On sharp lower bounds for strong approximation of SDEs with Hölder continuous drift coefficient

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based on joint work with

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Introduction

Given a scalar autonomous SDE

$$\begin{aligned} dX_t &= \mu(X_t)\,dt + dW_t, \quad t \in [0,1], \\ X_0 &= x_0 \in \mathbb{R} \end{aligned} \tag{1}$$

with drift coefficient $\mu\colon \mathbb{R} \to \mathbb{R}$ and scalar Brownian motion W.

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with drift coefficient $\mu\colon \mathbb{R} \to \mathbb{R}$ and scalar Brownian motion W.

Approximate X_1 by

$$\widehat{X}_{n,1} = u(W_{t_1}, \dots, W_{t_n}),$$

where $t_1, \dots, t_n \in [0,1]$ and $u\colon \mathbb{R}^n \to \mathbb{R}$ is measurable, w.r.t. L_p-error

$$\mathbb{E}\big[|X_1 - \widehat{X}_{n,1}|^p\big]^{1/p}.$$

Typical upper bound: There exists $c > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\big[|X_1 - \widehat{X}_{n,1}|^p\big]^{1/p} \le \frac{c}{n^\alpha}.$$

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$$\inf_{\substack{t_1,\dots,t_n \in [0,1], \\ u \text{ measurable}}} \mathbb{E}\big[|X_1 - u(W_{t_1}, \dots, W_{t_n})|^p\big]^{1/p} \geq \frac{c}{n^\beta}.$$

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Ideally: $\alpha = \beta$.

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In this talk: μ is Hölder continuous or of fractional Sobolev regularity.

I. Hölder continuous drift

Approximation of SDEs with Hölder continuous μ:

Gyöngy, Rásonyi (2011), Pamen, Taguchi (2017),
Ngo, Taguchi (2017, 2019), Bao, Huang, Yuan (2019),
Dareiotis, Gerencsér (2020), Butkovsky, Dareiotis, Gerencsér (2021),
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Upper error bounds

Euler scheme. For $n \in \mathbb{N}$ and $\ell = 0, \ldots, n-1$,

$$\widehat{X}^E_{n,0} = x_0,$$

$$\widehat{X}^E_{n,(\ell+1)/n} = \widehat{X}^E_{n,\ell/n} + \frac{1}{n}\,\mu(\widehat{X}^E_{n,\ell/n}) + W_{(\ell+1)/n} - W_{\ell/n}.$$

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Theorem 1 (Butkovsky, Dareiotis, Gerencsér 2021).
Let $\alpha \in (0,1]$ and assume that μ is bounded with $\mu \in C^\alpha$. Then for all $\varepsilon > 0$ and $p \in [1,\infty)$ there exists $c > 0$ such that for all $n \in \mathbb{N}$,

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Remark. Gerencsér, Lampl, Ling (2023+): L_p-error rate $\frac{1+\alpha}{2} - \varepsilon$ for the Milstein scheme for SDEs with μ as in Theorem 1 and diffusion coefficient $\sigma \in C^3_b$ with $|\sigma| \ge c > 0$.

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Question. Existence of a method $\widehat{X}_{n,1} = u(W_{t_1}, \ldots, W_{t_n})$ that achieves a better L_p-error rate than $\frac{1+\alpha}{2}$?

Lower error bounds

Theorem 2 (Müller-Gronbach 2004, Hefter, Herzwurm, Müller-Gronbach 2019).
Assume that SDE (1) has a strong solution X and that there exist an open interval $J \subset \mathbb{R}$ and $t_0 \in [0,1]$ such that

(i) $\mathbb{P}(X_{t_0} \in J) > 0$,

(ii) μ is three times continuously differentiable on J with $\inf_{x \in J} |\mu'(x)| > 0$.

Then there exists $c > 0$ such that for all $n \in \mathbb{N}$

$$\inf_{\substack{t_1,\ldots,t_n \in [0,1] \\ u\colon \mathbb{R}^n \to \mathbb{R} \text{ measurable}}} \mathbb{E}\big[|X_1 - u(W_{t_1}, \ldots, W_{t_n})|\big] \geq \frac{c}{n}.$$

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Remark. Assumptions of Theorem 2 are satisfied for the SDE (1) with $\mu = \cos$ and $x_0 = \pi/2$, $t_0 = 0$ and $J = (\pi/3, 2\pi/3)$.

Note: μ is bounded and $\mu \in C^\alpha$ for all $\alpha \in (0,1]$.

Theorem 3 (Ellinger, Müller-Gronbach, Y 2025+).
For every $\alpha \in (0,1)$ there exist $\mu_\alpha \in C^\alpha$ bounded and $c > 0$ such that for all $n \in \mathbb{N}$,

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Remark. Possible choice of μ_α in Theorem 3: the Weierstraß function

$$\mu_\alpha(x) = \sum_{j=1}^{\infty} 2^{-\alpha j} \cos(2^j x), \quad x \in \mathbb{R}.$$

Sketch of proof of Theorem 3

For simplicity: mean squared error and $t_i = i/n$, $i = 0, \dots, n$.

Coupling of noise (see Müller-Gronbach, Y 2023)

Construct a Brownian motion $\widetilde{W}^n$ such that $\widetilde{W}^n_{t_i} = W_{t_i}$ for $i = 1, \dots, n$ and

$$\mathbb{P}^{(W,\widetilde{W}^n)|W_{t_1},\dots,W_{t_n}} = \mathbb{P}^{W|W_{t_1},\dots,W_{t_n}} \times \mathbb{P}^{\widetilde{W}^n|W_{t_1},\dots,W_{t_n}}.$$

Let $\widetilde{X}^n$ denote the solution of SDE (1) with driving Brownian motion $\widetilde{W}^n$.

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Let $\widetilde{X}^n$ denote the solution of SDE (1) with driving Brownian motion $\widetilde{W}^n$. Then, for every measurable $u\colon \mathbb{R}^n \to \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\big[|X_1 - u(W_{t_1}, \ldots, W_{t_n})|^2\big]^{1/2} \\
&\quad = \frac{1}{2}\big(\mathbb{E}\big[|X_1 - u(W_{t_1}, \ldots, W_{t_n})|^2\big]^{1/2} + \mathbb{E}\big[|\widetilde{X}^n_1 - u(\widetilde{W}^n_{t_1}, \ldots, \widetilde{W}^n_{t_n})|^2\big]^{1/2}\big) \\
&\quad \geq \frac{1}{2}\,\mathbb{E}\big[|X_1 - \widetilde{X}^n_1|^2\big]^{1/2}.
\end{aligned}$$

Analysis of $\mathbb{E}\big[|X_1 - \widetilde{X}_1^n|^2\big]$.

1. $$\mathbb{E}\big[|X_1 - \widetilde{X}_1^n|^2\big] = \mathbb{E}\Big[\Big|\int_0^1 (\mu_\alpha(X_t) - \mu_\alpha(\widetilde{X}_t^n))dt\Big|^2\Big]$$
$$\succeq \sum_i \mathbb{E}\Big[\Big|\int_{t_{i-1}}^{t_i} (\mu_\alpha(X_t) - \mu_\alpha(\widetilde{X}_t^n))dt\Big|^2\Big].$$

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2. For $i \geq n/2$,

$$\mathbb{E}\Big[\Big|\int_{t_{i-1}}^{t_i} (\mu_\alpha(X_t) - \mu_\alpha(\widetilde{X}_t^n))dt\Big|^2\Big]$$
$$\succeq \mathbb{E}\Big[\Big|\int_{t_{i-1}}^{t_i} \big(\mu_\alpha(X_{t_{i-1}} + W_t - W_{t_{i-1}}) - \mu_\alpha(X_{t_{i-1}} + \widetilde{W}_t^n - \widetilde{W}_{t_{i-1}}^n)\big)dt\Big|^2\Big] - o\big(\tfrac{1}{n^{2+\alpha}}\big)$$
$$\succeq \int_{\mathbb{R}} \varphi(x)\,\mathbb{E}\Big[\Big|\int_{t_{i-1}}^{t_i} \big(\mu_\alpha(x + W_t - W_{t_{i-1}}) - \mu_\alpha(x + \widetilde{W}_t^n - \widetilde{W}_{t_{i-1}}^n)\big)dt\Big|^2\Big]dx - o\big(\tfrac{1}{n^{2+\alpha}}\big)$$
$$\succeq \frac{1}{n^2}\,\mathbb{E}\Big[\int_{-\pi}^{\pi}\Big|\int_0^1 \big(\mu_\alpha\big(x + \tfrac{1}{\sqrt{n}}W_t\big) - \mu_\alpha\big(x + \tfrac{1}{\sqrt{n}}\widetilde{W}_t^1\big)\big)dt\Big|^2 dx\Big] - o\big(\tfrac{1}{n^{2+\alpha}}\big).$$

3. Recall: $\mu_\alpha(y) = \sum_{j=1}^{\infty} 2^{-\alpha j}\cos(2^j y) = \frac{1}{2}\sum_{j=1}^{\infty} 2^{-\alpha j}(e^{\mathbf{i}2^j y} + e^{-\mathbf{i}2^j y})$. Thus,

$$\mathbb{E}\Big[\int_{-\pi}^{\pi}\Big|\underbrace{\int_0^1 \big(\mu_\alpha\big(x+\tfrac{1}{\sqrt{n}}W_t\big) - \mu_\alpha\big(x+\tfrac{1}{\sqrt{n}}\widetilde{W}_t^1\big)\big)\,dt}_{=\sum_{j\in\mathbb{Z}\setminus\{0\}} \frac{1}{2}A_j\cdot e^{\mathrm{sgn}(j)\mathbf{i}2^{|j|}x}}\Big|^2 dx\Big]$$

$$= \mathbb{E}\Big[2\pi \sum_{j\in\mathbb{Z}\setminus\{0\}} \big|\tfrac{1}{2}A_j\big|^2\Big]$$

$$\succeq \mathbb{E}[|A_{j^*}|^2]$$

$$= 2^{-2\alpha j^*}\underbrace{\int_0^1\int_s^1 2e^{-\frac{1}{2}\left(\frac{2^{j^*}}{\sqrt{n}}\right)^2(u-s)}\Big(1-e^{-\frac{1}{2}\left(\frac{2^{j^*}}{\sqrt{n}}\right)^2(2s-2su)}\Big)\,du\,ds}_{\geq c>0}$$

$$\succeq \frac{1}{n^\alpha},$$

where $A_j = 2^{-\alpha|j|}\int_0^1 \big(e^{\mathrm{sgn}(j)\mathbf{i}\frac{2^{|j|}}{\sqrt{n}}W_t} - e^{\mathrm{sgn}(j)\mathbf{i}\frac{2^{|j|}}{\sqrt{n}}\widetilde{W}_t^1}\big)\,dt$ and $j^* = \lfloor \log_2(\sqrt{n})\rfloor$.

II. Drift of fractional Sobolev regularity

Fractional Sobolev regularity. For $\alpha \in (0,1)$ and $p \in [1,\infty)$,

$$\mathcal{W}^{\alpha,p} = \left\{ \mu\colon \mathbb{R} \to \mathbb{R} \text{ measurable} \;\middle|\; \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\mu(x)-\mu(y)|^p}{|x-y|^{1+\alpha p}} \, dx \, dy < \infty \right\}.$$

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Remark. If $\mu \in C^\alpha$ and has compact support then

$$\mu \in \bigcap_{0<s<\alpha} \mathcal{W}^{s,p} =: \mathcal{W}^{\alpha-,p}$$

for all $p \in [1,\infty)$.

Upper error bounds

Theorem 4 (Dareiotis, Gerencsér, Lê 2023).

Let $\alpha \in (0,1)$ and $p \in [2,\infty)$. Assume that μ is bounded with $\mu \in \mathcal{W}^{\alpha,p}$. Then for every $\varepsilon > 0$ there exists $c > 0$ such that for all $n \in \mathbb{N}$,

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Remark. Neuenkirch, Szölgyenyi (2021): L_2-error rate as in Theorem 4 for $\mu = a + b$ with $a \in C_b^2$, b bounded and $b \in \mathcal{W}^{\alpha,2} \cap L_1$ for (non-equidistant) Euler scheme.

Lower error bounds

Theorem 5 (Ellinger, Müller-Gronbach, Y 2023+).
For every $\alpha \in (1/2, 1)$ there exist $\mu_\alpha \in \mathcal{W}^{\alpha-,2}$ bounded and $c > 0$ such that for all $n \in \mathbb{N}$,

$$\inf_{\substack{t_1,\dots,t_n \in [0,1] \\ u\colon \mathbb{R}^n \to \mathbb{R} \text{ measurable}}} \mathbb{E}\Big[|X_1 - u(W_{t_1}, \dots, W_{t_n})|^2\Big]^{1/2} \geq \frac{c}{n^{\frac{1+\alpha}{2}}}.$$

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Remark. Possible choice of μ_α in Theorem 5: $\mu_\alpha = \mathscr{F} h_\alpha$, where

$$h_\alpha \colon \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{1}{(1+|x|)^{\alpha+1/2}}.$$

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See also Altmeyer (2021) for lower error bounds for approximation of occupation time functionals $\int_0^1 \mu(W_t)\, dt$.

Note: $X_1 = x_0 + \int_0^1 \mu(X_t)\, dt + W_1$.

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Question: a matching lower bound for $\alpha \in (0, 1/2]$ and $p > 2$?

Lower error bounds

Theorem 6 (Ellinger, Müller-Gronbach, Y 2025+).

For every $\alpha \in (0,1)$ and every $p \in [1,\infty)$ there exist $\mu_\alpha \in \mathcal{W}^{\alpha-,p}$ bounded and $c > 0$ such that for all $n \in \mathbb{N}$,

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Remark. Possible choice of μ_α in Theorem 6:

$$\mu_\alpha(x) = 1_{[-3\pi/2,3\pi/2]}(x) \cdot \sum_{j=1}^{\infty} 2^{-\alpha j} \cos(2^j x), \quad x \in \mathbb{R}.$$