Randomised Quadratures and Randomised Numerical Schemes

Workshop "Milstein's method: 50 years on", Nottingham.

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RANDOMISED NUMERICAL SCHEMES

– ODES

An example

Consider a one-dimensional ODE of Carathéodory type

$$\dot{x}(t) = -\frac{|x(t)|}{1000} + \left|\sin(2^9\pi t)\right| := f(t, x(t)), \qquad \text{(ODE1)}$$

with $x(0) = 1.1$ and $t \in [0, 1]$. Easier to check that $\exists C > 0$ s.t.

$$|f(t, x) - f(t, y)| \leq C|x - y|. \qquad \text{(Lip cond)}$$

✓ (ODE1) admits a unique global solution over $[0, 1]$.[2]

Aim: to simulate the solution trajectory via (forward) Euler method.

[2]Note that if the coefficient f is Hölder only, it does not admit unique solution. For instance, $\dot{x}(t) = x^{1/3}$ subject to $x(0) = 0$ with two solutions, a trivial one and another one $x(t) = (2t/3)^{3/2}$.

An example

Step 0. Fix an equidistant partition $\mathcal{T}^k$ of $[0,1]$ of the form

$$\mathcal{T}^k = \{t_0 = 0 < t_1 < \cdots < t_j < \cdots < t_{N_k} = 1\} \text{ with } t_j = jk.$$

Euler Scheme

Step 1. *Set* $x_0 = 1.1$;

Step 2. *For* j *in* $\{1, \ldots, N_k\}$, *iteratively evaluate*

$$\begin{aligned} x_j &= x_{j-1} + kf(x_{j-1}, t_{j-1}) \\ &= x_{j-1} + k\Big(- \frac{|x_{j-1}|}{1000} + \big|\sin(2^9 \pi t_{j-1})\big|\Big). \end{aligned} \tag{Euler}$$

The divergence

If choosing stepsize $k = 1/2^i$, with $i \leq 9$, then in (Euler)

$$\left|\sin(2^9 \pi t_{j-1})\right| = \left|\sin\left(2^9 \pi \frac{j-1}{2^i}\right)\right| = \left|\sin(2^{9-i}(j-1)\pi)\right| = 0.$$

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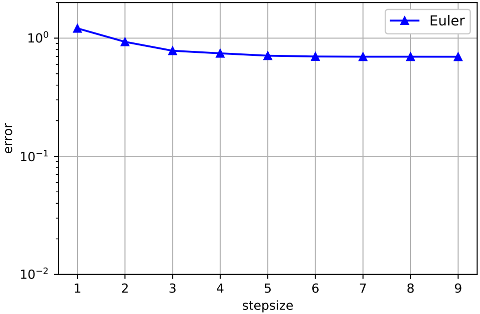

Figure: Euler scheme for (ODE1), stepsize vs error.

Randomized Euler scheme (Kruse& W., 2017)

Step 2. *Sample* $\tau_j \sim \mathcal{U}(0,1)$, *then*

$$\hat{t}_{j-1} = t_{j-1} + k\tau_j \in [t_{j-1}, t_j],$$

$$x_j = x_{j-1} + k\Big(- \frac{|x_{j-1}|}{1000} + \big|\sin(2^9 \pi \hat{t}_{j-1})\big|\Big).$$

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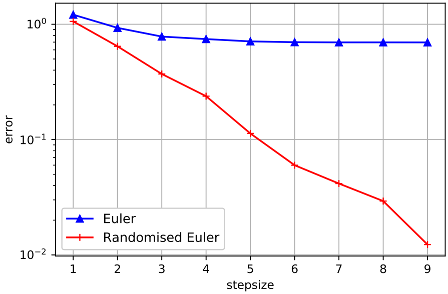

Figure: Euler vs randomized Euler for (ODE1), stepsize vs error.

The general case

Consider numerical approximation of

$$\dot{x}(t) = f(t, x(t)), \quad t \in [0, T], \quad x(0) = x_0, \qquad \text{(ODE)}$$

where $f\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is γ-Hölder continuous in t, $0 < \gamma \leq 1$; Lipschitz in state variable, ie,

$$|f(t, x) - f(s, x)| < C|t - s|^{\gamma}, \quad \forall x \in \mathbb{R}^d. \qquad \text{(Hölder)}$$

Remark (Heinrich&Milla, 2008)

The minimum error of any deterministic method depending only on $N \in \mathbb{N}$ point evaluations of f is of order $\mathcal{O}(N^{-\gamma})$.

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Strategy: Deterministic method '$\times$' Stratified Monte-Carlo

Stratified Monte-Carlo integration

Sampling

$(\tau_j)_j$: a seq. of $\mathcal{U}(0,1)$-distributed IID r.v. on $(\Omega_\tau, \mathcal{F}^\tau, \mathbb{P}_\tau)$.

To estimate a targeted integral:

$$\int_a^b z(t)dt = (b-a)\int_a^b z(t) \underbrace{\frac{1}{(b-a)}}_{\text{Density of } \mathcal{U}(a,b)} dt$$

$$= (b-a)\mathbb{E}_\tau\left[z\big(a+\tau(b-a)\big)\right]$$

$$\overset{\text{Classic}}{\approx} (b-a)\frac{1}{N}\sum_{i=1}^{N} z\underbrace{\big(a+\tau_i(b-a)\big)}_{\in[a,b]}$$

$$\overset{\text{Stratified}}{\approx} (b-a)\frac{1}{N}\sum_{i=0}^{N-1} z\Big(\underbrace{a+\frac{(i+\tau_{i+1})}{N}(b-a)}_{\in[a+i(b-a)/N,\, a+(i+1)(b-a)/N]}\Big).$$

Remark

Monte-Carlo convergence rate: $\mathcal{O}(N^{-\frac{1}{2}})$ if $z \in L^2$.

Randomized quadrature rule (1D)

$z\colon [0,T] \to \mathbb{R}^d$, measurable, in $L^p([0,T];\mathbb{R}^d)$ with $p \in [2,\infty)$.
Define *randomized Riemann sum approx.* $Q^n_{\tau,h}[z]$ of $\int_0^{t_n} z(s)ds$

$$Q^n_{\tau,k}[z] := k \sum_{j=1}^{n} z(t_{j-1} + k\tau_j), \quad n \in \{1,\dots,N_k\}.$$

Theorem (Kruse&W., 2017)

$Q^n_{\tau,k}[z] \in L^p(\Omega_\tau;\mathbb{R}^d)$ *is an unbiased estimator. For all* $h \in (0,1)$,

$$\Big\| \max_{n\in\{1,\dots,N_k\}} \Big| \int_0^{t_n} z(s)ds - Q^n_{\tau,k}[z] \Big| \Big\|_{L^p(\Omega_\tau;\mathbb{R})} \le C \|z\|_{\mathcal{C}^\gamma([0,T])} k^{\frac{1}{2}+\gamma}.$$

Sketch of proof

Define the error term

$$E^n := \int_0^{t_n} g(s)\,\mathrm{d}s - Q^n_{\tau,h}[g] = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(g(s) - g(t_{j-1} + k\tau_j)\big)\,\mathrm{d}s$$

$$\Downarrow$$

$(E^n)_{n\in\{1,\dots,N_k\}}$ is a discrete time L^p-martingale

$\Downarrow$ Burkholder-Davis-Gundy (BDG)

$$\Big\| \max_{n\in\{1,\dots,N_k\}} |E^n| \Big\|_{L^p(\Omega;\mathbb{R})} \leq C_p \Big\| [E]^{\frac{1}{2}}_{N_k} \Big\|_{L^p(\Omega;\mathbb{R})}$$

$$\leq C_p \Big(\sum_{j=1}^{N_k} \underbrace{\Big\| \int_{t_{j-1}}^{t_j} \big(g(s) - g(t_{j-1} + k\tau_j)\big)\,\mathrm{d}s \Big\|^2_{L^p(\Omega;\mathbb{R}^d)}}_{\leq \|g\|^2_{\mathcal{C}^\gamma([0,T])} k^{2+2\gamma}} \Big)^{\frac{1}{2}}$$

$$\leq C_p T^{\frac{1}{2}} \|g\|_{\mathcal{C}^\gamma([0,T])} k^{\frac{1}{2}+\gamma}.$$

Randomised Runge-Kutta

Step 2. Sample $\tau_j \sim \mathcal{U}(0,1)$, then

$$\begin{aligned} x_j^{\tau} &= x_{j-1} + \tau_j k f(t_{j-1}, x_{j-1}), \\ x_j &= x_{j-1} + k f(t_{j-1} + \tau_j k, x_j^{\tau}), \end{aligned} \qquad \text{(Randomised RK)}$$

$$\Downarrow$$

$$\sum x_j = \sum x_{j-1} + \sum k f(t_{j-1} + \tau_j k, x_j^{\tau})$$

Randomised Runge-Kutta

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Remark (Stengle, 1990,1995; Jentzen&Neuenkirch, 2009)

$$x_j^e = x_{j-1}^e + k f(t_{j-1} + \tau_j k, x_{j-1}^e). \qquad \text{(Randomised Euler)}$$

Convergences

Theorem (L^p convergence (Kruse&W., 2017))

For given $k \in (0,1)$, we have [3]

$$\Big\| \max_{j \in \{0,1,\dots,N_k\}} \big| x(t_j) - x_j \big| \Big\|_{L^p(\Omega_\tau;\mathbb{R})} \le C k^{\frac{1}{2}+\gamma}.$$

[3] $\| \max_{j \in \{0,1,\dots,N_k\}} |x(t_j) - x_j^e| \|_{L^p(\Omega_\tau;\mathbb{R})} \le C k^{(\frac{1}{2}+\gamma)\wedge 1}$.

Convergences

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Theorem (Almost-sure convergence (Kruse&W., 2017))

Let $(k_m)_{m\in\mathbb{N}}$ be an arbitrary sequence of stepsizes with $\sum_m k_m < \infty$. For every $\epsilon \in (0, \frac{1}{2})$ there exist a random variable $m_\epsilon^u : \Omega_\tau \to \mathbb{N}$ and a measurable set $A_\epsilon^u \in \mathcal{F}$ with $\mathbb{P}(A_\epsilon^u) = 1$ such that for every $\omega \in A_\epsilon^u$ and $m \geq m_\epsilon^u(\omega)$ we have

$$\max_{n \in \{0,1,\dots,N_{k_m}\}} \big|x(t_n) - x_n^{k_m}(\omega)\big| \leq k_m^{\gamma+\frac{1}{2}-\epsilon}.$$

---

[3] $\| \max_{j\in\{0,1,\dots,N_k\}} |x(t_j) - x_j^e| \|_{L^p(\Omega_\tau;\mathbb{R})} \leq C k^{(\frac{1}{2}+\gamma)\wedge 1}$.

L^2 convergence for an ODE with jumps

Consider the following ODE with a non-continuous coefficient function:

$$\begin{cases} \dot{u}(t) &= g(t)u, \quad t \in [0, T], \\ u(0) &= 1, \end{cases}$$

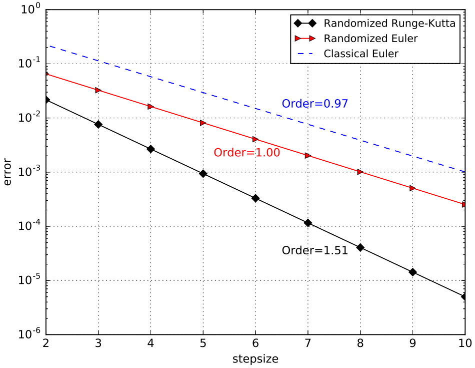
where $g(t) := \left[-\frac{1}{10}\mathrm{sgn}(\frac{1}{4}T - t) - \frac{1}{5}\mathrm{sgn}(\frac{1}{2}T - t) - \frac{7}{10}\mathrm{sgn}(\frac{3}{4}T - t)\right]$ and

$$\mathrm{sgn}(t) := \begin{cases} -1, & \text{if } t < 0, \\ 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases}$$

It can be verified that equals $u(T) = \exp(-\frac{3}{10}T)$.

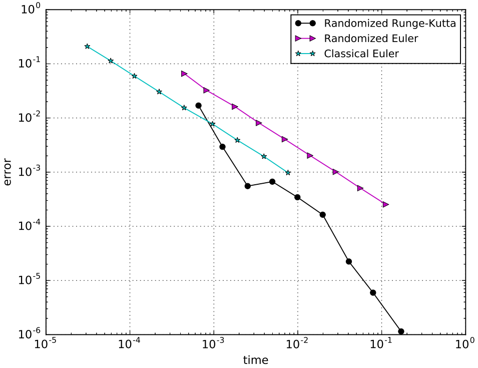
Error plot

Figure: L^2-errors versus step sizes for ODE with jumps



Time costs

Figure: L^2-errors versus CPU time



RANDOMISED NUMERICAL SCHEMES

– ELLIPTIC PDES

Elliptic equation

Consider the elliptic boundary value problem on $\mathcal{D} \subset \mathbb{R}^2$

$$\begin{cases} -\operatorname{div}\big(\sigma \nabla u\big) = f, & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D}, \end{cases} \tag{Elliptic}$$

where $\sigma, f : \mathcal{D} \to \mathbb{R}$ are coefficients with $\sigma(x) \geq \sigma_0 > 0$ for all $x \in \mathcal{D}$.

Fact (Existence of the weak solution u)

The variational form:

$$a(u, v) = F(v) \quad \text{for all } v \in H_0^1(\mathcal{D}), \tag{VF}$$

with

$$a(u, v) := \int_{\mathcal{D}} \sigma(x) \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x, \quad F(v) := \int_{\mathcal{D}} f(x) v(x) \, \mathrm{d}x,$$

for $\sigma \in L^\infty(\mathcal{D})$ and $f \in L^2(\mathcal{D})$.

Galerkin finite element method

- $\star$ $(\mathcal{T}_h)_{h\in(0,1]}$: finite subdivisions of $\mathcal{D}$ into triangles;
- $\star$ $S_h \subset H_0^1(\mathcal{D})$: the associated FES of piecewise linear functions.

Fact (Galerkin approximation)

For $h \in (0,1]$, find $u_h \in S_h$ s.t. for all $v_h \in S_h$

$$\int_{\mathcal{D}} \sigma(x)\nabla u_h(x) \cdot \nabla v_h(x)\,\mathrm{d}x = \int_{\mathcal{D}} f(x)v_h(x)\,\mathrm{d}x. \qquad \text{(Galerkin)}$$

Galerkin finite element method

- $\star$ $(\varphi_j)_{j=1}^{N_h}$: a basis of S_h;
- $\star$ the representation $u_h = \sum_{j=1}^{N_h} u_j \varphi_j$: $\mathbf{u} = [u_1, \ldots, u_{N_h}]^\top \in \mathbb{R}^{N_h}$.

Fact (FEM linear system)

For $h \in (0,1]$, find $\mathbf{u}$ s.t.

$$A_h \mathbf{u} = f_h, \qquad \text{(FEM)}$$

where the stiffness matrix $A_h \in \mathbb{R}^{N_h \times N_h}$ *is given by*

$$[A_h]_{i,j} = \int_{\mathcal{D}} \sigma(x) \nabla \varphi_i(x) \cdot \nabla \varphi_j(x) \, \mathrm{d}x$$

for all $i, j \in \{1, \ldots, N_h\}$, and the load vector $f_h \in \mathbb{R}^{N_h}$ *has the entries*

$$[f_h]_i = \int_{\mathcal{D}} f(x) \varphi_i(x) \, \mathrm{d}x, \quad i \in \{1, \ldots, N_h\}. \qquad \text{(Load)}$$

Why randomised quadrature formula?

- ⋆ For general $\sigma \in L^\infty(\mathcal{D})$ and $f \in L^2(\mathcal{D})$, no closed form for the stiffness matrix and the load vector.
- ⋆ Standard methods for numerical integration need additional smoothness of σ and f.

Randomised quadrature rule (2D)

Fix h and $\mathcal{T}_h$:

$$Q_{MC}[v] = \sum_{T \in \mathcal{T}_h} |T| v(Z_T) \approx \int_{\mathcal{D}} v(x) \, \mathrm{d}x. \qquad \text{(MC)}$$

Here $(Z_T)_{T \in \mathcal{T}_h}$ denotes an independent family of random variables such that for each triangle $T \in \mathcal{T}_h$ the random variable Z_T is uniformly distributed on T, that is $Z_T \sim \mathcal{U}(T)$.

Lemma (Kruse, Polydorides & W., 19)

Q_{MC} is unbiased, i.e., for every $v \in L^1(\mathcal{D})$ it holds that

$$\mathbb{E}\big[Q_{MC}[v]\big] = \int_{\mathcal{D}} v(x)\,\mathrm{d}x.$$

Moreover, if $v \in L^2(\mathcal{D})$ then it holds that

$$\mathbb{E}\Big[\Big|\int_{\mathcal{D}} v(x)\,\mathrm{d}x - Q_{MC}[v]\Big|^2\Big] \le \sqrt{3}h^2\|v\|^2_{L^2(\mathcal{D})}.$$

In addition, if $v \in W^{s,2}(\mathcal{D})$ [4] for some $s \in (0,1)$

$$\mathbb{E}\Big[\Big|\int_{\mathcal{D}} v(x)\,\mathrm{d}x - Q_{MC}[v]\Big|^2\Big] \le h^{2+2s}|v|^2_{W^{s,2}(\mathcal{D})}.$$

[4] For a domain $\mathcal{D} \subset \mathbb{R}^2$ and $s \in (0,1)$ the norm $\|\cdot\|_{W^{s,2}(\mathcal{D})}$ is given by

$$\|v\|_{W^{s,2}(\mathcal{D})} = \Big(\|v\|^2_{L^2(\mathcal{D})} + \int_{\mathcal{D}}\int_{\mathcal{D}} \frac{|v(x_1)-v(x_2)|^2}{|x_1-x_2|^{2+2s}}\,\mathrm{d}x_2\,\mathrm{d}x_1\Big)^{\frac{1}{2}}.$$

Approximated variational problem

Define $a_{MC}\colon S_h \times S_h \to L^2(\Omega)$ and $F_{MC}\colon S_h \to L^2(\Omega)$, then

$$a_{MC}(v_h, w_h) := Q_{MC}[\sigma \nabla v_h \cdot \nabla w_h] = \sum_{T \in \mathcal{T}_h} |T| \sigma(Z_T) \nabla v_h(Z_T) \cdot \nabla w_h(Z_T), \tag{aMC}$$

$$F_{MC}(v_h) := Q_{MC}[f v_h] = \sum_{T \in \mathcal{T}_h} |T| f(Z_T) v_h(Z_T) \tag{FMC}$$

for all $v_h, w_h \in S_h$.

Formulation (FEM)

In terms of a_{MC} and F_{MC} the problem is stated as follows:

$$\begin{cases} \text{Find } u_h^{MC}\colon \Omega \to S_h \text{ such that } \mathbb{P}\text{-almost surely} \\ a_{MC}(u_h^{MC}, v_h) = F_{MC}(v_h) \text{ for all } v_h \in S_h. \end{cases} \tag{aGarlerkin}$$

Convergence

Theorem (Kruse, Polydorides & W., 19)

If $f \in L^p(\mathcal{D})$, $p \in [2,\infty]$, *and* $\sigma \in L^\infty(\mathcal{D}) \cap W^{s,q}(\mathcal{D})$, $s \in (0,1]$, $q \in (2,\infty)$, *then it holds*

$$\|u - u_h^{MC}\|_{L^2(\Omega;H_0^1(\mathcal{D}))} \le \underbrace{Ch\|u\|_{H^2(\mathcal{D})}}_{\text{Error from FEM}} + \underbrace{Ch^{1-\frac{2}{p}}\|f\|_{L^p(\mathcal{D})}}_{\text{Error from RQ}} + \underbrace{Ch^s\|u\|_{H^2(\mathcal{D})}|\sigma|_{W^{s,q}(\mathcal{D})}}_{\text{Error from RQ}}$$

for $\forall h \in (0,1]$, *and* u *and* u_h^{MC} *denote the solutions to* (VF) *and* (aGarlerkin), *respectively.*

Approximating the load vector

Recall that the entries of the load vector $f_h \in \mathbb{R}^{N_h}$ defined in (Load):

$$[f_h]_j = \int_{\mathcal{D}} f(x)\varphi_j(x)\,\mathrm{d}x, \quad j \in \{1, \dots, N_h\}.$$

For each triangle $T \in \mathcal{T}_h$ with $T \cap \operatorname{supp}(\varphi_j) \neq \emptyset$, we can rewrite it as

$$\int_T f(x)\varphi_j(x)\,\mathrm{d}x = \int_T \frac{f(x)\varphi_j(x)}{p_{T,j}(x)} p_{T,j}(x)\,\mathrm{d}x \approx \frac{|T| f(Y_{T,j})}{3},$$

where $p_{T,j}(x) := \frac{\varphi_j(x)}{\int_T \varphi_j(x)\,\mathrm{d}x} = 3|T|^{-1}\varphi_j(x)$ and $Y_{T,j} \sim p_{T,j}$.

Formulation (Importance-sampling RQR)

The load vector can be approximated via

$$[f_h]_j \approx F_{IS}(\varphi_j) := \frac{1}{3} \sum_{\substack{T \in \mathcal{T}_h \\ T \cap \operatorname{supp}(\varphi_j) \neq \emptyset}} |T| f(Y_{T,j}), \quad Y_{T,j} \sim p_{T,j}.$$

Solving the Poisson Equation

Consider the Poisson boundary value problem on $\mathcal{D}$

$$\begin{cases} -\Delta u = f, & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D}, \end{cases} \qquad \text{(Poisson)}$$

where $f \in L^p(\mathcal{D})$ for some $p \in [2, \infty]$.

Remark

Poisson equation is a special case of (Elliptic) *with* $\sigma \equiv 1$. *The stiffness matrix* A *in* (FEM) *does not require any quadrature rule.*

In terms of F_{IS} the problem is stated as follows:

$$\begin{cases} \text{Find } u_h^{IS} \colon \Omega \to S_h \text{ such that } \mathbb{P}\text{-almost surely} \\ a(u_h^{IS}, v_h) = F_{IS}(v_h) \text{ for all } v_h \in S_h. \end{cases} \qquad \text{(aGarlerkin2)}$$

Convergences

Theorem (Kruse, Polydorides & W., 19)

Define $\ell_h = \max(1, \log(1/h))$ *and* u_h^{IS} *the solution to* (aGarlerkin2)*.*

1. *If* $f \in L^p(\mathcal{D})$*,* $p \in (2, \infty]$*, then* $\exists\, C \in (0, \infty)$ *s.t.* $\forall h \in (0, 1]$

$$\left\| u_h^{IS} - u \right\|_{L^2(\Omega; H_0^1(\mathcal{D}))} \leq Ch \|u\|_{H^2(\mathcal{D})} + C\ell_h^{\frac{1}{2}+\frac{1}{p}} h^{1-\frac{2}{p}} \|f\|_{L^p(\mathcal{D})}.$$

2 *If* $f \in W^{s,2}(\mathcal{D})$*,* $s \in [0, 1)$*, then* $\exists\, C \in (0, \infty)$ *s.t.* $\forall h \in (0, 1]$

$$\left\| u_h^{IS} - u \right\|_{L^2(\Omega; L^2(\mathcal{D}))} \leq Ch^2 \|u\|_{H^2(\mathcal{D})} + C\ell_h h^{1+s} |f|_{W^{s,2}(\mathcal{D})}.$$

An experiment

Consider the Poisson equation on $\mathcal{D} = [0,1]^2$ with f as

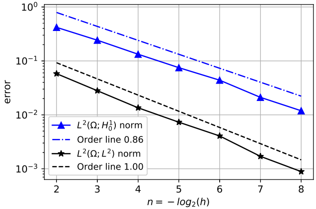
$$f(x,y) := |x-y|^{-q} + 10\sin(2^3\pi x)\mathrm{sgn}(2y-x),$$

where $q = 0.49$ and

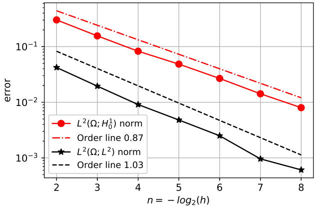
$$\mathrm{sgn}(x) := \begin{cases} -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0, \\ 1, & \text{if } x > 0. \end{cases}$$

- Discretised solution based on the uniform meshes with mesh size $h = 2^{-n}$, where $n \in \{3, \dots, 8\}$;
- Record $\|u_h^{MC} - u_h\|_{L^2(\Omega;H_0^1(\mathcal{D}))}$, $\|u_h^{MC} - u_h\|_{L^2(\Omega;L^2(\mathcal{D}))}$, $\|u_h^{IS} - u_h\|_{L^2(\Omega;H_0^1(\mathcal{D}))}$, $\|u_h^{IS} - u_h\|_{L^2(\Omega;L^2(\mathcal{D}))}$ through $M = 5000$ independent solution realisations.

Error plots



(a) MC estimator



(b) IS estimator

Error vs time cost

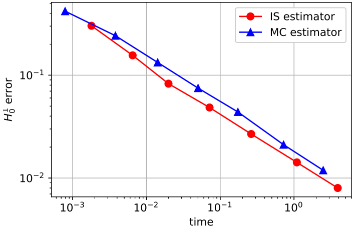


Figure: Error in H_0^1 norm against time cost for MC estimator and IS estimator.

Error vs time cost (smoother case)

Consider $f_2 : \mathcal{D} \to \mathbb{R}$:

$$f_2(x, y) := 8x(1 - x)y(1 - y), \quad \text{for } (x, y) \in \mathcal{D}.$$

In fact, it can be easily verified that $f_2 \in H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$.

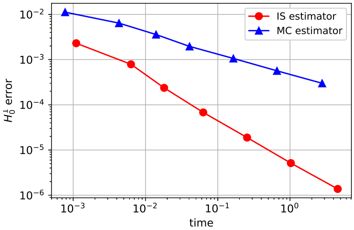


Figure: Error in H_0^1 norm against time cost for MC estimator and IS estimator.

RANDOMISED NUMERICAL SCHEMES

– ADDITIVE SDES

An ODE regularised by noise

Consider the $\mathbb{R}^d$-valued SDE,

$$\begin{cases} \mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + \mathrm{d}B(t), & t \in [0,1], \\ x(0) = x_0 \in \mathbb{R}^d, \end{cases} \tag{aSDE}$$

where $B = (B(t))_{0 \le t \le 1}$ is a d-dim BM on a probability space $(\Omega_B, \mathcal{F}^B, \mathbb{P}_B)$, and the drift coefficient $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed to be α-Hölder continuous in time and bounded β-Hölder continuous in space with $\alpha, \beta \in (0,1]$, i.e.,

$$\|f\|_{\mathcal{C}_b^{\alpha,\beta}} := \sup_{t\in[0,1],x\in\mathbb{R}^d} |f(t,x)| + \sup_{x\in\mathbb{R}^d, s\neq t} \frac{|f(t,x) - f(s,x)|}{|t-s|^\alpha} + \sup_{t\in[0,1], x\neq y} \frac{|f(t,x) - f(t,y)|}{|x-y|^\beta} < \infty.$$

Euler-Maruyama

The EM method on a fixed stepsize $1/n$:

$$\bar{X}^{(n)}_{t_{i+1}} = \bar{X}^{(n)}_{t_i} + \frac{1}{n} f\left(t_i, \bar{X}^{(n)}_{t_i}\right) + \underbrace{B(t_{i+1}) - B(t_i)}_{\sim \mathcal{N}(0, t_{i+1} - t_i)}, \tag{EM-dis}$$

and for $t \in (t_i, t_{i+1}]$,

$$\begin{aligned} \bar{X}^{(n)}_t &= x_0 + \int_0^t f\left(\kappa_n(s), \bar{X}^{(n)}_{\kappa_n(s)}\right) \mathrm{d}s + B(t) \\ &= \bar{X}^{(n)}_{t_i} + \int_{t_i}^t f\left(t_i, \bar{X}^{(n)}_{t_i}\right) \mathrm{d}s + B(t) - B(t_i) \end{aligned} \tag{EM-cont}$$

where $\kappa_n(s) := \lfloor ns \rfloor / n$.

Existing numerical work on EM

- Convergence in probability [Gyöngy & Krylov, 1996]
- Order of strong convergence
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    $\Downarrow$
    $\alpha \wedge ((\beta + 1)/2 - \epsilon), \epsilon \in (0, 1/2)$ for (aSDE).

Remark (Neuenkirch & Szölgyeny, 2021)

The error of the EM scheme can be reduced to a quadrature problem (for instance, via a Zvonkin-type transformation [Zvonkin, 1974]).

Two quadratures

Two quadratures (Pamen & Taguchi, 2017)

$$\mathbb{I}_1(\bar{X}^{(n)}) := \mathbb{E}_B\left[\sup_{s\le u\le t}\left|\int_s^u \left(f(r,\bar{X}_r^{(n)}) - f(\kappa_n(r),\bar{X}_{\kappa_n(r)}^{(n)})\right)\,\mathrm{d}r\right|^p\right]$$

and

$$\mathbb{I}_2(\bar{X}^{(n)},g) := \mathbb{E}_B\left[\sup_{s\le u\le t}\left|\int_s^u g(r,\bar{X}_r^{(n)})\cdot\left(f(r,\bar{X}_r^{(n)}) - f(\kappa_n(r),\bar{X}_{\kappa_n(r)}^{(n)})\right)\,\mathrm{d}r\right|^p\right]$$

where $g \in \mathcal{C}_b^{0,1}([0,1]\times\mathbb{R}^d;\mathbb{R}^d)$.

Quadratures through direct calculation

Note that

$$\|\bar{X}_t - \bar{X}_{\kappa_n(t)}\|_{L^p(\Omega_B;\mathbb{R}^d)} \leq Cn^{-1/2}.$$

Thus

$$\begin{aligned}
&\|f(r,\bar{X}_r) - f(\kappa_n(r),\bar{X}_{\kappa_n(r)})\|_{L^p(\Omega_B;\mathbb{R}^d)} \\
&\leq \|f\|_{\mathcal{C}_b^{\alpha,\beta}}(|t-\kappa_n(t)|^\alpha + \|\bar{X}_t - \bar{X}_{\kappa_n(t)}\|^\beta_{L^p(\Omega_B;\mathbb{R}^d)}) \\
&\leq C\|f\|_{\mathcal{C}_b^{\alpha,\beta}}(n^{-\alpha} + n^{-\beta/2}) \leq Cn^{-\gamma}.
\end{aligned}$$

Two quadratures (Pamen & Taguchi, 2017)

$$\mathbb{I}_1(\bar{X}^{(n)}) \propto n^{-\gamma p} \quad \textit{and} \quad \mathbb{I}_2(\bar{X}^{(n)},g) \propto n^{-\gamma p}.$$

Quadratures through stochastic sewing

To show that

$$\mathbb{I}_1^s(\bar{X}^{(n)}) := \mathbb{E}_B\left[\sup_{s\le u\le t}\left|\int_s^u \left(f(\bar{X}_r^{(n)}) - f(\bar{X}_{\kappa_n(r)}^{(n)})\right)\,\mathrm{d}r\right|^p\right] \propto n^{-(\beta/2+1/2-\epsilon)p}.$$

Steps of [Butkovsky et al, 2021].

- ⋆ showed $\mathbb{I}_1^s(B) \propto n^{-(\beta/2+1/2-\epsilon)p}$ via SSL [Lê, 2020].
- ⋆ Applied Girsanov's Theorem.

! Cannot handle the time-variable in $\mathbb{I}_1(\bar{X}^{(n)})$.

Randomised EM: two quadratures

The randomised EM method on a fixed stepsize $1/n$:

$$X_t^{(n)} = x_0 + \int_0^t f\left(\kappa_n^\tau(s), X_{\kappa_n(s)}^{(n)}\right) \mathrm{d}s + B_t, \qquad \text{(rEM)}$$

where $\kappa_n^\tau(s) := (\lfloor ns \rfloor + \tau)/n$ and $\tau \sim \mathcal{U}(0,1)$.

Two quadratures (Bao and W., 2025)

$$\mathbb{I}_1^\tau(X^{(n)}) := \mathbb{E}\left[\sup_{s \le u \le t}\left|\int_s^u \left(f(r, X_r^{(n)}) - f(\kappa_n^\tau(r), X_{\kappa_n(r)}^{(n)})\right) \mathrm{d}r\right|^p\right]$$

and

$$\mathbb{I}_2^\tau(X^{(n)}, g) := \mathbb{E}\left[\sup_{s \le u \le t}\left|\int_s^u g(r, X_r^{(n)}) \cdot \left(f(r, X_r^{(n)}) - f(\kappa_n^\tau(r), X_{\kappa_n(r)}^{(n)})\right) \mathrm{d}r\right|^p\right].$$

The convergence

Adopt the framework of [Pamen & Taguchi, 2017]. Recall that $\gamma := \alpha \wedge (\beta/2)$. using SSL:

✓ $\mathbb{I}_1^\tau(X^{(n)}), \mathbb{I}_2^\tau(X^{(n)}, g) \propto n^{-(\gamma+1/2-\epsilon)p}$.

Theorem (Bao and W., 2025)

For any $p \geq 1$ and $\epsilon \in (0, 1/2)$, there exists a positive constant C depending on $m, M, d, p, x_0, \alpha, \beta$ and $\|f\|_{\mathcal{C}_b^{\alpha,\beta}}$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq 1}\left|X(t) - X_t^{(n)}\right|^p\right] \leq Cn^{-(\gamma+1/2-\epsilon)p}.$$

The convergence (α-stable process)

$$\mathrm{d}X(t) = f(t, X(t))\mathrm{d}t + \mathrm{d}L(t) \qquad (\alpha\text{-stable})$$

Adopt the framework of [Chen et al, 2018] and utilise conditional shifted version of SSL [Butkovsky et al, 2024].

Theorem (Bao, Wang and W., 2025)

Assume that the drift coefficient $f \in \mathcal{C}_b^{\eta,\beta}$ and L is a d-dimensional symmetric α-stable process, where $\alpha \in (1,2)$ and suppose that

$$2\beta + \alpha > 2 \text{ and } (\eta + 1)\alpha + \beta > 2.$$

Then for any $p \geq 1$ and $\varepsilon \in (0, 1/2)$, $\exists C > 0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le 1} \left|X(t) - X^{(n)}(t)\right|^p\right] \le Cn^{-(1/2+\gamma-\varepsilon)p},$$

where $\gamma := \eta \wedge (\beta/\alpha) \wedge (1/2)$.

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