A fully discretized domain decomposition approach for semi-linear SPDEs

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joint work with Eskil Hansen, Marvin Jans

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Milstein's method: 50 years on

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Outline

Underlying problem

Method: Domain decomposition integrator

Error bound

Numerical example

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Underlying problem

Equation

$$\begin{cases} \mathrm{d}X(t) = \big[-AX(t) + f(t, X(t)) \big] \, \mathrm{d}t + B(t, X(t)) \, \mathrm{d}W(t), \quad t \in (0, t_f), \\ X(0) = X_0 \end{cases}$$

Underlying problem

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Unbounded operator A

- A is a positive, linear operator on a Hilbert space H
- A is densely defined on H and a sectorial operator

Example: $A = -\Delta \colon L^2 \to L^2$.

Underlying problem

Equation

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Initial condition X_0

- filtered probability space by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- $X_0 \colon \Omega \to H$
- X_0 is $\mathcal{F}_0$-measurable
- for $p \in [2, \infty)$ and $\theta_{X_0} \in [0, 1)$: $\|A^{\theta_{X_0}} X_0\|_{L^p(\Omega;H)} \leq C$.

Example: $X_0(\mathbf{x}, \omega) = \sin(\pi x_1)\sin(\pi x_2)$.

Underlying problem

Equation

$$\begin{cases} \mathrm{d}X(t) = \big[-AX(t) + f(t,X(t))\big]\,\mathrm{d}t + B(t,X(t))\,\mathrm{d}W(t), \quad t \in (0,t_f), \\ X(0) = X_0 \end{cases}$$

Non-linearity f

- $f\colon [0,t_f] \times H \to H$
- f is $\frac{1}{2}$-Hölder continious w.r.t time
- f is Lipschitz w.r.t $X(t)$, bounded for $X(t) = 0$
- for a $\theta_f \in [0,\frac{1}{2})$, $\|A^{\theta_f} f(t,v)\|_H \leq C\big(1 + \|A^{\theta_f} v\|_H\big)$ $\forall v \in \mathrm{dom}(A^{\theta_f})$

Example: $f(t,X) = \sqrt{t}\sin(X)$.

Underlying problem

Equation

$$\begin{cases} \mathrm{d}X(t) = \big[-AX(t) + f(t, X(t))\big]\,\mathrm{d}t + B(t, X(t))\,\mathrm{d}W(t), \quad t \in (0, t_f), \\ X(0) = X_0 \end{cases}$$

Stochastic perturbation B

- $B\colon [0, t_f] \times H \to L_2^0$, with $L_2^0 = \mathcal{HS}(Q^{\frac{1}{2}} H, H)$
- B is $\frac{1}{2}$-Hölder continious w.r.t time
- B is Lipschitz w.r.t $X(t)$, bounded for $X(t) = 0$
- for a $\theta_B \in [0, \frac{1}{2})$, $\|A^{\theta_B} B(t, v)\|_{L_2^0} \leq C\big(1 + \|A^{\theta_B} v\|_H\big)$ $\forall v \in \mathrm{dom}(A^{\theta_B})$

Example: $B(t, X) = 1 + X$.

Underlying problem

Equation

$$\begin{cases} \mathrm{d}X(t) = \big[-AX(t) + f(t, X(t)) \big] \, \mathrm{d}t + B(t, X(t)) \, \mathrm{d}W(t), \quad t \in (0, t_f), \\ X(0) = X_0 \end{cases}$$

Q-Wiener process W

Karhunen–Loève expansion of W

$$W(t) = \sum_{k=1}^{\infty} \sqrt{q_k} e_k \beta_k(t),$$

where

- $\{q_k\}_{k \in \mathbb{N}}$ and $\{e_k\}_{k \in \mathbb{N}} \subset H$ are the eigenvalues and eigenfunctions of of Q,
- $\{\beta_k\}_{k \in \mathbb{N}}$ are real valued, independent and identically distributed $\mathcal{F}_t$-Wiener processes.

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Operator splitting

A time stepping method to approximate $X^n \approx X(t_n)$ with

- grid points $t_n = n\tau$
- step size $\tau = \frac{t_f}{N}$
- Wiener increment $\Delta W^n = W(t_{n+1}) - W(t_n)$

is given by

$$X^{n+1} - X^n = -\tau A X^{n+1} + \tau f(t_n, X^n) + B(t_n, X^n)\Delta W^n.$$

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is given by

$$X^{n+1} = \big(I + \tau A\big)^{-1}\big(X^n + \tau f(t_n, X^n) + B(t_n, X^n)\Delta W^n\big).$$

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is given by

$$X^{n+1} = \big(I + \tau(A_1 + A_2)\big)^{-1}\big(X^n + \tau f(t_n, X^n) + B(t_n, X^n)\Delta W^n\big).$$

Operator splitting

A time stepping method to approximate $X^n \approx X(t_n)$ with

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Then we solve

$$X^{n+1} = \big(I + \tau A_1\big)^{-1}\big(X^n + \tau f(t_n, X^n) + B(t_n, X^n)\Delta W^n\big)$$

and

$$X^{n+1} = \big(I + \tau A_2\big)^{-1}\big(X^n + \tau f(t_n, X^n) + B(t_n, X^n)\Delta W^n\big)$$

individually and "glue" them together to a suitable solution.

Different splittings

Examples:

- Lie splitting

$$\big(I + \tau(A_1 + A_2)\big)^{-1} \approx \big(I + \tau A_2\big)^{-1}\big(I + \tau A_1\big)^{-1}$$

- Douglas–Rachford

$$\big(I + \tau(A_1 + A_2)\big)^{-1} \approx \big(I + \tau A_2\big)^{-1}\big(I + \tau A_1\big)^{-1}\big(I + \tau^2 A_1 A_2\big)$$

Motivation Douglas–Rachford

Exact flow:

$$\mathrm{e}^{-\tau(A_1+A_2)} = I - \tau(A_1 + A_2) + \frac{\tau^2}{2}(A_1 + A_2)^2 + \mathcal{O}(\tau^3)$$

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Splitting scheme:

- Lie splitting:

$$(I + \tau A_1)^{-1}(I + \tau A_2)^{-1} = I - \tau(A_1 + A_2) + \tau^2(A_1 + A_2)^2 - \tau^2 A_2 A_1 + \mathcal{O}(\tau^3)$$

- Douglas–Rachford splitting:

$$(I + \tau A_1)^{-1}(I + \tau A_2)^{-1}(I + \tau^2 A_1 A_2) = I - \tau(A_1 + A_2) + \tau^2(A_1 + A_2)^2 + \mathcal{O}(\tau^3)$$

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Error: $\frac{\tau^2}{2}(A_1 + A_2)^2 - \tau^2 A_2 A_1 + \mathcal{O}(\tau^3)$

- Douglas–Rachford splitting:

$$(I + \tau A_1)^{-1}(I + \tau A_2)^{-1}(I + \tau^2 A_1 A_2) = I - \tau(A_1 + A_2) + \tau^2(A_1 + A_2)^2 + \mathcal{O}(\tau^3)$$

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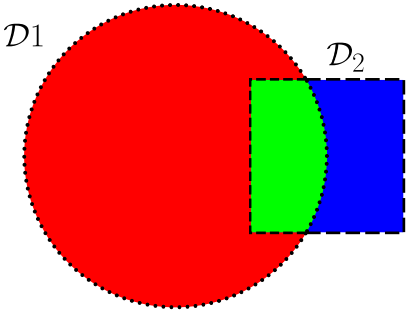
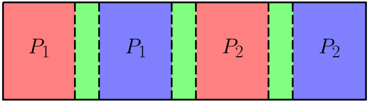
Domain Decomposition

- Overlapping Non-Iterative Domain Decomposition



Domain Decomposition

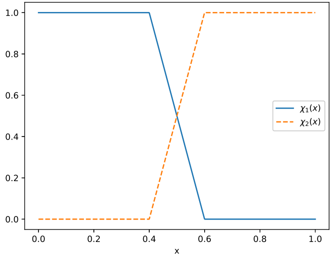
- Overlapping Non-Iterative Domain Decomposition
- Parallelization



P_1: Processor 1, P_2: Processor 2

Domain Decomposition

- Overlapping Non-Iterative Domain Decomposition
- Parallelization
- Weight functions



$$A = -\Delta$$

$$A_1 = -\nabla \cdot (\chi_1 \nabla), \quad A_2 = -\nabla \cdot (\chi_2 \nabla)$$

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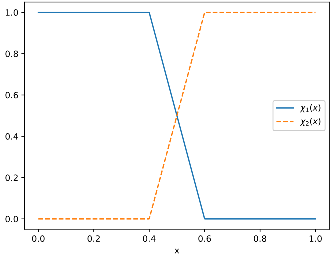

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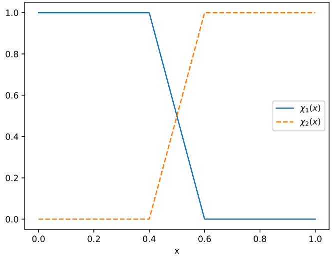

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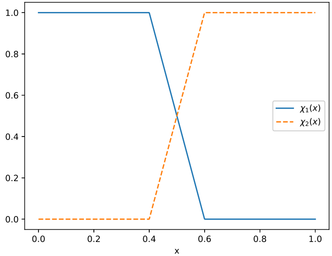
Domain Decomposition



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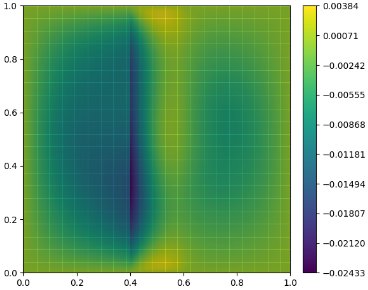
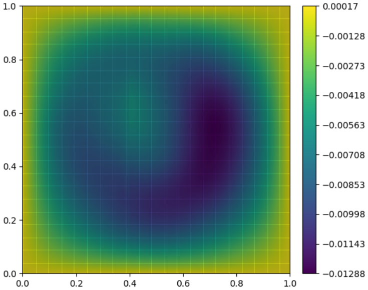
- E. & Hansen (2018, 2021):
  Porous medium, Lie/sum, no convergence rate
- Carelli et al. (2012): Different approach, less general equation
- Buckwar, Djurdjevac & E. (2024): Time discretization, Lie, variational setting but stronger regularity assumption

Lie vs. Modified Douglas-Rachford



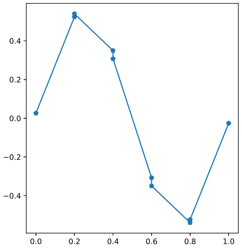
Lie splitting



Modified Douglas-Rachford splitting

Discontinious Galerkin

- Piecewise polynomials
- Block-diagonal mass matrix
- h is the mesh size
- $\mathcal{T}_h$ is set of elements T
- F_h is set of edges e
- jump $[v]|_e = v|_{T_e^+} - v|_{T_e^-}$
- average $\{v\}|_e = \frac{1}{2}\left(v_{T_e^+} + v_{T_e^-}\right)$



Then we have

$$(A_h v_h, w_h)_{L^2(\mathcal{D})} := \sum_{T \in \mathcal{T}_h} (\nabla v_h, \nabla w_h)_{(L^2(T))^d}$$

$$- \sum_{e \in F_h} \int_e \Big([\{\nabla v_h \cdot \mathbf{n}_e\}[w_h] + \{\nabla w_h \cdot \mathbf{n}_e\}[v_h]] - \frac{\sigma}{h}[v_h][w_h] \Big) \, \mathrm{d}\xi$$

while f_h and B_h are suitable projections of f and B.

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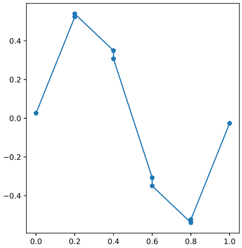

Then we have

$$
\begin{aligned}
(A_{h,1} v_h, w_h)_{L^2(\mathcal{D})} &:= \sum_{T \in \mathcal{T}_h} (\chi_1 \nabla v_h, \nabla w_h)_{(L^2(T))^d} \\
&- \sum_{e \in F_h} \int_e \chi_1 \left([\{\nabla v_h \cdot \mathbf{n}_e\}[w_h] + \{\nabla w_h \cdot \mathbf{n}_e\}[v_h]] - \frac{\sigma}{h}[v_h][w_h] \right) \mathrm{d}\xi
\end{aligned}
$$

while f_h and B_h are suitable projections of f and B.

Modified Douglas-Rachford splitting

We consider the scheme

$$\begin{cases} X_{h,\tau}^0 = P_h X_0, \\ X_{h,\tau}^1 = \alpha_{h,2}\alpha_{h,1}\big(X_{h,\tau}^0 + \tau f_h(t_0, X_{h,\tau}^0) + B_h(t_0, X_{h,\tau}^0)\Delta W^1\big), \\ X_{h,\tau}^n = S_{h,\tau} X_{h,\tau}^{n-1} + \alpha_{h,2}\alpha_{h,1}\big(\tau f_h(t_{n-1}, X_{h,\tau}^{n-1}) + B_h(f_h(t_{n-1}, X_{h,\tau}^{n-1}))\Delta W^n\big). \end{cases}$$

with

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1} \text{ and } \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

and

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(I + \tau^2 A_{h,1}A_{h,2})$$

- Hansen & Henningson (2016): Parabolic PDE, higher spatial regularity
- Hansen et al. (2016): Parabolic PDE, higher temporal regularity
- Hochbruck & Köhler (2022): 1^{st} order wave-like operators, Peacemond-Rachford, higher regularity

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Result

Theorem

If the previous mentioned assumptions hold for a $\theta_{X_0} \in [\theta_B, \theta_B + \frac{1}{2})$ and $\theta_f \in [0, \min\{\theta_{X_0}, \frac{1}{2}\}]$. Then it holds that

$$\begin{aligned}&\|X(t_f) - X_{h,\tau}^N\|_{L^p(\Omega;H)}\\ &\leq C\big(h^{2\min\{\theta_{X_0},\theta_f,\theta_B\}+1} + \tau^{\min\{\theta_{X_0},\theta_f,\theta_B,\frac{1}{2}\}}\big)(1 + \ln(N)).\end{aligned}$$

Convergence proof

Exact solution

$$X(t_n) = \mathrm{e}^{-t_n A} X_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathrm{e}^{-(t_n - s)A} f(s, X(s))\,\mathrm{d}s$$
$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathrm{e}^{-(t_n - s)A} B(s, X(s))\,\mathrm{d}W(s)$$

Numerical approximation

$$X_{h,\tau}^n = S_{h,\tau}^{n-1} \alpha_{h,2}\alpha_{h,1} P_h X_0 + \tau \sum_{k=0}^{n-1} S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} f_h(t_k, X_{h,\tau}^k)$$
$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} B_h(t_k, X_{h,\tau}^k)\,\mathrm{d}W(s).$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1} A_{h,2})$$
$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Error representation

$$X(t_n) - X^n_{h,\tau} = \big(\mathrm{e}^{-t_n A} - S^{n-1}_{h,\tau}\alpha_{h,2}\alpha_{h,1}P_h\big)X_0$$
$$+ \sum_{k=0}^{n-1}\Big[\int_{t_k}^{t_{k+1}} \mathrm{e}^{-(t_n-s)A} f(s,X(s))\,\mathrm{d}s - \tau S^{n-k-1}_{h,\tau}\alpha_{h,2}\alpha_{h,1} f_h(t_k, X^k_{h,\tau})\Big]$$
$$+ \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \big[\mathrm{e}^{-(t_n-s)A} B(s,X(s)) - S^{n-k-1}_{h,\tau}\alpha_{h,2}\alpha_{h,1} B_h(t_k, X^k_{h,\tau})\big]\,\mathrm{d}W(s).$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1}A_{h,2})$$
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Convergence proof

Error representation

$$X(t_n) - X_{h,\tau}^n = \left(\mathrm{e}^{-t_n A} - S_{h,\tau}^{n-1}\alpha_{h,2}\alpha_{h,1}P_h\right)X_0$$

$$+ \sum_{k=0}^{n-1}\Big[\int_{t_k}^{t_{k+1}} \mathrm{e}^{-(t_n-s)A} f(s, X(s))\,\mathrm{d}s - \tau S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1} f_h(t_k, X_{h,\tau}^k)\Big]$$

$$+ \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left[\mathrm{e}^{-(t_n-s)A} B(s, X(s)) - S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1} B_h(t_k, X_{h,\tau}^k)\right]\mathrm{d}W(s).$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1}A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Error representation

$$X(t_n) - X_{h,\tau}^n = \big(\mathrm{e}^{-t_n A} - S_{h,\tau}^{n-1}\alpha_{h,2}\alpha_{h,1}P_h\big)X_0$$

$$+ \sum_{k=0}^{n-1} \Big[\int_{t_k}^{t_{k+1}} \mathrm{e}^{-(t_n-s)A} f(s,X(s))\,\mathrm{d}s - \tau S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1} f_h(t_k, X_{h,\tau}^k)\Big]$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big[\mathrm{e}^{-(t_n-s)A} B(s,X(s)) - S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1} B_h(t_k, X_{h,\tau}^k)\big]\,\mathrm{d}W(s).$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1+\tau^2 A_{h,1}A_{h,2})$$

$$\alpha_{h,1} = (I+\tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I+\tau A_{h,2})^{-1}$$

Convergence proof

Error representation

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$$+ \sum_{k=0}^{n-1}\Big[\int_{t_k}^{t_{k+1}} \mathrm{e}^{-(t_n-s)A} f(s,X(s))\,\mathrm{d}s - \tau S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1} f_h(t_k, X_{h,\tau}^k)\Big]$$

$$+ \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left[\mathrm{e}^{-(t_n-s)A} B(s,X(s)) - S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1} B_h(t_k, X_{h,\tau}^k)\right]\mathrm{d}W(s).$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1}A_{h,2})$$

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Convergence proof

Error coming from stochastic perturbation B

Apply Burkholder–Davis–Gundy inequality

$$\Big\| \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big[\mathrm{e}^{-(t_n - s)A} B(s, X(s)) - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} B_h(t_k, X_{h,\tau}^k) \big] \,\mathrm{d}W(s) \Big\|_{L^p(\Omega;H)}^2$$

$$\leq C \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big\| \mathrm{e}^{-(t_n - s)A} B(s, X(s)) - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} B_h(t_k, X_{h,\tau}^k) \big\|_{L^p(\Omega;L_2^0)}^2 \,\mathrm{d}s$$

$$S_{h,\tau} = \alpha_{h,2} \alpha_{h,1} (1 + \tau^2 A_{h,1} A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Error coming from stochastic perturbation B

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \mathrm{e}^{-(t_n-s)A} B(s, X(s)) - S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} B_h(t_k, X_{h,\tau}^k) \right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s$$

$$\leq C\Big(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \mathrm{e}^{-(t_n-s)A} B(s, X(s)) - \mathrm{e}^{-(t_n-t_k)A} B(t_k, X(t_k)) \right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \big(\mathrm{e}^{-(t_n-t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} P_h\big) B(t_k, X(t_k)) \right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} \big(P_h B(t_k, X(t_k)) - B_h(t_k, X_{h,\tau}^k)\big) \right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s \Big)$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1}A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Error coming from stochastic perturbation B

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \mathrm{e}^{-(t_n-s)A} B(s, X(s)) - S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} B_h(t_k, X_{h,\tau}^k) \right\|_{L^p(\Omega;L_2^0)}^2 \,\mathrm{d}s$$

$$\leq C \Big(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \mathrm{e}^{-(t_n-s)A} B(s, X(s)) - \mathrm{e}^{-(t_n-t_k)A} B(t_k, X(t_k)) \right\|_{L^p(\Omega;L_2^0)}^2 \,\mathrm{d}s$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| \big(\mathrm{e}^{-(t_n-t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} P_h \big) B(t_k, X(t_k)) \right\|_{L^p(\Omega;L_2^0)}^2 \,\mathrm{d}s$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left\| S_{h,\tau}^{n-k-1} \alpha_{h,2}\alpha_{h,1} \big(P_h B(t_k, X(t_k)) - B_h(t_k, X_{h,\tau}^k) \big) \right\|_{L^p(\Omega;L_2^0)}^2 \,\mathrm{d}s \Big)$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1} A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Error coming from stochastic perturbation B

$$\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big\| \mathrm{e}^{-(t_n - s)A} B(s, X(s)) - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} B_h(t_k, X_{h,\tau}^k) \big\|^2_{L^p(\Omega; L_2^0)} \,\mathrm{d}s$$

$$\leq C \Big(\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big\| \mathrm{e}^{-(t_n - s)A} B(s, X(s)) - \mathrm{e}^{-(t_n - t_k)A} B(t_k, X(t_k)) \big\|^2_{L^p(\Omega; L_2^0)} \,\mathrm{d}s$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big\| \big(\mathrm{e}^{-(t_n - t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} P_h \big) B(t_k, X(t_k)) \big\|^2_{L^p(\Omega; L_2^0)} \,\mathrm{d}s$$

$$+ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \big\| S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} \big(P_h B(t_k, X(t_k)) - B_h(t_k, X_{h,\tau}^k) \big) \big\|^2_{L^p(\Omega; L_2^0)} \,\mathrm{d}s \Big)$$

$$S_{h,\tau} = \alpha_{h,2} \alpha_{h,1} (1 + \tau^2 A_{h,1} A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Error coming from stochastic perturbation B

$$\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left\| \mathrm{e}^{-(t_n-s)A}B(s,X(s)) - S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1}B_h(t_k,X_{h,\tau}^k)\right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s$$

$$\leq C\Big(\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left\| \mathrm{e}^{-(t_n-s)A}B(s,X(s)) - \mathrm{e}^{-(t_n-t_k)A}B(t_k,X(t_k))\right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s$$

$$+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left\| \big(\mathrm{e}^{-(t_n-t_k)A} - S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1}P_h\big)B(t_k,X(t_k))\right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s$$

$$+\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left\| S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1}\big(P_hB(t_k,X(t_k)) - B_h(t_k,X_{h,\tau}^k)\big)\right\|_{L^p(\Omega;L_2^0)}^2 \mathrm{d}s\Big)$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1+\tau^2A_{h,1}A_{h,2})$$

$$\alpha_{h,1} = (I+\tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I+\tau A_{h,2})^{-1}$$

Convergence proof

Splitting error

$$\left\| \left(\mathrm{e}^{-(t_n - t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} P_h \right) B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)}$$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1 + \tau^2 A_{h,1} A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Splitting error

$$\begin{aligned}
&\left\| \left(\mathrm{e}^{-(t_n - t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} P_h \right) B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)} \\
&\leq \left\| \left(\mathrm{e}^{-t_n A} - S_{h,\tau}^{n-1} \alpha_{h,2} \alpha_{h,1} P_h \right) A^{-\theta_B} \right\|_{\mathcal{L}(H)} \left\| A^{\theta_B} B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)}
\end{aligned}$$

$$S_{h,\tau} = \alpha_{h,2} \alpha_{h,1} (1 + \tau^2 A_{h,1} A_{h,2})$$

$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Splitting error

$$
\begin{aligned}
&\left\| \left(\mathrm{e}^{-(t_n - t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} P_h \right) B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)} \\
&\leq \Big(\| \left(\mathrm{e}^{-t_n A} - \mathrm{e}^{-t_n A_h} P_h \right) A^{-\theta_B} \|_{\mathcal{L}(H)} + \| \left(\mathrm{e}^{-t_n A_h} - S_{h,\tau}^{n-1} \alpha_{h,2} \alpha_{h,1} \right) P_h A^{-\theta_B} \|_{\mathcal{L}(H)} \Big) \\
&\qquad \times \left\| A^{\theta_B} B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)}
\end{aligned}
$$

$$
S_{h,\tau} = \alpha_{h,2} \alpha_{h,1} (1 + \tau^2 A_{h,1} A_{h,2})
$$

$$
\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}
$$

Convergence proof

Splitting error

$$\left\| \left(\mathrm{e}^{-(t_n - t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} P_h \right) B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)}$$
$$\leq \Big(\| \left(\mathrm{e}^{-t_n A} - \mathrm{e}^{-t_n A_h} P_h \right) A^{-\theta_B} \|_{\mathcal{L}(H)} + \| \left(\mathrm{e}^{-t_n A_h} - S_{h,\tau}^{n-1} \alpha_{h,2} \alpha_{h,1} \right) P_h A^{-\theta_B} \|_{\mathcal{L}(H)} \Big)$$
$$\times \left\| A^{\theta_B} B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)}$$

- Crouzeix: $\| \left(\mathrm{e}^{-t_n A} - \mathrm{e}^{-t_n A_h} P_h \right) A^{-\theta_B} \|_{\mathcal{L}(H)} \leq C t_n^{-\frac{1}{2}} h^{2\theta_B + 1}$

$$S_{h,\tau} = \alpha_{h,2} \alpha_{h,1} (1 + \tau^2 A_{h,1} A_{h,2})$$
$$\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}$$

Convergence proof

Splitting error

$$\begin{aligned}&\left\|\left(\mathrm{e}^{-(t_n-t_k)A} - S_{h,\tau}^{n-k-1}\alpha_{h,2}\alpha_{h,1}P_h\right)B(t_k,X(t_k))\right\|_{L^p(\Omega;L_2^0)}\\ &\leq \left(Ct_n^{-\frac{1}{2}}h^{2\theta_B+1} + \left\|\left(\mathrm{e}^{-t_nA_h} - S_{h,\tau}^{n-1}\alpha_{h,2}\alpha_{h,1}\right)P_hA^{-\theta_B}\right\|_{\mathcal{L}(H)}\right)\\ &\qquad \times \left\|A^{\theta_B}B(t_k,X(t_k))\right\|_{L^p(\Omega;L_2^0)}\end{aligned}$$

- Splitting: $\left\|\left(\mathrm{e}^{-t_nA_h} - S_{h,\tau}^{n-1}\alpha_{h,2}\alpha_{h,1}\right)P_hA^{-\theta_B}\right\|_{\mathcal{L}(H)} \leq C\tau^{\theta_B}(1+\ln(n))$

$$S_{h,\tau} = \alpha_{h,2}\alpha_{h,1}(1+\tau^2A_{h,1}A_{h,2})$$

$$\alpha_{h,1} = (I+\tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I+\tau A_{h,2})^{-1}$$

Convergence proof

Splitting error

$$
\begin{aligned}
&\left\| \left(\mathrm{e}^{-(t_n - t_k)A} - S_{h,\tau}^{n-k-1} \alpha_{h,2} \alpha_{h,1} P_h \right) B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)} \\
&\leq \left(C t_n^{-\frac{1}{2}} h^{2\theta_B + 1} + \left\| \left(\mathrm{e}^{-t_n A_h} - S_{h,\tau}^{n-1} \alpha_{h,2} \alpha_{h,1} \right) P_h A^{-\theta_B} \right\|_{\mathcal{L}(H)} \right) \\
&\qquad \times \left\| A^{\theta_B} B(t_k, X(t_k)) \right\|_{L^p(\Omega; L_2^0)}
\end{aligned}
$$

- Splitting: $\left\| \left(\mathrm{e}^{-t_n A_h} - S_{h,\tau}^{n-1} \alpha_{h,2} \alpha_{h,1} \right) P_h A^{-\theta_B} \right\|_{\mathcal{L}(H)} \leq C \tau^{\theta_B} (1 + \ln(n))$
- If $A_{h,1}$ and $A_{h,2}$ are self-adjoint and commute: $\left\| \left(\mathrm{e}^{-t_n A_h} - S_{h,\tau}^{n-1} \alpha_{h,2} \alpha_{h,1} \right) P_h \right\|_{\mathcal{L}(H)} \leq C t_n^{-\frac{1}{2}} \tau^{\frac{1}{2}} (1 + \ln(n))^{\frac{1}{2}}$ and we could improve our result
[Ichinose, Tamura, 2001]

$$
S_{h,\tau} = \alpha_{h,2} \alpha_{h,1} (1 + \tau^2 A_{h,1} A_{h,2})
$$

$$
\alpha_{h,1} = (I + \tau A_{h,1})^{-1}, \quad \alpha_{h,2} = (I + \tau A_{h,2})^{-1}
$$

Outline

Underlying problem

Method: Domain decomposition integrator

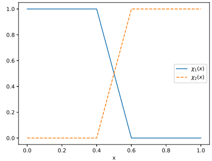
Error bound

Numerical example

Numerical example

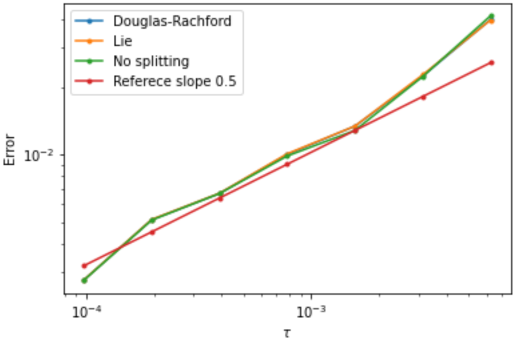
$$\begin{cases} \mathrm{d}X(t,\mathbf{x}) = \big(\Delta X(t,\mathbf{x}) + \pi^2(1+X(t,\mathbf{x}))\sin(\pi x_1)\sin(\pi x_2)\big)\,\mathrm{d}t \\ \qquad +10X(t,\mathbf{x})\,\mathrm{d}W(t,\mathbf{x}), & (t,\mathbf{x}) \in (0,0.1)\times\mathcal{D}; \\ X(t,\mathbf{x}) = 0, & (t,\mathbf{x}) \in (0,0.1)\times\partial\mathcal{D}; \\ X(0,\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2), & \mathbf{x}\in\mathcal{D}. \end{cases}$$

- DUNE-FEM (Dedner et al. (2020))
- $\mathcal{D} = (0,1)^2$
- 100 Monte Carlo iterations
- $\beta_{k,\ell}$ are i.i.d. Brownian motions
- $W(t,\mathbf{x}) = \sum_{(k,\ell)\in\mathbb{N}^2}(k^2+\ell^2)^{-2-\varepsilon}\sin(k\pi x_1)\sin(\ell\pi x_2)\beta_{k,\ell}(t)$



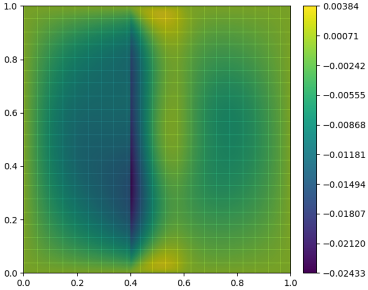
Numerical example

- Temporal error at the final time for $\tau = 0.1 \cdot 2^{-j}$ with $j = \{2, \ldots, 7\}$ and fixed spatial parameter $h = 1/200$

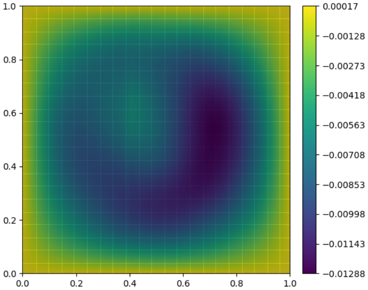


Lie vs. Modified Douglas–Rachford

- Comparison of splittings with $h = 1/200$ and $\tau = 0.1 \cdot 2^{-6}$



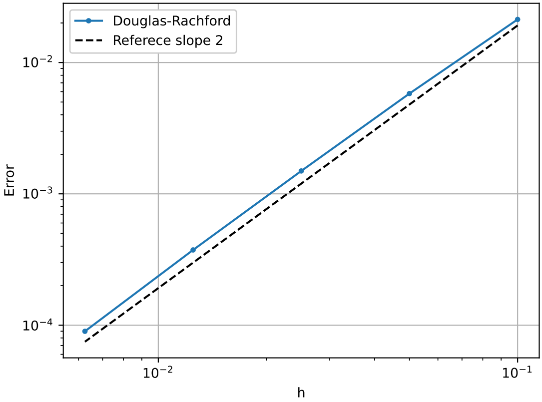
Lie splitting



Modified Douglas-Rachford splitting

Numerical example

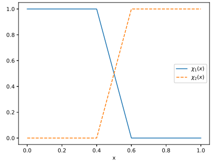
- Spatial error at the final time for $\tau = 10^{-4}$ and different space discretization with $h = \frac{1}{5} \cdot 2^{-j}$ and $j = \{1, \ldots, 5\}$



Numerical example

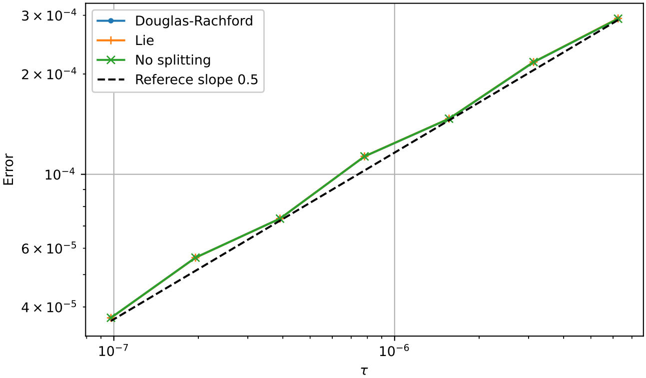
$$\begin{cases} \mathrm{d}X(t,\mathbf{x}) = \Delta X^4(t,\mathbf{x})\,\mathrm{d}t + X(t,\mathbf{x})\,\mathrm{d}W(t,\mathbf{x}), & (t,\mathbf{x}) \in [0,0.01] \times \mathcal{D}; \\ X(t,\mathbf{x}) = 0, & (t,\mathbf{x}) \in [0,0.01] \times \partial\mathcal{D}; \\ X(0,\mathbf{x}) = S^{-\frac{1}{5}}\left(\frac{1}{10} - \frac{3}{40}\frac{4(x-\frac{1}{2})^2}{S^{\frac{2}{5}}}\right), & \mathbf{x} \in \mathcal{D}, \end{cases}$$

- DUNE-FEM (Dedner et al. (2020))
- $\mathcal{D} = (0,1)$, $S = 0.02$,
- 100 Monte Carlo iterations
- β_k are i.i.d. Brownian motions
- $W(t,\mathbf{x}) = \sum_{k=1}^{\infty} k^{-\frac{5}{2}-2\varepsilon} \sin(k\pi\mathbf{x})\beta_k(t)$



Numerical example

- Temporal error at the final time for $\tau = 10^{-4} \cdot 2^{-j}$ with $j = \{4, \ldots, 10\}$ and fixed spatial parameter $h = 1/200$



Thank you for your attention!

Related preprint: https://doi.org/10.48550/arXiv.2412.10125

Underlying problem

Existence of a solution

- In the above setting, there exists a unique mild solution up to modifications.
- For $\theta_{X_0} \in [\theta_B, \theta_B + \frac{1}{2})$, fulfills

$$\sup_{t \in (0,t_f)} \|A^{\theta_{X_0}} X(t)\|_{L^p(\Omega;H)} \leq C, \quad t \in (0, t_f)$$

and

$$X \in C^{\min\{\theta_{X_0}, \frac{1}{2}\}}([0, t_f]; L^p(\Omega; H)).$$

Underlying problem

Existence of a solution

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$$\sup_{t \in (0,t_f)} \|A^{\theta_{X_0}} X(t)\|_{L^p(\Omega;H)} \leq C, \quad t \in (0, t_f)$$

and

$$X \in C^{\min\{\theta_{X_0}, \frac{1}{2}\}}([0, t_f]; L^p(\Omega; H)).$$

In short

We simplify $\theta_{X_0} = \frac{1}{2}$:

- $$\sup_{t \in (0,t_f)} \|A^{\frac{1}{2}-\varepsilon} X(t)\|_{L^p(\Omega;H)} \leq C, \quad t \in (0, t_f)$$

- $$X \in C^{\frac{1}{2}}([0, t_f]; L^p(\Omega; H)).$$