur le développement de Taylor d'un semi-groupe, et nous avons cherché à retrouver les conclusion de Milshtein d'une manière peut-être plus naturelle ; ce faisant nous avons affiné ses hypothèses pour l'erreur au sens de l'espérance.

Aussi, après avoir présenté quelques résultats sur les intégrales et les équations différentielles stochastiques (chapitre 1), nous présentons la méthode de Milshtein et proposons deux nouvelles démonstrations de ses estimations (chapitres 2 et 3).

Un autre type d'erreur restait négligé : l'erreur "uniforme sur chaque trajectoire", c'est à dire, à $\varpi \in \Omega$ fixé, la quantité $\sup_k |\bar{X}_k(\varpi) - X_{t_k}(\varpi)|$.

En utilisant un théorème dû à Halim Doss, nous avons pu construire (chapitre 4) un algorithme pour lequel nous montrons :

$$\sup_k |\bar{X}_k(\varpi) - X_{t_k}(\varpi)| \leq C^{te} \cdot \sqrt{\Delta t \log(1/\Delta t)} + O(\Delta t).$$

Ceci est illustré par des tests numériques.

Enfin nous nous sommes intéressé à une équation différentielle stochastique particulière, pour laquelle le résultat de Doss nous a suggéré une variante de l'algorithme précédent dont il a été possible d'estimer l'erreur, bien que l'équation ne satisfasse pas les hypothèses justifiant la précédente inégalité.

27 years ago

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From: "Milstein" <Grigori.Milstein@usu.ru>
Subject: from Milstein
Date: 11 November 1998 at 08:11:15 CET
To: <Denis.Talay@sophia.inria.fr>

Dear Denis,
How is life treating you?
It is already more than two months as I am in Ekaterinburg.
My address now:
                          Prof. G.N. Milstein
                          Department of Mathematics
                          Ural State University
                          Lenin str., 51
                          620083 Ekaterinburg, Russia

I would like to receive your preprints and reprints.
Best regards,
Grigori
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I. INTRODUCTION

Motivations

Motivation: Critical parameters for complex particle systems with singular McKean-Vlasov interaction kernels

$$dX_t^{i,N} = \int \beta_\chi(x,z)\ \mu_s^N(dz) + \int\int \gamma_\chi(x,z)\ \mu_s^N(dz)\ dW_t^i$$

where β_χ and γ_χ are singular kernels and χ is a critical parameter to determine

- Most often, inaccessible accurate tail estimates on the limit probability distribution μ_s of $\mu_s^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_s^{i,N}}$
- Simulations may give intuition on the finiteness of the expectations $\int |\beta(x,z)|\ \mu_s(dz)$ and $\int |\gamma(x,z)|\ \mu(dz)$ (or of other integrals useful to the construction of solutions)

Our strategy

A naive strategy: Regularize the model by removing singularities. Regularization of interaction kernels, simulation of truncated laws, etc.
Risk: Lose explosion and condensation times and get inaccurate approximations of critical parameters

Our strategy: Develop a test to detect simulations with possibly infinite expectation or variance

- The test needs to have a very low numerical cost
- Not developed on classical limit theorems because they do not lead to effective procedures (see a next slide)

Preliminary difficulty: How to set-up the problem?

The interesting Hawkins'result

'Does there exist a test, which makes the right decision with arbitrarily high probability if given sufficient data, of the hypothesis that a given r.v. has finite expectation?'
Answer: NO!

Theorem (**Hawkins**)
Let $\mathcal{G}$ (resp. $\mathcal{H}$) be the set of densities with finite (resp. infinite) means. Let $\mathcal{T}$ be the class of sequential tests which terminate in finite time, whatever is the density of the data.
$\mathcal{G}$ and $\mathcal{H}$ would be distinguishable in $\mathcal{T}$ if: $\forall \epsilon$ it would exist a test in $\mathcal{T}$ s.t.

$$\begin{cases} \mathbb{E}_F(\phi) < \epsilon & \text{if } \; F \in \mathcal{G} \\ \mathbb{E}_F(\phi) > 1 - \epsilon & \text{if } \; F \in \mathcal{H} \end{cases}$$

...But $\mathcal{G}$ and $\mathcal{H}$ are NOT distinguishable.

Illustration 1: Fluctuations of the Maxima

Many limit theorems describing the behaviour of $\frac{S_N}{N} := \frac{1}{N}\sum_{j=1}^{N}\Upsilon_j$ suppose that the Υ_i's are centered. They cannot be applied in our situation since the expectation is unknown .

An a priori interesting result for non centered r.v.'s:
Let $\Upsilon, \Upsilon_1, \ldots, \Upsilon_n$ be i.i.d. positive random variables.
Set $M_n := \max\{\Upsilon_1, \ldots, \Upsilon_n\}$.

Proposition

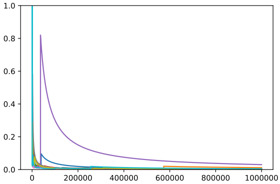
1. If $\mathbb{E}(\Upsilon) = \infty$ then $\limsup_n \frac{M_n}{n} = \infty$ *a.s.*
2. If $\mathbb{E}(\Upsilon) < \infty$ then $\lim_n \frac{M_n}{n} = 0$ *a.s.*

However,

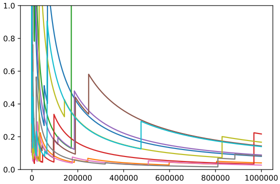
Example:

Let $\Upsilon = |G|^{-r}$ with $G \sim \mathcal{N}(0,1)$.
For $r < 1$ one has $\mathbb{E}(\Upsilon) < \infty$ whereas for $r \geq 1$ one has $\mathbb{E}(\Upsilon) = \infty$. In that second case,

- Very big values are too rare in the samples to lead to significantly excessive values of $\frac{M_n}{n}$, even when n is very large
- The larger is n, the larger needs to be Υ_n to make $\frac{M_n}{n}$ significantly larger than $\frac{M_{n-1}}{n-1}$:
- Jumps occur at times T_n where new upper record values appear. For any i.i.d. sequence the probability law of $T_{n+1} - T_n$ does not depend on the law of the subjacent random variable and has infinite expectation (**Nevzorov**)

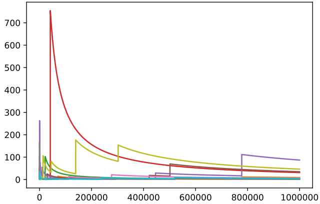


(a) $r = 0.6$

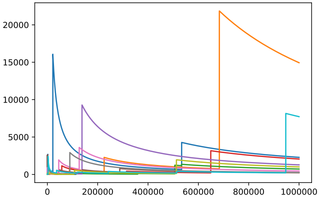


(b) $r = 0.85$

Fig. 1: Trajectories of $\frac{M_n}{n}$ in the finite expectation of Υ case.



(a) $r = 1.1$



(b) $r = 1.5$

Fig. 2: Trajectories of $\frac{M_n}{n}$ in the infinite expectation of Υ case.

Illustration 2: Self-normalized Iterated Logarithm Law

Suppose that $X \in \mathrm{DA}(2)$, i.e. there exist A_n and B_n s.t.

$$B_n^{-1} \sum_{i=1}^{n} X_i - A_n$$

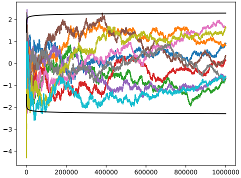
weakly converges to a Gaussian law.
Suppose also that $\mathbb{E}(X) = 0$.
Then

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^{n} X_j}{V_n \sqrt{2 \log \log(n)}} = 1 \quad \text{a.s.}$$

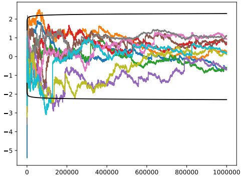
First issue: In our case, $\mathbb{E}(X)$ is unknown .

Second issue: Unfortunately, ineffective to test if $X \in \mathrm{DA}(2)$.

Example: Again, let $X = |G|^{-r}$. For $r \leq 0.5$, $X \in \mathrm{DA}(2)$, whereas for $r > 0.5$, $X \in \mathrm{DA}(1/r)$. In both cases, most of the self-normalized empirical means remain strictly confined between the curves $-\sqrt{2 \log \log(n)}$ and $\sqrt{2 \log \log(n)}$.

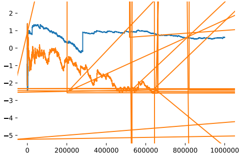


(a) $r = 0.3$

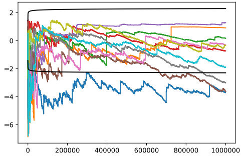


(b) $r = 0.49$

Fig. 3: Paths of the self-normalized empirical mean for laws in DA(2)



(a) $r = 0.6$



(b) $r = 0.7$

Fig. 4: Paths of the Self-normalized empirical mean for laws NOT in DA(2)

II. From reminders on stable laws to our strategy

A few reminders on stable laws

Stable law with parameters $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$:
Its characteristic function is

$$\exp(i\ a\ \lambda - b\ |\lambda|^{\alpha}(1 + i\ \beta\ \mathrm{sign}(\lambda)\ w(\lambda, \alpha)),$$

where

$$w(\lambda, \alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{if} \quad \alpha \neq 1 \\ \frac{2}{\pi}\ \log(|\lambda|) & \text{if} \quad \alpha = 1 \end{cases}$$

If $\alpha = 2$: Gaussian. If $\alpha = 1$ and $\beta = 0$: Cauchy.

Domain of attraction $DA(\alpha)$ of an α-stable law F_α:
The law F belongs to $DA(\alpha)$ if there exist A_n and B_n s.t.

$$B_n^{-1} \sum_{i=1}^{n} X_i - A_n$$

weakly converges to F_α, where the X_i are i.i.d. with probability distribution function F. MANY INTERESTING F's!

Domains of attraction and tails:

Let F be a probability distribution function. For $R > 0$ denote the truncated second moment of F by

$$U(R) := \int_{-R}^{R} x^2 F(dx).$$

- The probability distribution F belongs to the domain of attraction DA(2) of the Gaussian distribution if and only if the function U is slowly varying at infinity, that is, $\lim_{s\to\infty} U(sx)/U(s) = 1$ for any $x > 0$
- It belongs to some other domain of attraction DA(α) with $0 < \alpha < 2$ if there exist a slowly varying at infinity function $H : \mathbb{R}_+ \to \mathbb{R}$ and positive numbers p and q such that

$$\begin{cases} 1 - F(x) + F(-x) \sim \frac{2-\alpha}{\alpha}\, x^{-\alpha}\, H(x), \quad x \to \infty, \\ \lim_{x\to\infty} \frac{1-F(x)}{1-F(x)+F(-x)} = p, \quad \lim_{x\to\infty} \frac{F(-x)}{1-F(x)+F(-x)} = q \end{cases}$$

Domains of attraction and variance:

If the random variable X belongs to $D(\alpha)$ then

$$\mathbb{E}(|X|^{\delta}) < \infty \quad \text{for} \quad 0 < \delta < \alpha,$$
$$\mathbb{E}(|X|^{\delta}) = \infty \quad \text{for} \quad \delta > \alpha \quad \text{and} \quad \alpha < 2.$$

In particular, $\mathbb{E}(X^2) = \infty$ for $\alpha < 2$.

Remark.

If $\mathbb{E}(X^2) < \infty$ then X belongs to DA(2). The converse is not true.

For example, $X = \sqrt{|Y|}$ with Y Cauchy belongs to DA(2) and has an infinite second moment.

This observation explains our choice of the null hypothesis in a next slide.

Slow convergence of estimators of α

The estimation of α is difficult.
Consider the **Meerschaert-Scheffler** estimator of the index α:

$$\hat{\alpha}_n := \frac{\max\{\log\left(\sum_{i=1}^n (X_i - \bar{X}_n)^2\right), 0\}}{2\log(n)}.$$

Meerschaert and Scheffler proved that the estimator $\hat{\alpha}_n$ is asymptotically consistent when the data belong to some $DA(\alpha)$: $\hat{\alpha}_n \xrightarrow[n\to\infty]{\mathbb{P}} \frac{1}{\alpha}$. However, the convergence rate of $\hat{\alpha}_n$ is low.

Typical behaviour of $\hat{\alpha}_n$:

Random Variable	α	$n = 10^5$	$n = 10^6$	$n = 10^7$
$\|G\|^{-0.6},\ G \sim \mathcal{N}(0,1)$	0.6	0.67540532	0.66530621	0.65674121
$\|G\|^{-0.7},\ G \sim \mathcal{N}(0,1)$	0.7	0.76454733	0.75509289	0.74727714
$\|G\|^{-0.8},\ G \sim \mathcal{N}(0,1)$	0.8	0.85825199	0.84867939	0.84254414

Table 1: Mean value of the Meerschaert-Scheffler estimator over 10000 simulations of samples with size n.

The estimator $\hat{\alpha}_n$ is not reliable to detect $\alpha < 2$:
Now, $X \sim |G|^{-0.45}$ with $G \sim N(0,1)$. Thus X belongs to $D(2)$.

m	Empirical mean of $\hat{\gamma}$	Empirical sd of $\hat{\gamma}$	Empirical min of $\hat{\gamma}$
10^5	0.55740888	0.0169447807	0.5331192
10^6	0.55160767	0.0113382256	0.53850901
10^7	0.54669192	0.00862631795	0.53852758
10^8	0.54221338	0.00650946694	0.53628999
10^9	0.53842022	0.00491848875	0.53440811

Table 2: Descriptive statistics of the Meerschaert-Scheffler estimator of the tail index of $X = |G^{-0.45}|$ over 10000 simulations of samples with different sizes.

Based on these empirical results one would conclude that X does not belong to DA(2).

Our objective:

To provide an asymptotic statistical test under the additional assumption that $X := \sqrt{|V|}$ belongs to some domain of attraction $\mathrm{DA}(\alpha)$ of a stable law of index $0 < \alpha \leq 2$.

The null and alternative hypotheses of our hypothesis test respectively are:

$$\boxed{\mathbf{H_0} : X \in \mathrm{DA}(2)}$$

and

$$\boxed{\mathbf{H_1} : \exists 0 < \alpha < 2, \quad X \in \mathrm{DA}(\alpha)}$$

Our key observation is that X cannot have a finite second moment when H_0 is rejected (and therefore H_1 is accepted).

Key known result: A generalization of Donsker's theorem

Theorem

Let $X, X_1, X_2, \dots$ be a sequence of non-degenerate i.i.d. random variables such that $X \in \mathrm{DA}(\alpha)$.

Let

$$S_n := \sum_{j=1}^{n} X_j$$

There exist centering constants μ_m and normalizing constants c_m such that

$$L^m := \left(\frac{S_{\lfloor mt \rfloor} - \mu_m t}{c_m}, t \geq 0 \right) \xrightarrow[n\to\infty]{\mathcal{D}} L,$$

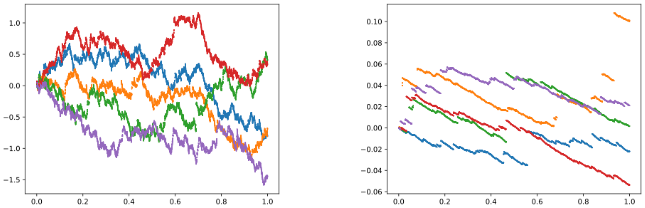
where

- L is a standard α-stable Lévy process if $\alpha < 2$
- L is a standard Brownian motion if $\alpha = 2$

Consequences

- For $\alpha < 2$ the trajectories of α-stable processes are a.s. discontinuous, whereas for $\alpha = 2$ the trajectories of the Brownian motion are a.s. continuous
- For m large enough the trajectories of $(S_{\lfloor mt \rfloor} - \mu_m t)/c_m$ should resemble the trajectories of the limit process

Conclusion:
Testing for jumps in the trajectories of $(S_{\lfloor mt \rfloor} - \mu_m t)/c_m$ should allow to discriminate between $X \in \mathrm{DA}(2)$ and $X \in \mathrm{DA}(\alpha)$.



(a) $r = 0.2$ (b) $r = 0.8$

Fig. 5: Trajectories of L^m when the subjacent random variable $X \sim |G|^{-r}$, where $G \sim \mathcal{N}(0, 1)$. In the left column $r = 0.2$, therefore the limit of L^m is a Brownian motion and the trajectories of L^m seem to be continuous. In the right column, $r = 0.8$, hence the limit of L^m is a Lévy process and the trajectories of L^m seem to have jumps.

For $r = 0.2$: We are under $\mathbf{H}_0$. $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$ are known explicitly.
For $r = 0.8$: X is in the normal domain of attraction of stable distribution with index $1/r$, hence the normalizing constant is $c_m = m^{1/r}$.

Conclusions from the reminders:

- We propose to deal with the restricted class of r.v. in the domain of attraction of a stable law
- For m big enough, the trajectories of $(S_{\lfloor mt \rfloor} - \mu_m t)/c_m$ should resemble the trajectories of the limit process
- For $\alpha < 2$, the trajectories of α-stable processes are almost surely discontinuous, whereas for $\alpha = 2$ the trajectories of the Brownian motion are almost surely continuous
- Testing for jumps the trajectories of $(S_{\lfloor mt \rfloor} - \mu_m t)/c_m$ could allow to discriminate between $X \in \text{DA}(2)$ and $X \in \text{DA}(\alpha)$

A severe drawback: We do not know the values of μ_m and c_m, in particular because we do not know α.

III. Our hypothesis test

To bypass the fact that μ_m is unknown:

Lemma.

Let $D[0,1]$ be the space of *cadlag* functions on $[0,1]$ and Ψ be the map

$$\forall x(\cdot) \in D[0,1], \quad \boxed{\Psi(x)(t) := x(t) - tx(1)}$$

The mapping Ψ is continuous for the Skorokhod topology .

Corollary.

$$\text{Let} \quad \boxed{Z^m := \left(\frac{S_{\lfloor mt \rfloor} - tS_m}{c_m}, t \geq 0 \right)}$$

If $X \in \mathrm{DA}(\alpha)$, then

$$\boxed{Z^m \xrightarrow[m \to \infty]{\mathcal{D}} Z := \Psi(L)}$$

- $\alpha < 2$: L is an α-stable Lévy process ($\Psi(L)$ is discontinuous)
- $\alpha = 2$: L is a Brownian motion ($\Psi(L)$ is a (continuous) Brownian bridge)

To bypass the fact that c_m is unknown:

For any stochastic process $(Y_t)_{0\leq t\leq 1}$ set $\boxed{\Delta_i^n Y := Y_{i/n} - Y_{(i-1)/n}}$.

The realized bivariation and realized quadratic variation are

$$\boxed{\widehat{B}(Y,n) := \sum_{i=1}^{n-1} |\Delta_i^n Y||\Delta_{i+1}^n Y|} \quad \text{and} \quad \boxed{\widehat{Q}(Y,n) := \sum_{i=1}^{n} |\Delta_i^n Y|^2}$$

The normalized bivariation is

$$\boxed{\widehat{\mathscr{S}_n}(Y) := \frac{\widehat{B}(Y,n)}{\widehat{Q}(Y,n)}}$$

The normalized bivariation was used by **Barndorff-Nielsen and Shephard** who considered (Y_t) of the form

$$Y_t = Y_0 + \int_0^t a_s \, ds + \int_0^t \sigma_s \, dW_s + \sum_{j=1}^{N_t} c_j, \tag{1}$$

where W is a Brownian motion and (N_t) is a counting process.
They notably assumed that c_j are non-zero random variable, σ is pathwise bounded away from zero and (a, σ) is independent of (W_t).
They provided a test to decide '$\mathcal{H}_0$: $(N_t) \equiv 0$' against '$\mathcal{H}_1$: $(N_t) \not\equiv 0$'.
The test is based on the statistic

$$\frac{1}{\sqrt{n}} \left(\frac{2}{\pi} \widehat{\mathscr{S}_n}(Y) - 1 \right) \frac{\int_0^t \sigma_s^2 \, ds}{\sqrt{\int_0^t \sigma_s^4 \, ds}}$$

In our case, the process σ of $\Psi(L)$ would be null and the number of jumps before any time $t > 0$ would be infinite under $\mathbf{H_1}$.

Ait-Sahalia and Jacod used p-variation with $p > 2$ in their test to determine whether a semimartingale Y is continuous or not from the observation of one single path at discrete times.
As Barndorff-Nielsen and Shephard they used the non-degeneracy assumption on the integrand σ.

In our case, Y is, either an transformed Lévy process, or a Brownian bridge. In both cases, the p-variation is asymptotically infinite since a Brownian bridge satisfies

$$Y_t = x + B_t + \int_0^t \frac{-Y_s}{1-s}\, ds$$

for some Brownian motion (B_t).

Recall

$$Z^m := \left(\frac{S_{\lfloor mt \rfloor} - tS_m}{c_m}, t \geq 0 \right)$$

and

$$Z^m \xrightarrow[m \to \infty]{\mathcal{D}} Z := \Psi(L)$$

- $\alpha < 2$: L is an α-stable Lévy process ($\Psi(L)$ is discontinuous)
- $\alpha = 2$: L is a Brownian motion ($\Psi(L)$ is a (continuous) Brownian bridge)

Barndorff-Nielsen and Shephard's test gave us the idea to consider the statistic

$$\widehat{\mathscr{S}}_n^m := \widehat{\mathscr{S}}_n(Z^m)$$

An important property.

The map $\widehat{\mathscr{S}_n}$ is invariant under normalizations:

$$\widehat{\mathscr{S}_n}(Z^m) = \frac{\sum_{i=1}^{n-1} |Z^m_{i/n} - Z^m_{(i-1)/n}||Z^m_{(i+1)/n} - Z^m_{i/n}|}{\sum_{i=1}^{n} |Z^m_{i/n} - Z^m_{(i-1)/n}|^2}$$

$$= \frac{\sum_{i=1}^{n-1} \left|\sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor +1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \frac{S_m}{m})\right| \left|\sum_{j=\lfloor \frac{mi}{n} \rfloor +1}^{\lfloor \frac{m(i+1)}{n} \rfloor} (X_j - \frac{S_m}{m})\right|}{\sum_{i=1}^{n} \left|\sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor +1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \frac{S_m}{m})\right|^2}$$

Therefore, to compute $\widehat{\mathscr{S}_n}(Z^m)$ we need to know neither μ_m nor c_m.

Our main result:

Theorem.

Assume that X belongs to some $D(\alpha)$. Consider and i.i.d. sample $X_1, \ldots, X_m$ of X, and the statistic

$$\widehat{\mathscr{S}}_n^m := \widehat{\mathscr{S}_n}(Z^m) = \frac{\sum_{i=1}^{n-1} \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor+1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \frac{S_m}{m}) \right| \left| \sum_{j=\lfloor \frac{mi}{n} \rfloor+1}^{\lfloor \frac{m(i+1)}{n} \rfloor} (X_j - \frac{S_m}{m}) \right|}{\sum_{i=1}^{n} \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor+1}^{\lfloor \frac{mi}{n} \rfloor} (X_j - \frac{S_m}{m}) \right|^2}$$

Let z_q denote the q-quantile of a standard normal random variable and let $\sigma_\pi^2 := 1 + \frac{4}{\pi} - \frac{20}{\pi^2}$. The rejection region

$$C_{n,m} := \left\{ \left| \widehat{\mathscr{S}}_n^m - \frac{2}{\pi} \right| > z_{1-q/2} \sqrt{\frac{\sigma_\pi^2}{n}} \right\}$$

satisfies

1. $\lim_{n\to\infty} \lim_{m\to\infty} \mathbb{P}\left(C_{n,m} | \mathbf{H_1}\right) = 1$
2. $\limsup_{n\to\infty} \limsup_{m\to\infty} \mathbb{P}\left(C_{n,m} | \mathbf{H_0}\right) \leq q$

Consistency of the statistic

Proposition.
For any $\epsilon > 0$ one has

$$\lim_{n\to\infty} \lim_{m\to\infty} \mathbb{P}\left(|\widehat{\mathscr{S}}_n^m - \kappa| \le \epsilon\right) = 1$$

with

$$\kappa := \begin{cases} 0 & \text{when} \quad X \in \mathrm{DA}(\alpha),\ \alpha < 2 \\ \frac{2}{\pi} & \text{when} \quad X \in \mathrm{DA}(2) \end{cases}$$

Consequence:
This proposition easily leads to the first part of our main theorem since $\kappa = 0$ under $\mathbf{H_1}$ and therefore

$$\forall \epsilon > 0, \quad \lim_{n\to\infty} \lim_{m\to\infty} \mathbb{P}\left(\widehat{\mathscr{S}}_n^m \le \epsilon\right) = 1$$

Sketch of the proof of the consistency of our statistic

Recall

$$\boxed{\widehat{\mathscr{S}_n}(Y) := \frac{\widehat{B}(Y,n)}{\widehat{Q}(Y,n)}} \quad \text{and} \quad \boxed{\widehat{\mathscr{S}_n^m} := \widehat{\mathscr{S}_n}(Z^m)}$$

Lemma 1.

For any $n \in \mathbb{N}$,

$$\widehat{\mathscr{S}_n^m} = \widehat{\mathscr{S}_n}(Z^m) \xrightarrow[m\to\infty]{\mathcal{D}} \widehat{\mathscr{S}_n}(\Psi(L))$$

Lemma 2.

1. Let L be an α-stable process starting from 0. For $1 < \alpha < 2$,

$$\widehat{\mathscr{S}_n}(\Psi(L)) \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

   For $\alpha \leq 1$ the convergence holds in probability.

2. Let L be a Brownian motion, then

$$\widehat{\mathscr{S}_n}(\Psi(L)) \xrightarrow[n\to\infty]{\text{a.s.}} \frac{2}{\pi}$$

Lemma 2-1.

1. If L is an α-stable process with $\alpha < 2$, then $\widehat{Q}(\Psi(L), n)$ converges in distribution to a non-degenerate random variable
2. If L is a Brownian motion, then

$$\widehat{Q}(\Psi(L), n) \xrightarrow[n\to\infty]{\text{a.s.}} 1$$

Lemma 2-2.

Let L be, either a Brownian motion or an α-stable process starting from 0 with $\alpha > 1$. Then,

$$\widehat{B}(\Psi(L), n) - \widehat{B}(L, n) \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

For $\alpha \leq 1$ the convergence holds in probability.

Lemma 2-3.

1. Let L be an α-stable process starting from 0. For $\alpha > 1$,

$$\widehat{B}(L, n) \xrightarrow[n\to\infty]{\text{a.s.}} 0$$

   For $\alpha \leq 1$ the convergence holds in probability.
2. Let L be a Brownian motion, then

$$\widehat{B}(L, n) \xrightarrow[n\to\infty]{\text{a.s.}} \frac{2}{\pi}$$

A Central Limit Theorem for our statistic

Proposition If X belongs to $D(2)$, for any bounded and continuous function $\psi : \mathbb{R} \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \lim_{m\to\infty} \mathbb{E}\left[\psi\left(\frac{\sqrt{n}}{\sigma_\pi}(\widehat{\mathscr{S}}_n^m - \frac{2}{\pi})\right)\right] = \mathbb{E}\left[\psi(\mathcal{G})\right]$$

where $\sigma_\pi^2 := 1 + \frac{4}{\pi} - \frac{20}{\pi^2}$ and $G \sim \mathcal{N}(0,1)$.

Consequence:

This proposition leads to the second part of our main theorem: For

$$\psi_\delta(x) := \begin{cases} 1 & \text{for} \quad x < z_q \\ 1 - \frac{1}{\delta}(x - z_q) & \text{for} \quad z_q \le x \le z_q + \delta \\ 0 & \text{for} \quad x > z_q + \delta \end{cases}$$

one has

$$\forall 0 < \delta < 1, \quad \mathbb{P}\left(C_{n,m} | \mathbf{H_0}\right) \le \mathbb{E}\left(\psi_\delta(\widehat{\mathscr{S}}_n^m) | \mathbf{H_0}\right)$$

from which

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \mathbb{P}\left(C_{n,m} | \mathbf{H_0}\right) \le \mathbb{E}\left[\psi_\delta(G)\right] \le q$$

Sketch of the proof of our CLT

Lemma.
Let L be a Brownian motion, then:

$$\left(\sqrt{n}\left[\widehat{B}(\Psi(L),n)-\frac{2}{\pi}\right],\sqrt{n}\left[\widehat{Q}(\Psi(L),n)-1\right]\right)\xrightarrow[n\to\infty]{\mathcal{D}}\mathcal{N}_2\left(0,\Sigma\right)$$

where

$$\Sigma := \begin{pmatrix} 1+\frac{4}{\pi}-\frac{12}{\pi^2} & \frac{4}{\pi} \\ \frac{4}{\pi} & 2 \end{pmatrix}$$

A funny use of Tanaka's formula:
Let $G_i \sim \mathcal{N}(0,1)$ be i.i.d. and set $\bar{G}^n := \frac{1}{n}\sum_{i=1}^n G_i$. We use the formula to prove

$$\mathbb{E}\Big[\left(|G_i-\bar{G}^n||G_{i+1}-\bar{G}^n|-|G_i||G_{i+1}|\right)\left(|G_k-\bar{G}^n||G_{k+1}-\bar{G}^n|-|G_k||G_{k+1}|\right)\Big] \leq \frac{C}{n^2}$$

IV. Insights on the choices of m and n

On the choice of n (α close to 2)

Let X be symmetric α-stable with $\alpha \approx 2$ so that $\mathbb{E}|X| < \infty$.
Assume that m is large enough to have $\bar{X}_m \approx 0$ a.s.
Set

$$\widetilde{\mathscr{S}}_n^m := \frac{\sum_{i=1}^{n-1} \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor + 1}^{\lfloor \frac{mi}{n} \rfloor} X_j \right| \left| \sum_{j=\lfloor \frac{mi}{n} \rfloor + 1}^{\lfloor \frac{m(i+1)}{n} \rfloor} X_j \right|}{\sum_{i=1}^{n} \left| \sum_{j=\lfloor \frac{m(i-1)}{n} \rfloor + 1}^{\lfloor \frac{mi}{n} \rfloor} X_j \right|^2}$$

$$= \frac{\sum_{i=1}^{n-1} |W_{i-1}||W_i|}{\sum_{i=1}^{n} W_i^2}$$

with

$$W_i := \left(\frac{n}{m}\right)^{1/\alpha} \sum_{j=\lfloor \frac{mi}{n} \rfloor + 1}^{\lfloor \frac{m(i+1)}{n} \rfloor} X_j$$

Notice that W_i has the same α-stable distribution as X .

1. For n large enough,

$$\frac{1}{n}\sum_{i=1}^{n-1}|W_{i-1}||W_i| \approx (\mathbb{E}|W_1|)^2$$

2. (**Embrechts and Goldie**) For some slowly varying function $\ell_0(\cdot)$,

$$\frac{1}{n^{2/\alpha}\ell_0(n)}\sum_{i=1}^{n-1}|W_i|^2 \Longrightarrow \mathcal{W}, \ \frac{\alpha}{2} - \text{stable}$$

Consequently, in distribution

$$\widetilde{\mathscr{S}}_n^m \approx \frac{1}{n^{2/\alpha-1}\ell_0(n)}\frac{(\mathbb{E}|W_1|)^2}{\mathcal{W}}$$

Now use estimates on densities of positive stable random variables (see e.g. **Nolan**) to get

$$\mathbb{P}(C_{n,m}|\mathbf{H}_1) = \mathbb{P}\left(\widehat{\mathscr{S}}_n^m < \frac{2}{\pi} - z_{1-q/2}\sqrt{\frac{\sigma_\pi^2}{n}}\right) + \mathbb{P}\left(\widehat{\mathscr{S}}_n^m > \frac{2}{\pi} + z_{1-q/2}\sqrt{\frac{\sigma_\pi^2}{n}}\right)$$

$$\simeq 1 - \frac{C(\alpha,q)}{n^{\frac{2}{\alpha}-1}}$$

On the choice of m

On the probability density v_m of the normalized sum $Z_m := \frac{A}{m^{1/\alpha}} \sum_{i=1}^{m} X_i$:

Theorem (**Basu, Maejima and Patra**)
Let (X_j) be a sequence of i.i.d. random variables with common probability density v_1. Suppose that they are centered and belong to the domain of attraction of a stable law of index $1 < \alpha < 2$ whose probability density is denoted by v_α. Suppose that their characteristic function belongs to $L^r(\mathbb{R})$ with $r \geq 1$. Finally, suppose that $\int_{\mathbb{R}} x^2 \, |v_1(x) - v_\alpha(x)| \, dx < \infty$.

Then, for some positive number A, for any m large enough,

$$\sup_x (1 + |x|^\alpha) \, |v_m(x) - v_\alpha(x)| = \mathcal{O}\left(\frac{1}{m^{\frac{2}{\alpha}-1}}\right)$$

In our case we deduce that

$$\begin{aligned}\mathbb{P}(|\overline{X}_m| > \epsilon) &= \mathbb{P}(|Z_m| > A\epsilon\ m^{1-\frac{1}{\alpha}}) \\ &\leq \int_{|x|>A\epsilon m^{1-\frac{1}{\alpha}}} v_\alpha(x)\ dx + \mathcal{O}(\frac{1}{m^{\frac{2}{\alpha}-1}}),\end{aligned}$$

from which, by using **Gairing and Imkeller**'s tail estimates for centered stable random variables,

$$\mathbb{P}(|\overline{X}_m| > \epsilon) \leq \frac{C}{\epsilon^\alpha\ m^{\alpha-1}} + \frac{C}{m^{\frac{2}{\alpha}-1}}$$

Conclusion: m needs to be chosen much larger than n , especially when α is close to 2. Actually, $\overline{X}_m$ can be seen as a random perturbation term in $\widehat{\mathscr{S}}_n^m$ whose expectation needs to be small enough.

Remark: Numerical experiments tend to show that the preceding estimates are sub-optimal .

IV. Empirical evidence

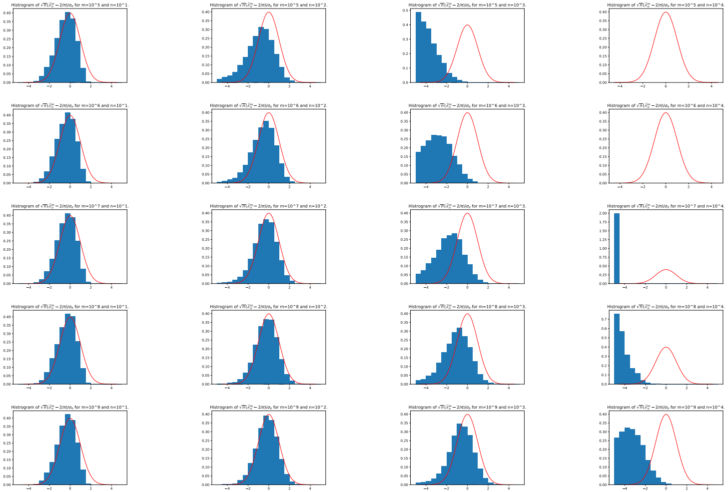


Fig. 6: Empirical distribution of standardized $\widehat{\mathscr{S}_n^m}$ for $r = 0.4$. The value of m increases from 10^5 on the top row up to 10^9 in the bottom row. The value of n increases from 10^1 on the left column up to 10^4 in the right column.

Comments on the figure

1. Under H_0, when n is too large one is zooming in too much and sees the discontinuities that Z^m has by construction.
2. Under H_0 and H_1 the test provides satisfying ersults for moderately large n and m.
3. Our theoretical and numerical results apply to some cases of weakly dependent data .

On weak dependence cases

Using limit theorems due to Tyran-Kamińska (2010) and Shao (1993) one can check that our preceding results hold true e.g. when the sample is stationary, k-dependent and satisfies the two following conditions:

$$\sup\left\{|\,\mathrm{Corr}(f,g)| : f \in L^2(\sigma(X_1)), g \in L^2(\sigma(X_2, X_3, \ldots))\right\} < 1$$

and

$$\forall \epsilon > 0, \quad \forall 2 \leq j \leq k, \quad \lim_{m \to \infty} \mathbb{P}\left(|X_j| > \epsilon c_m / |X_1| > \epsilon c_m\right) = 0$$

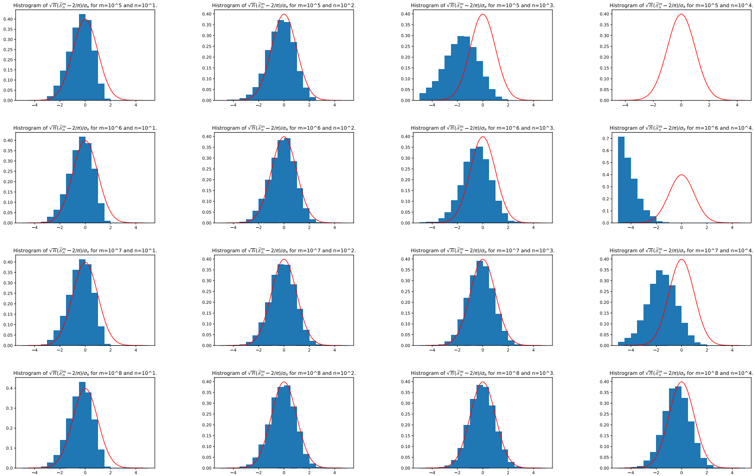


Fig. 7: Empirical distribution of standardized $\widehat{\mathscr{S}}_n^m$ for a 1-dependent sequence of random variables when $r = 0.3$. The value of m increases from 10^5 on the top row up to 10^8 in the bottom row. The value of n increases from 10^1 on the left column up to 10^4 in the right column.

To conclude

Suppose you observe a sample with large empirical expectation. Our test helps you to decide whether the sampled probability distribution belongs to the domain of attraction of a Gaussian law or of a stable law with index lower than 2 (then, its second moment is infinite.)

Our non stringent condition: The observations belong to the domain of attraction of a stable law.

To define our statistics we use the sample to construct a discretized path of a stochastic process. We have analyzed the convergence and convergence rate of the discretized process to its limit in the weak sens in the spirit of the inspiring G.N. Milstein's works.