Network inference via approximate Bayesian computation. Illustration on a stochastic multi-population neural mass model

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joint work with
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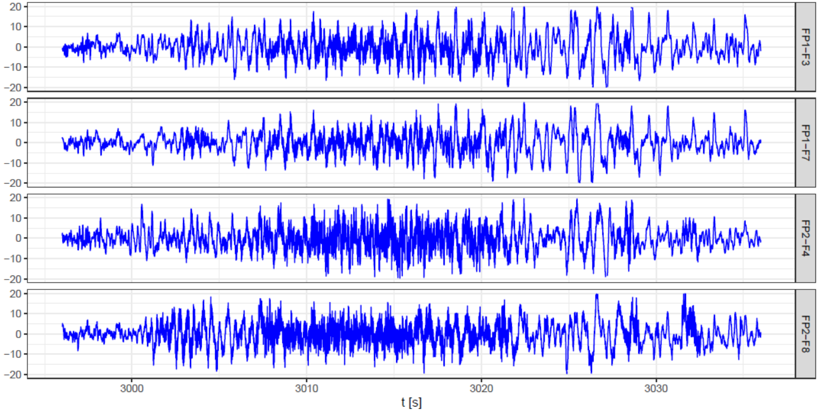
Massimiliano Tamborrino (University of Warwick)

JOHANNES KEPLER UNIVERSITY LINZ

EEG data with epileptic activity[1]



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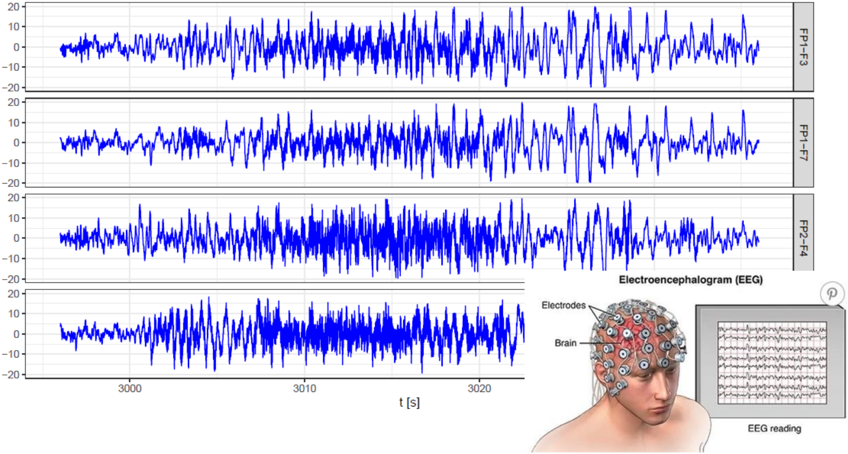

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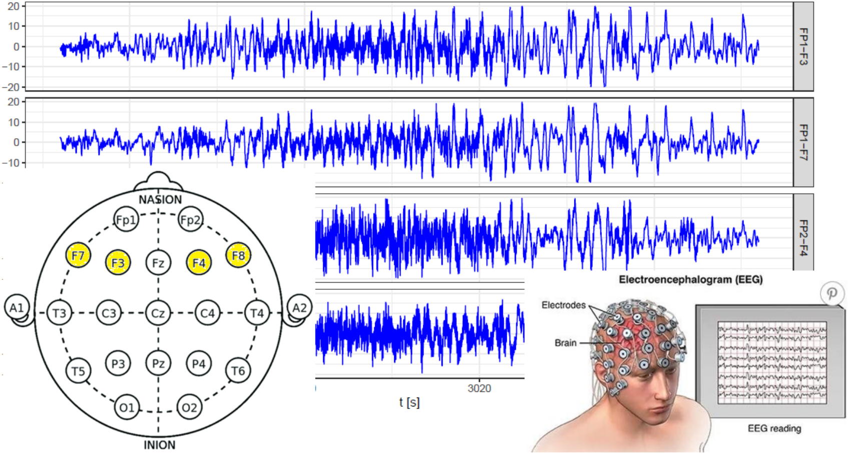

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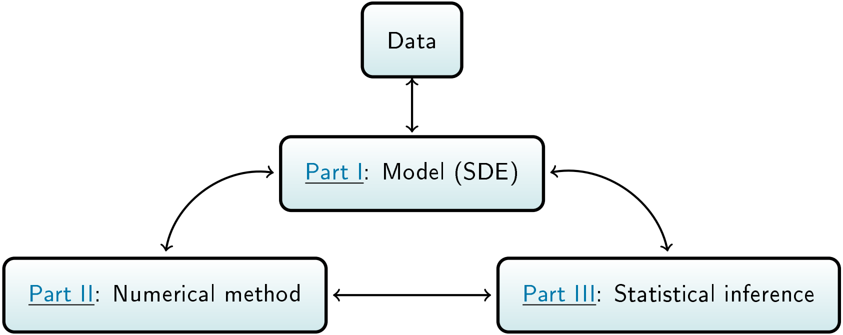
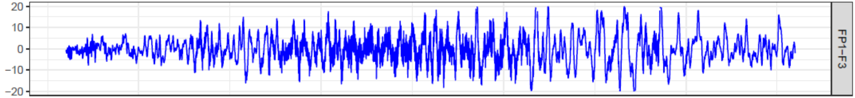
Overview


Data
Part I: Model (SDE)
Part II: Numerical method
Part III: Statistical inference

Part I

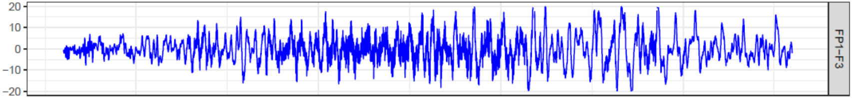
Multi-population stochastic neural mass model

Modelling one neural population[2]



[2] Jansen & Rit (*Biol. Cybern.*, 1995), Ableidinger, Buckwar & Hinterleitner (*J. Math. Neurosci.*, 2017)

Modelling one neural population[2]



- 6-dimensional SDE, describing the activity of one population of neurons:

$$
\begin{aligned}
dX_1(t) &= X_4(t)dt \\
dX_2(t) &= X_5(t)dt \\
dX_3(t) &= X_6(t)dt \\
dX_4(t) &= \left[Aa\left(\mathrm{sig}\left(X_2(t) - X_3(t)\right)\right) - 2aX_4(t) - a^2 X_1(t)\right] dt + \tau dW_4(t) \\
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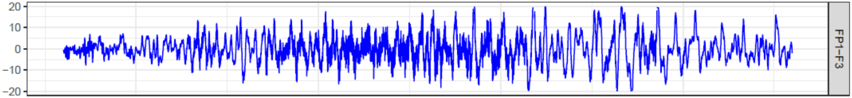

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- Observed process: $Y(t) := X_2(t) - X_3(t)$ (*EEG signal*)

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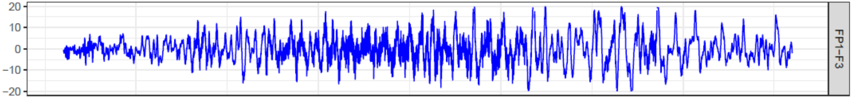

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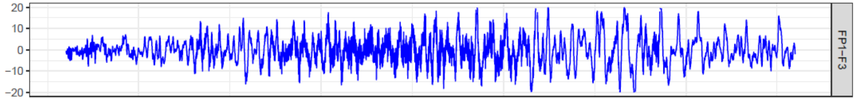

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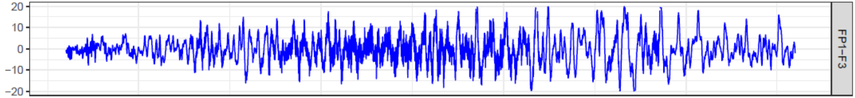

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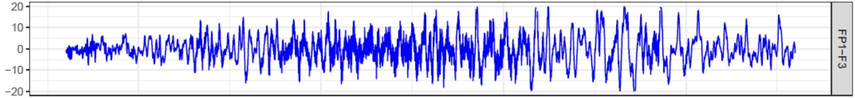

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dX_5(t) &= \left[Aa\left(\mu + C_2\mathrm{sig}\left(C_1X_1(t)\right)\right) - 2aX_5(t) - a^2X_2(t)\right] dt + \sigma d W_5(t) \\
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Modelling N neural populations

$$dX_1(t) = X_4(t)dt$$

$$dX_2(t) = X_5(t)dt$$

$$dX_3(t) = X_6(t)dt$$

$$dX_4(t) = \left[A\ a\ \left(\mathrm{sig}\left(X_2(t) - X_3(t)\right)\right) - 2a\ X_4(t) - a^2 X_1(t)\right] dt + \tau\ dW_4(t)$$

$$dX_5(t) = \left[A\ a\ \left(\mu\ + C_2\ \mathrm{sig}(C_1\ X_1(t))\right) - 2a\ X_5(t) - a^2 X_2(t)\right] dt + \sigma\ dW_5(t)$$

$$dX_6(t) = \left[B\ b\ C_4\ \mathrm{sig}\left(C_3\ X_1(t)\right) - 2b\ X_6(t) - b^2 X_3(t)\right] dt + \tau\ dW_6(t)$$

Modelling N neural populations

- 6-dimensional SDE, describing the k-th, $k = 1, \ldots, N$, population of neurons:

$$
\begin{aligned}
dX_1^k(t) &= X_4^k(t)dt \\
dX_2^k(t) &= X_5^k(t)dt \\
dX_3^k(t) &= X_6^k(t)dt \\
dX_4^k(t) &= \left[A\ a\ \left(\mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right)\right) - 2a\ X_4^k(t) - a^2 X_1^k(t)\right] dt + \tau\ dW_4^k(t) \\
dX_5^k(t) &= \left[A\ a\ \left(\mu\ + C_2\ \mathrm{sig}\left(C_1\ X_1^k(t)\right)\right) - 2a\ X_5^k(t) - a^2 X_2^k(t)\right] dt + \sigma\ dW_5^k(t) \\
dX_6^k(t) &= \left[B\ b\ C_4\ \mathrm{sig}\left(C_3\ X_1^k(t)\right) - 2b\ X_6^k(t) - b^2 X_3^k(t)\right] dt + \tau\ dW_6^k(t)
\end{aligned}
$$

- Observed process: $Y^k(t) := X_2^k(t) - X_3^k(t)$ (k-*th EEG signal*)

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dX_5^k(t) &= \left[A_k a_k (\mu_k + C_{2k}\mathrm{sig}(C_{1k} X_1^k(t))) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t) \\
dX_6^k(t) &= \left[B_k b_k C_{4k} \mathrm{sig}\left(C_{3k} X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)
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Modelling N coupled neural populations[3]

- $6N$-dimensional SDE, describing N interacting populations of neurons, where the k-th, $k = 1, \dots, N$, population is defined via:

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dX_5^k(t) &= \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k}X_1^k(t)\right) \qquad \right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt \\
&\quad + \sigma_k dW_5^k(t) \\
dX_6^k(t) &= \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k}X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t).
\end{aligned}
$$

- Observed process (*N EEG signals*):

$$
Y(t) := (Y^1(t), \dots, Y^N(t))^\top = (X_2^1(t) - X_3^1(t), \dots, X_2^N(t) - X_3^N(t))^\top
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[3] Wendling et al. (*Biol. Cybern.*, 2000)

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dX_5^k(t) &= \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt \\
&\quad + \sigma_k dW_5^k(t) \\
dX_6^k(t) &= \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k}X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t).
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dX_5^k(t) &= \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k} X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt \\
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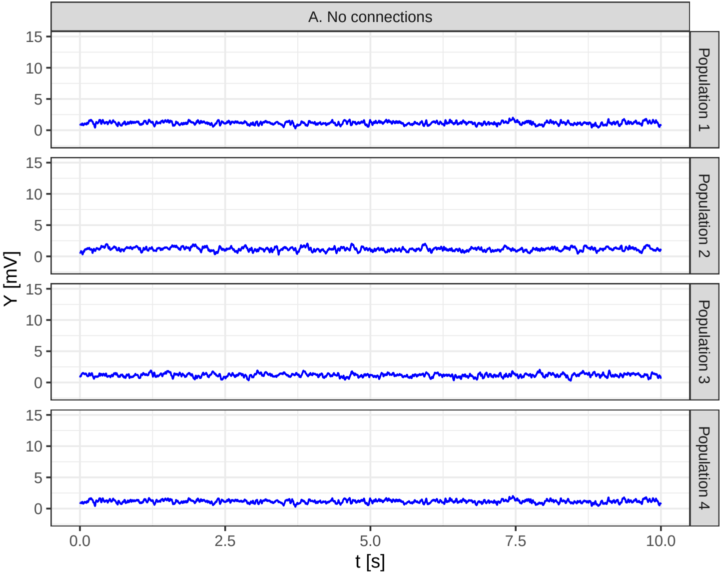
- Network: Coupling direction parameters $\rho_{jk} \in \{0, 1\}$

[3] Wendling et al. (*Biol. Cybern.*, 2000)

No connections: $\rho_{jk} = 0$



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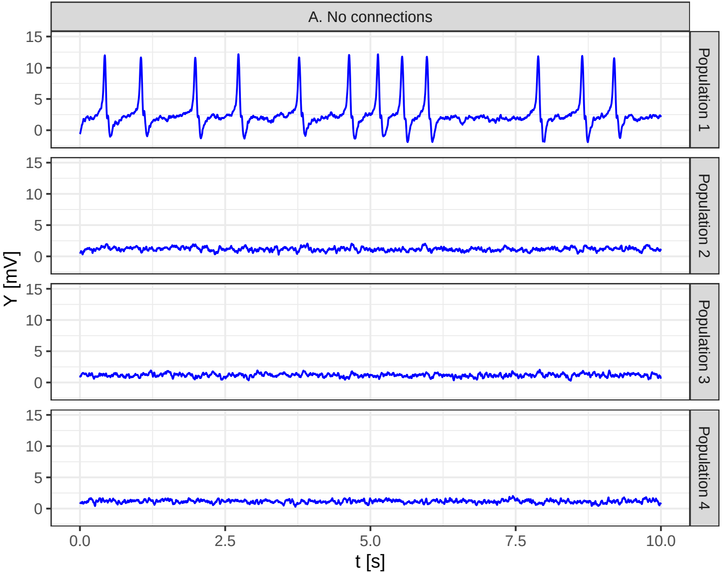
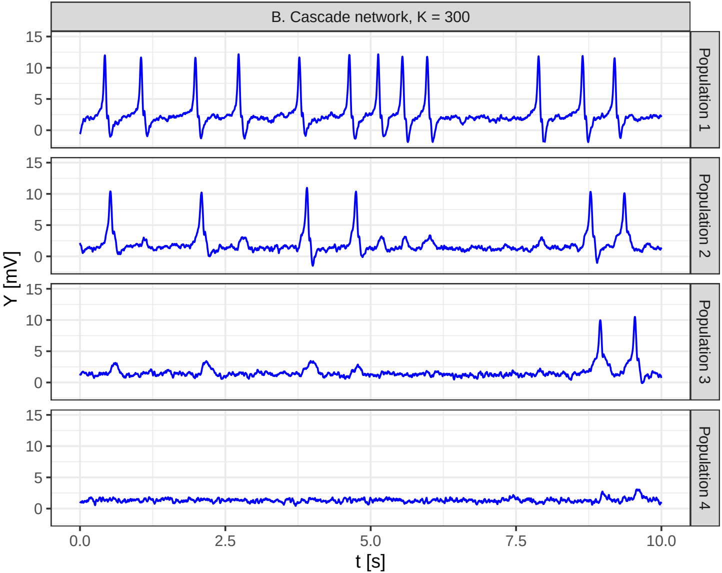


Cascade network: $\rho_{12} = \rho_{23} = \rho_{34} = 1$



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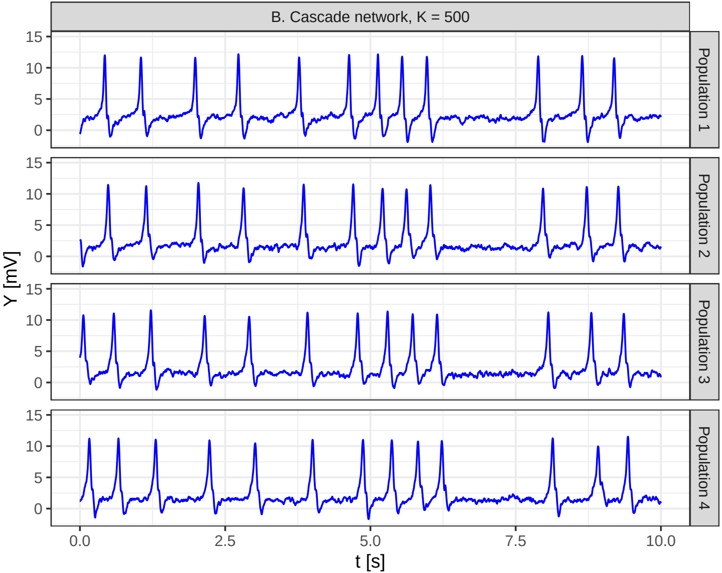

Part II
Numerical simulation method: Splitting

Part II

Numerical simulation method: Splitting[4]

[4] Ableidinger, Buckwar & Hinterleitner (*J. Math. Neurosci.*, 2017)

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$
\begin{aligned}
dX_1^k(t) &= X_4^k(t)dt \\
dX_2^k(t) &= X_5^k(t)dt \\
dX_3^k(t) &= X_6^k(t)dt \\
dX_4^k(t) &= \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t) \\
dX_5^k(t) &= \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k} X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt \\
&\quad + \sigma_k dW_5^k(t) \\
dX_6^k(t) &= \left[B_k b_k C_{4k} \mathrm{sig}\left(C_{3k} X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)
\end{aligned}
$$

- Re-formulation:

$$
d\left(\quad\right) = \left(\qquad\qquad\qquad\right) dt \; + \left(\quad\right) dW^k(t)
$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$
$$dX_2^k(t) = X_5^k(t)dt$$
$$dX_3^k(t) = X_6^k(t)dt$$
$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$
$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k} X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt$$
$$+ \sigma_k dW_5^k(t)$$
$$dX_6^k(t) = \left[B_k b_k C_{4k} \mathrm{sig}\left(C_{3k} X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix} dt + \begin{pmatrix} \quad \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$

$$dX_2^k(t) = X_5^k(t)dt$$

$$dX_3^k(t) = X_6^k(t)dt$$

$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$

$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k} X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t)$$

$$dX_6^k(t) = \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k} X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \Big(\qquad\qquad\qquad \Big) dt + \Big(\quad \Big) dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$
$$dX_2^k(t) = X_5^k(t)dt$$
$$dX_3^k(t) = X_6^k(t)dt$$
$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$
$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k} X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t)$$
$$dX_6^k(t) = \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k} X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ \end{pmatrix} dt + \begin{pmatrix} \\ \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$

$$dX_2^k(t) = X_5^k(t)dt$$

$$dX_3^k(t) = X_6^k(t)dt$$

$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$

$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t)$$

$$dX_6^k(t) = \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k}X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ -\Gamma_k^2 Q^k(t) \end{pmatrix} dt + \begin{pmatrix} \\ \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$
$$dX_2^k(t) = X_5^k(t)dt$$
$$dX_3^k(t) = X_6^k(t)dt$$
$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$
$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t)$$
$$dX_6^k(t) = \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k}X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ -\Gamma_k^2 Q^k(t) - 2\Gamma_k P^k(t) \end{pmatrix} dt + \begin{pmatrix} \\ \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$

$$dX_2^k(t) = X_5^k(t)dt$$

$$dX_3^k(t) = X_6^k(t)dt$$

$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$

$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t)$$

$$dX_6^k(t) = \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k}X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ -\Gamma_k^2 Q^k(t) - 2\Gamma_k P^k(t) + G_k(Q(t)) \end{pmatrix} dt + \begin{pmatrix} \\ \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- k-th population of the $6N$-dimensional SDE:

$$dX_1^k(t) = X_4^k(t)dt$$
$$dX_2^k(t) = X_5^k(t)dt$$
$$dX_3^k(t) = X_6^k(t)dt$$
$$dX_4^k(t) = \left[A_k a_k \mathrm{sig}\left(X_2^k(t) - X_3^k(t)\right) - 2a_k X_4^k(t) - a_k^2 X_1^k(t)\right] dt + \tau_k dW_4^k(t)$$
$$dX_5^k(t) = \left[A_k a_k \left(\mu_k + C_{2k}\mathrm{sig}\left(C_{1k}X_1^k(t)\right) + \sum_{j=1, j\neq k}^{N} \rho_{jk} K_{jk} X_1^j(t)\right) - 2a_k X_5^k(t) - a_k^2 X_2^k(t)\right] dt + \sigma_k dW_5^k(t)$$
$$dX_6^k(t) = \left[B_k b_k C_{4k}\mathrm{sig}\left(C_{3k}X_1^k(t)\right) - 2b_k X_6^k(t) - b_k^2 X_3^k(t)\right] dt + \tau_k dW_6^k(t)$$

- Re-formulation:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ -\Gamma_k^2 Q^k(t) - 2\Gamma_k P^k(t) + G_k(Q(t)) \end{pmatrix} dt + \begin{pmatrix} \mathbb{O}_3 \\ \Sigma_k \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- Re-formulation of the k-th population:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ -\Gamma_k^2 Q^k(t) - 2\Gamma_k P^k(t) + G_k(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_3 \\ \Sigma_k \end{pmatrix} dW^k(t)$$

Re-formulation: stochastic Hamiltonian type system

- Re-formulation of the k-th population:

$$d\begin{pmatrix} Q^k(t) \\ P^k(t) \end{pmatrix} = \begin{pmatrix} P^k(t) \\ -\Gamma_k^2 Q^k(t) - 2\Gamma_k P^k(t) + G_k(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_3 \\ \Sigma_k \end{pmatrix} dW^k(t)$$

- Re-formulation of the $6N$-dimensional SDE:

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t)$$

  - $Q = (Q^1, \dots, Q^N)^\top$, $P = (P^1, \dots, P^N)^\top$
  - $W = (W^1, \dots, W^N)^\top$, $G(Q) = (G_1(Q), \dots, G_N(Q))^\top$
  - $\Gamma = \text{diag}[a_1, a_1, b_1, \dots, a_N, a_N, b_N]$, $\Sigma = \text{diag}[\tau_1, \sigma_1, \tau_1, \dots, \tau_N, \sigma_N, \tau_N]$

- $6N$-dimensional SDE:

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t)$$

Splitting

- $6N$-dimensional SDE:

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t)$$

Splitting

- $6N$-dimensional SDE:

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t)$$

Splitting

- $6N$-dimensional SDE:

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \;+\; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t)$$

Splitting

- $6N$-dimensional SDE:

$$d\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} P(t) \\ -\Gamma^2 Q(t) - 2\Gamma P(t) + G(Q(t)) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t)$$

- Split it into two exactly solvable subsystems:

$$d\begin{pmatrix} Q^{[1]}(t) \\ P^{[1]}(t) \end{pmatrix} = \begin{pmatrix} P^{[1]}(t) \\ -\Gamma^2 Q^{[1]}(t) - 2\Gamma P^{[1]}(t) \end{pmatrix} dt \; + \; \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix} dW(t) \tag{1}$$

$$d\begin{pmatrix} Q^{[2]}(t) \\ P^{[2]}(t) \end{pmatrix} = \begin{pmatrix} 0_{3N} \\ G(Q^{[2]}(t)) \end{pmatrix} dt \tag{2}$$

Exact solution of subsystem (1)

- Subsystem (1):

$$dX^{[1]}(t) = \bar{A}X^{[1]}(t)dt + \bar{\Sigma}dW(t), \qquad \bar{A} = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}$$

Exact solution of subsystem (1)

- Subsystem (1):

$$dX^{[1]}(t) = \bar{A}X^{[1]}(t)dt + \bar{\Sigma}dW(t), \qquad \bar{A} = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}$$

- Exact solution:

$$X^{[1]}(t_{i+1}) = \varphi_{\Delta}^{[1]}\left(X^{[1]}(t_i)\right) = e^{\bar{A}\Delta}X^{[1]}(t_i) + \xi_i(\Delta), \qquad \xi_i(\Delta) \sim \mathcal{N}\left(0_{6N}, \bar{C}(\Delta)\right)$$

Exact solution of subsystem (1)

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$$dX^{[1]}(t) = \bar{A}X^{[1]}(t)dt + \bar{\Sigma}dW(t), \qquad \bar{A} = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}$$

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$$dX^{[1]}(t) = \bar{A}X^{[1]}(t)dt + \bar{\Sigma}dW(t), \qquad \bar{A} = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}$$

- Exact solution:

$$X^{[1]}(t_{i+1}) = \varphi_{\Delta}^{[1]}\left(X^{[1]}(t_i)\right) = e^{\bar{A}\Delta}X^{[1]}(t_i) + \xi_i(\Delta), \qquad \xi_i(\Delta) \sim \mathcal{N}\left(0_{6N}, \bar{C}(\Delta)\right)$$

- $$e^{\bar{A}\Delta} = \begin{pmatrix} e^{-\Gamma\Delta}\left(\mathbb{I}_{3N} + \Gamma\Delta\right) & e^{-\Gamma\Delta}\Delta \\ -\Gamma^2 e^{-\Gamma\Delta}\Delta & e^{-\Gamma\Delta}\left(\mathbb{I}_{3N} - \Gamma\Delta\right) \end{pmatrix}$$

Exact solution of subsystem (1)

- Subsystem (1):

$$dX^{[1]}(t) = \bar{A}X^{[1]}(t)dt + \bar{\Sigma}dW(t), \qquad \bar{A} = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}$$

- Exact solution:

$$X^{[1]}(t_{i+1}) = \varphi_{\Delta}^{[1]}\left(X^{[1]}(t_i)\right) = e^{\bar{A}\Delta}X^{[1]}(t_i) + \xi_i(\Delta), \qquad \xi_i(\Delta) \sim \mathcal{N}\left(0_{6N}, \bar{C}(\Delta)\right)$$

- $$e^{\bar{A}\Delta} = \begin{pmatrix} e^{-\Gamma\Delta}\left(\mathbb{I}_{3N} + \Gamma\Delta\right) & e^{-\Gamma\Delta}\Delta \\ -\Gamma^2 e^{-\Gamma\Delta}\Delta & e^{-\Gamma\Delta}\left(\mathbb{I}_{3N} - \Gamma\Delta\right) \end{pmatrix} =: \begin{pmatrix} \vartheta(\Delta) & \kappa(\Delta) \\ \vartheta'(\Delta) & \kappa'(\Delta) \end{pmatrix}$$

Exact solution of subsystem (1)

- Subsystem (1):

$$dX^{[1]}(t) = \bar{A}X^{[1]}(t)dt + \bar{\Sigma}dW(t), \qquad \bar{A} = \begin{pmatrix} \mathbb{O}_{3N} & \mathbb{I}_{3N} \\ -\Gamma^2 & -2\Gamma \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \mathbb{O}_{3N} \\ \Sigma \end{pmatrix}$$

- Exact solution:

$$X^{[1]}(t_{i+1}) = \varphi_\Delta^{[1]}\left(X^{[1]}(t_i)\right) = e^{\bar{A}\Delta}X^{[1]}(t_i) + \xi_i(\Delta), \qquad \xi_i(\Delta) \sim \mathcal{N}\left(0_{6N}, \bar{C}(\Delta)\right)$$

- $$e^{\bar{A}\Delta} = \begin{pmatrix} e^{-\Gamma\Delta}\left(\mathbb{I}_{3N} + \Gamma\Delta\right) & e^{-\Gamma\Delta}\Delta \\ -\Gamma^2 e^{-\Gamma\Delta}\Delta & e^{-\Gamma\Delta}\left(\mathbb{I}_{3N} - \Gamma\Delta\right) \end{pmatrix} =: \begin{pmatrix} \vartheta(\Delta) & \kappa(\Delta) \\ \vartheta'(\Delta) & \kappa'(\Delta) \end{pmatrix}$$
- $$\bar{C}(\Delta) = \begin{pmatrix} \frac{1}{4}\Gamma^{-3}\Sigma^2\left(\mathbb{I}_{3N} + \kappa(\Delta)\vartheta'(\Delta) - \vartheta(\Delta)^2\right) & \frac{1}{2}\Sigma^2\kappa(\Delta)^2 \\ \frac{1}{2}\Sigma^2\kappa(\Delta)^2 & \frac{1}{4}\Gamma^{-1}\Sigma^2\left(\mathbb{I}_{3N} + \kappa(\Delta)\vartheta'(\Delta) - \kappa'(\Delta)^2\right) \end{pmatrix}$$

Exact solution of subsystem (2)

- Subsystem (2):

$$d\begin{pmatrix} Q^{[2]}(t) \\ P^{[2]}(t) \end{pmatrix} = \begin{pmatrix} 0_{3N} \\ G(Q^{[2]}(t)) \end{pmatrix} dt$$

Exact solution of subsystem (2)

- Subsystem (2):

$$d\begin{pmatrix} Q^{[2]}(t) \\ P^{[2]}(t) \end{pmatrix} = \begin{pmatrix} 0_{3N} \\ G(Q^{[2]}(t)) \end{pmatrix} dt$$

- Exact solution:

$$X^{[2]}(t_{i+1}) = \varphi_{\Delta}^{[2]}\left(X^{[2]}(t_i)\right) = X^{[2]}(t_i) + \Delta \begin{pmatrix} 0_{3N} \\ G(Q^{[2]}(t_i)) \end{pmatrix}$$

Part III

Adapted SMC-ABC algorithm for network inference: nSMC-ABC

Classical ABC method[5]

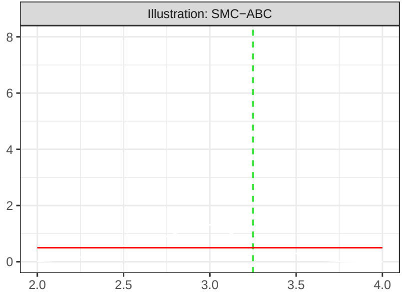
Acceptance-rejection ABC algorithm

Input: Observed data y, prior distribution $\pi(\theta)$, threshold δ.
Output: Samples from the ABC posterior distribution $\pi_{\mathrm{ABC}}(\theta|y)$.

1: **for** $j = 1$ to M **do**
2: Sample θ^* from the prior $\pi(\theta)$.
3: Conditioned on θ^*, simulate synthetic data y_{θ^*} from the SDE.
4: Compute summaries $s(y_{\theta^*})$ and a distance $D = d(s(y), s(y_{\theta^*}))$.
5: If $D < \delta$, keep θ^* as a sample from the ABC posterior.
6: **end for**

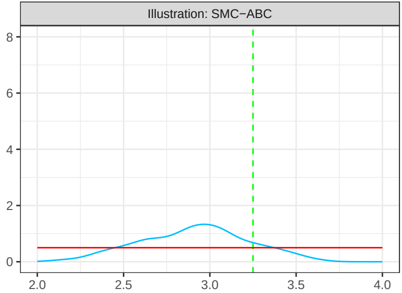
[5] Beaumont, Zhang & Balding (*Genetics*, 2002)

SMC-ABC method[6]



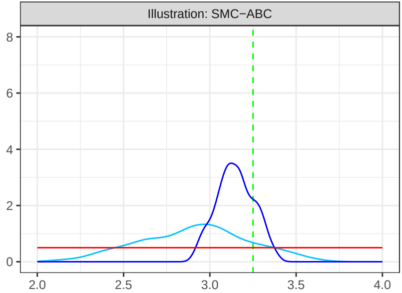
[6] Moral, Doucet & Jasra (*Stat. Comput.*, 2012)

SMC-ABC method[6]



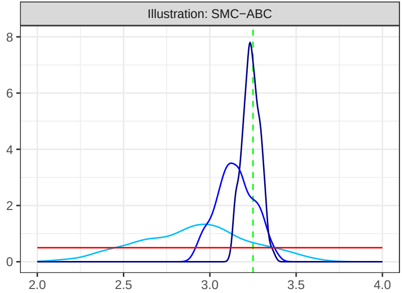
[6] Moral, Doucet & Jasra (*Stat. Comput.*, 2012)

SMC-ABC method[6]



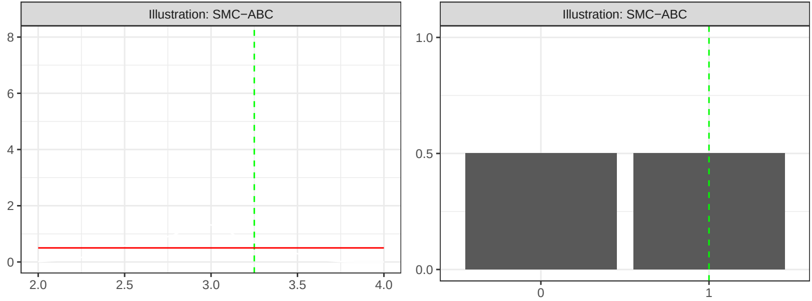
[6] Moral, Doucet & Jasra (*Stat. Comput.*, 2012)

SMC-ABC method[6]

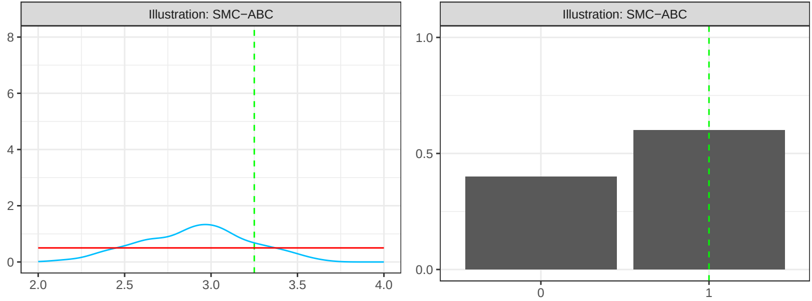


[6] Moral, Doucet & Jasra (*Stat. Comput.*, 2012)

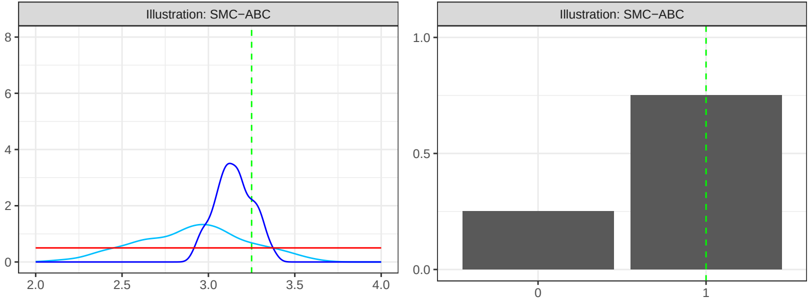
Adapted SMC-ABC method for network inference: nSMC-ABC



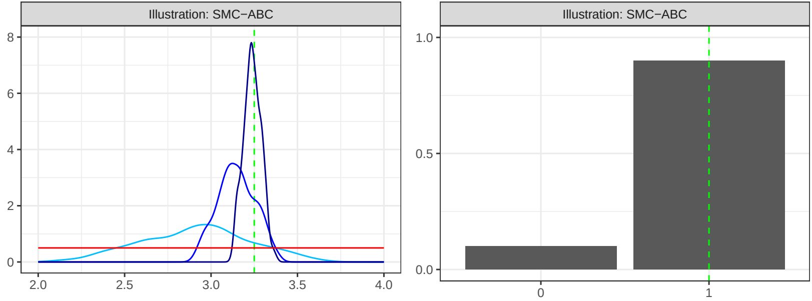
Adapted SMC-ABC method for network inference: nSMC-ABC



Adapted SMC-ABC method for network inference: nSMC-ABC



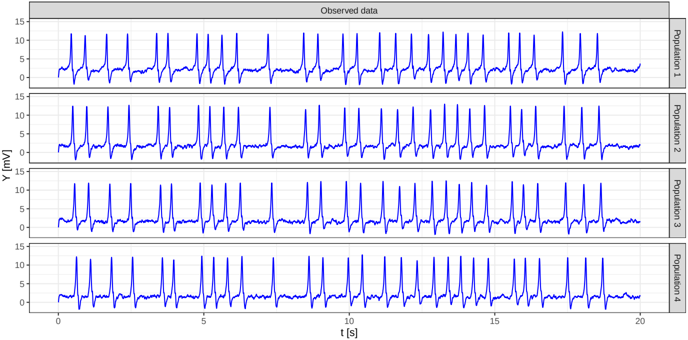
Adapted SMC-ABC method for network inference: nSMC-ABC



Inference results

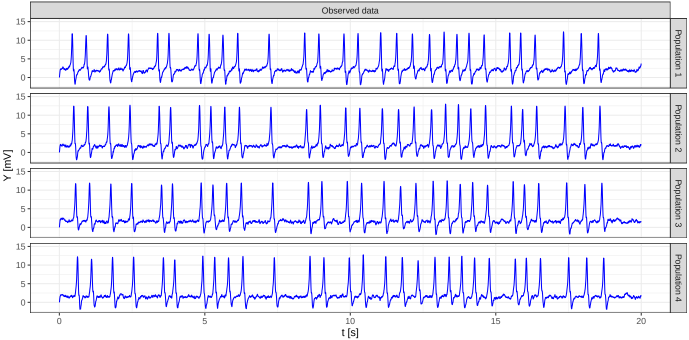
Inference results

- Observed data:



Inference results

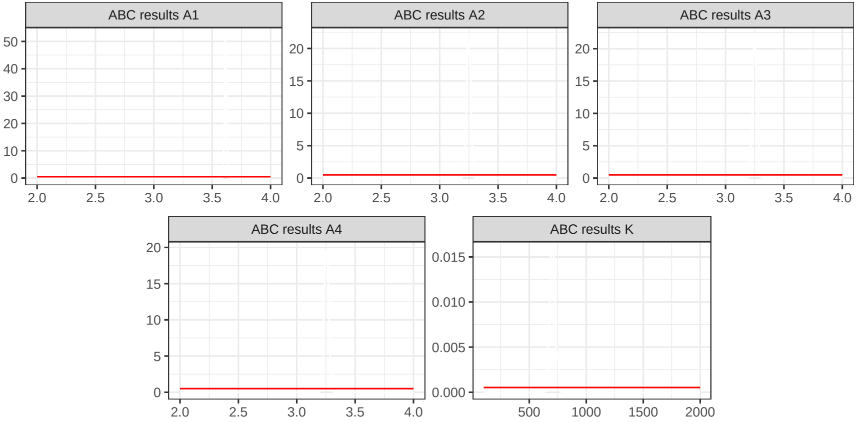
- Observed data:



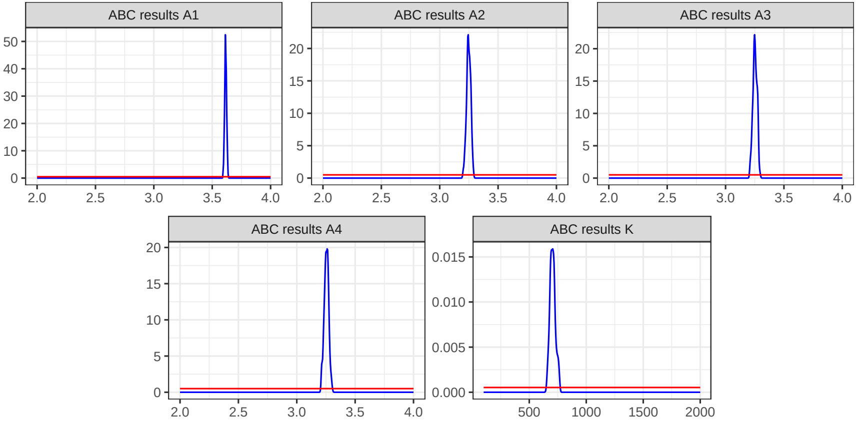
- Parameter vector:

$$\theta = (A_1, A_2, A_3, A_4, K, \text{vec}(\mathcal{P})), \qquad \mathcal{P} = \begin{pmatrix} - & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & - & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & - & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & - \end{pmatrix}$$

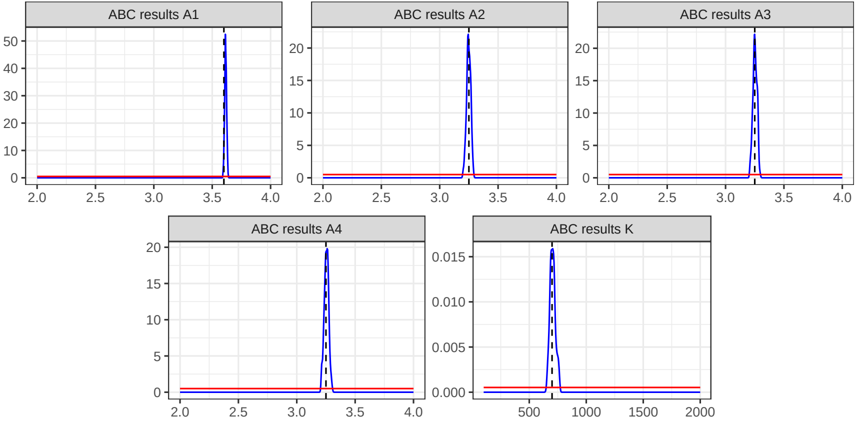
Inference results - continuous parameters



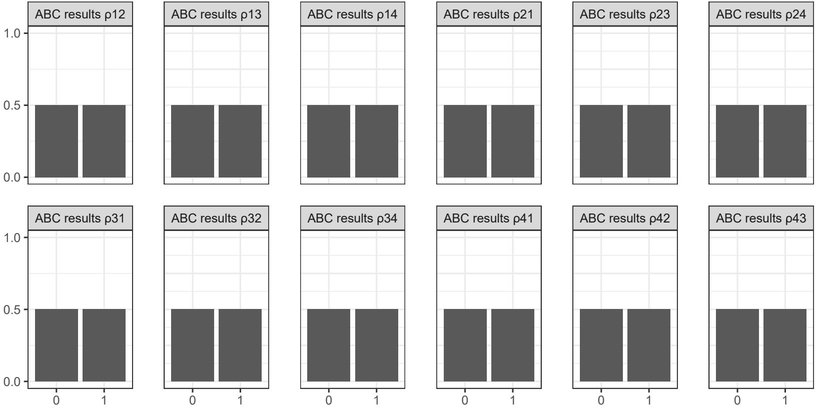
Inference results - continuous parameters



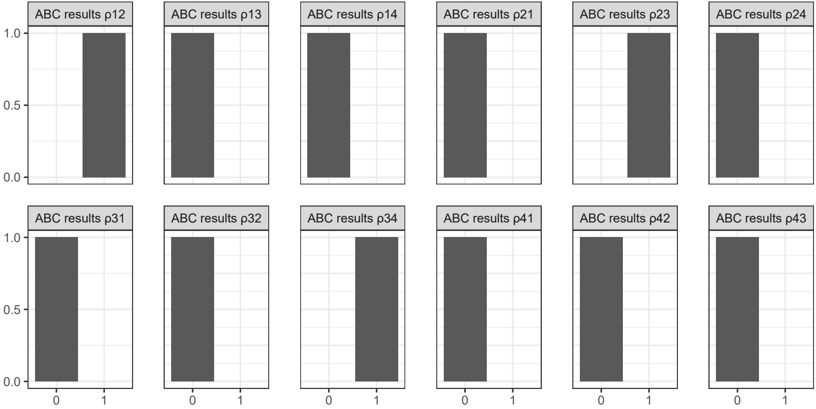
Inference results - continuous parameters



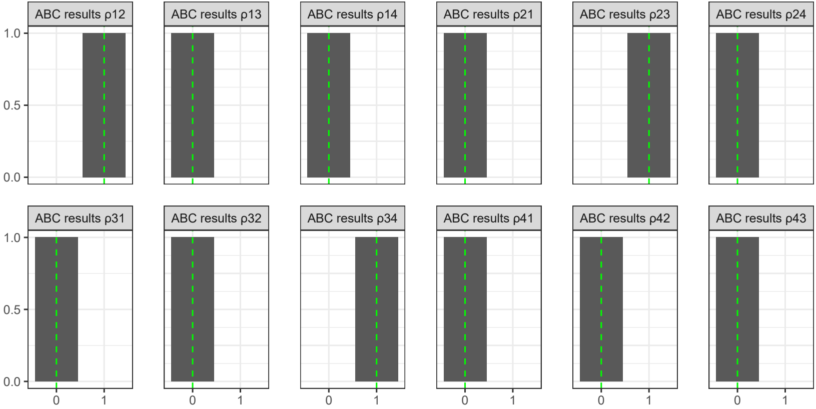
Inference results - discrete (network) parameters



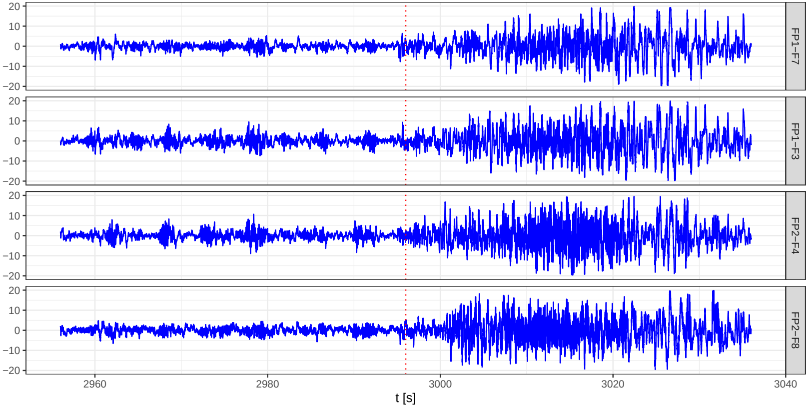
Inference results - discrete (network) parameters



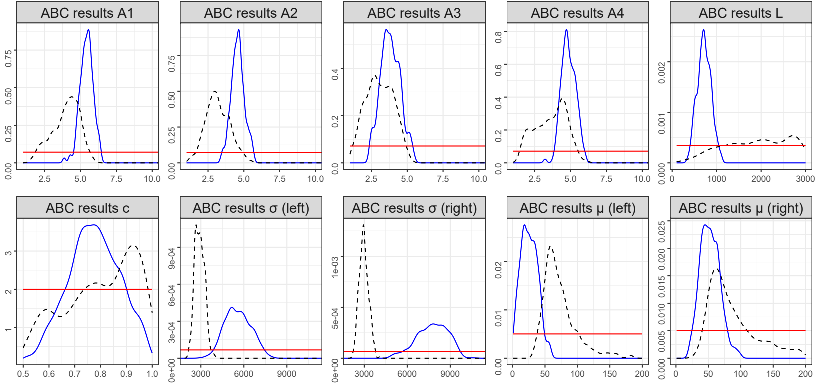
Inference results - discrete (network) parameters



Inference results - EEG data



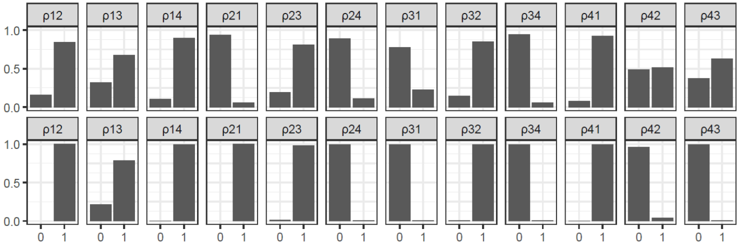
Inference results - EEG data

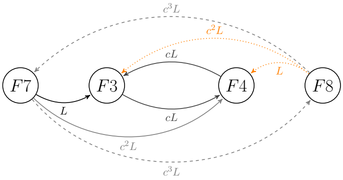


During seizure:

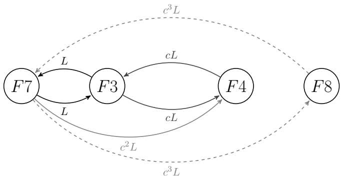
- larger activation in each of the individual neural populations
- larger noise intensity in both the left and the right brain hemisphere

Inference results - EEG data



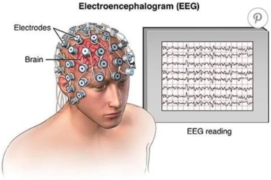


(a) Before seizure

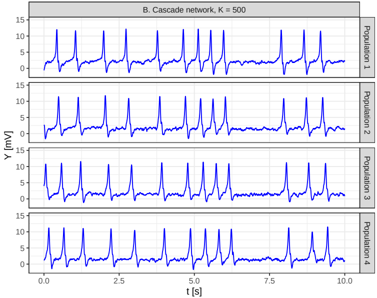


(b) During seizure

During seizure: stronger connectivity in the left brain hemisphere


Electroencephalogram (EEG)
Electrodes
Brain
EEG reading

Thank you!


B. Cascade network, K = 500
Population 1
Population 2
Population 3
Population 4
Y [mV]
t [s]

Literature

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