Fourier Coupled BSDE Method for Milstein and Second-Order Taylor Schemes

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Agenda

1. Coupled BSDEs
2. High-Order Discrete Scheme for the Forward Equation
3. Discrete Characteristic Function
4. BCOS Method
5. Numerical Experiments

BSDEs

- BSDEs provide a probabilistic representation of semi–linear/nonlinear PDEs.
- Our aim is for fast and accurate solvers.
- Some relevant forward SDEs are *non-affine* (CEV, local volatility), so characteristic functions are typically unknown.
- Coupled BSDEs pose a specific challenge!
- Goal: accurate, Fourier-based method for non-affine drift/volatility.

Problem Statement

Decoupled FBSDE on $[0, T]$

$$\begin{cases} \mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \ X_0 = x_0 \\ \mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \ Y_T = g(X_T) \end{cases} \Rightarrow \text{solve } (Y_0, Z_0)$$

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Coupled FBSDE:

$$\begin{cases} \mathrm{d}X_t = \mu(t, X_t, Y_t, Z_t)\,dt + \sigma(t, X_t, Y_t, Z_t)\,dW_t \\ \mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,dt + Z_t\,dW_t, \quad t \in [0, T], \\ X_0 = x_0, \quad Y_T = g(X_T). \end{cases}$$

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- Drift μ and volatility σ now depend on Y, Z $\Rightarrow$ fully–coupled system.

$\Rightarrow$ Uniform grid $\pi = \{t_n\}_{n=0}^{N}$, $\Delta t = t_{n+1} - t_n$.

Backward Scheme with θ-Parameters (Zhao et al. 2012)

$$y(t_N,x) = g(x), \quad z(t_N,x) = \partial_x g(x)\,\sigma(t_N, y(t_N,x), z(t_N,x)),$$

$$\begin{aligned} z(t_n,x) = \frac{1}{\theta_3 \Delta t_n}\Big(&\theta_4 \Delta t_n\, \mathbb{E}_n^x[z(t_{n+1},X_{n+1})] \\ &+ (\theta_3-\theta_4)\,\mathbb{E}_n^x[y(t_{n+1},X_{n+1})\,\Delta W_n] \\ &+ (1-\theta_2)\Delta t_n\, \mathbb{E}_n^x[f(t_{n+1}, y(t_{n+1},X_{n+1}), z(t_{n+1},X_{n+1}))\,\Delta W_n]\Big), \end{aligned}$$

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with $\theta_1, \theta_2 \in [0,1]$, $\theta_3 \in (0,1]$, $\theta_4 \leq \theta_3$

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with $\theta_1, \theta_2 \in [0, 1]$, $\theta_3 \in (0, 1]$, $\theta_4 \leq \theta_3$

$\theta_1 = \theta_2 = \theta_3 = 1/2; \theta_4 = -1/2$, 2^{nd}-order accuracy

Motivation and Objective

We aim to extend the COS method to fully coupled FBSDEs using higher-order Taylor schemes.

This enables improved convergence:

- Strong convergence of order 1 (Milstein)
- Weak convergence of order 2 (simplified 2.0 weak Taylor)

COS Approximation Equations

We use the COS method to approximate conditional expectations:

$$\mathbb{E}_n^x\left[v(t_{n+1}, X_{n+1})\right] \approx \sum_{k=0}^{K-1}{}' \mathcal{V}_k(t_{n+1})\, \mathrm{Re}\left\{\varphi_{X_{n+1}}\left(\frac{k\pi}{b-a}\middle| t_n, x\right) \exp\left(-i\frac{k\pi a}{b-a}\right)\right\}, \tag{14}$$

where the Fourier cosine expansion coefficients are

$$\begin{aligned} \mathcal{V}_k(t_{n+1}) &= \frac{2}{b-a}\int_a^b v(t_{n+1}, x)\cos\left(\frac{k\pi(x-a)}{b-a}\right)\mathrm{d}x. \\ v(t_{n+1}, x) &= \sum_{k=0}^{\infty}{}' \mathcal{V}_k(t_{n+1})\cos\left(\frac{k\pi(x-a)}{b-a}\right). \end{aligned}$$

Here, the notation $\sum{}'$ indicates that the $k = 0$ term has weight $1/2$.

When we don't know the ChF, Decoupled Case

Rewrite the Euler, Milstein, and 2.0 weak Taylor discretization schemes:

$$X_{n+1} = x + \bar{m}(x)\Delta t + \bar{s}(x)\Delta W_{n+1} + \bar{\kappa}(x)(\Delta W_{n+1})^2, \quad X_m = x.$$

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For the Euler scheme: $\bar{m}(x) = \mu(x), \quad \bar{s}(x) = \sigma(x), \quad \bar{\kappa}(x) = 0.$

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For the Milstein scheme:

$$\bar{m}(x) = \mu(x) - \frac{1}{2}\sigma\sigma_x(x), \quad \bar{s}(x) = \sigma(x), \quad \bar{\kappa}(x) = \frac{1}{2}\sigma\sigma_x(x).$$

For the order 2.0 weak Taylor scheme:

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For the order 2.0 weak Taylor scheme:

$$\bar{m}(x) = \mu(x) - \frac{1}{2}\sigma\sigma_x(x) + \frac{1}{2}\left(\mu\mu_x(x) + \frac{1}{2}\mu_{xx}\sigma^2(x)\right)\Delta t,$$

$$\bar{s}(x) = \sigma(x) + \frac{1}{2}\left(\mu_x\sigma(x) + \mu\sigma_x(x) + \frac{1}{2}\sigma_{xx}\sigma^2(x)\right)\Delta t,$$

$$\bar{\kappa}(x) = \frac{1}{2}\sigma\sigma_x(x).$$

Discrete Characteristic Function

$$X_{n+1} = x + \bar{m}(x)\Delta t + \bar{\kappa}(x)\left(\Delta W_{n+1} + \frac{1}{2}\frac{\bar{s}(x)}{\bar{\kappa}(x)}\right)^2 - \frac{1}{4}\frac{\bar{s}^2(x)}{\bar{\kappa}(x)}$$
$$\stackrel{d}{=} x + \bar{m}(x)\Delta t - \frac{1}{4}\frac{\bar{s}^2(x)}{\bar{\kappa}(x)} + \bar{\kappa}(x)\Delta t\left(U_{n+1} + \sqrt{\lambda(x)}\right)^2,$$

$\lambda(x) := \frac{1}{4}\frac{\bar{s}^2(x)}{\bar{\kappa}^2(x)\Delta t}$, $U_{n+1} \sim \mathcal{N}(0,1)$.

$(U_{n+1} + \sqrt{\lambda(x)})^2 \sim \chi_1'^2(\lambda(x))$ non-central chi-squared distributed.

The characteristic function of X_{n+1}, given $X_n = x$

$$\varphi(u|X_n = x) = \mathbb{E}\left[\exp\left(iuX_{n+1}\right)\Big|X_n = x\right]$$
$$= \exp\left(iux + iu\bar{m}(x)\Delta t - \frac{\frac{1}{2}u^2\bar{s}^2(x)\Delta t}{1 - 2iu\bar{\kappa}(x)\Delta t}\right)(1 - 2iu\bar{\kappa}(x)\Delta t)^{-1/2}.$$

Simplified 2.0 Weak Taylor Scheme, Coupled Case

- Explicit strategy (Huijskens et al., 2017; Negyesi et al. 2025): at step n use the already computed coefficients at t_{n+1}:

$$\xi(x) := y_{n+1}(x), \quad \eta(x) := z_{n+1}(x).$$

- Works for any forward discretisation once $\partial_x \xi$, $\partial_x^2 \xi$ are available

$\Rightarrow$ The forward discretization for the simplified weak 2.0 scheme reads:

$$X_{n+1} = x + \bar{m}_{\xi,\eta}(t_n, x)\Delta t + \bar{s}_{\xi,\eta}(t_n, x)\Delta W_n + \bar{\kappa}_{\xi,\eta}(t_n, x)(\Delta W_n)^2 \quad (28)$$

- The coefficients are:

$$\begin{aligned} \bar{m}_{\xi,\eta}(t,x) = \bar{\mu}_{\xi,\eta}(t,x) - \bar{\sigma}_{\xi,\eta}(t,x)\partial_x \bar{\sigma}_{\xi,\eta}(t,x)/2 \\ + (\partial_t \bar{\mu}_{\xi,\eta}(t,x) + \bar{\mu}_{\xi,\eta}(t,x)\partial_x \bar{\mu}_{\xi,\eta}(t,x) \\ + \partial_{xx}^2 \bar{\mu}_{\xi,\eta}(t,x)(\bar{\sigma}_{\xi,\eta}(t,x))^2/2)\Delta t_n/2, \quad \text{etc.} \end{aligned} \quad (9)$$

Derivatives via Fourier Cosine Expansion

The decoupling fields $\xi(t_{n+1}, x)$ and $\eta(t_{n+1}, x)$ are approximated as:

$$\xi(t_{n+1}, x) = \sum_{k=0}^{K-1}{}' \mathcal{Y}_k(t_{n+1}) \cos\left(\frac{k\pi(x-a)}{b-a}\right), \tag{1}$$

$$\eta(t_{n+1}, x) = \sum_{k=0}^{K-1}{}' \mathcal{Z}_k(t_{n+1}) \cos\left(\frac{k\pi(x-a)}{b-a}\right) \tag{2}$$

Their derivatives are available analytically:

$$\partial_x \xi(t_{n+1}, x) = -\sum_{k=0}^{K-1}{}' \left(\frac{k\pi}{b-a}\right) \mathcal{Y}_k(t_{n+1}) \sin\left(\frac{k\pi(x-a)}{b-a}\right)$$

$$\partial_{xx} \xi(t_{n+1}, x) = -\sum_{k=0}^{K-1}{}' \left(\frac{k\pi}{b-a}\right)^2 \mathcal{Y}_k(t_{n+1}) \cos\left(\frac{k\pi(x-a)}{b-a}\right)$$

Algorithm : BCOS Method

Goal: Compute (Y_0, Z_0) from the terminal condition $Y_T = g(X_T)$ using a Fourier-based approach.

Step 1. Initialize $y_N = g$, $z_N = 0$ on grid $\mathcal{X}_N \subset [a, b]$.

Step 2. For $n = N - 1, \ldots, 0$:

- Set $\xi := y_{n+1}$, $\eta := z_{n+1}$
- Compute COS coefficients of y_{n+1}, z_{n+1}
- Compute derivatives $\partial_x y_{n+1}, \partial_x^2 y_{n+1}$ from coefficients
- Evaluate $\bar{m}, \bar{s}, \bar{\kappa}$
- Use COS method to evaluate:

$$y_n(x) = \mathbb{E}_x \left[y_{n+1}(X_{n+1}) + f(\cdots)\Delta t \right], \tag{3}$$

$$z_n(x) = \mathbb{E}_x \left[y_{n+1}(X_{n+1})\Delta W_n \right] / \Delta t \tag{4}$$

Step 3. Output: $Y_0 = y_0(x_0)$, $Z_0 = z_0(x_0)$.

Derivation: COS Approximation of $\mathbb{E}[(\Delta W_n)^k v(X_{n+1})]$

Suppose:

$$\rightarrow h(t_{n+1}, X_{n+1}) = \sum_{k=0}^{K-1}{}' \hat{\mathcal{H}}_k(t_{n+1}) \cos\left(\frac{k\pi(X_{n+1}-a)}{b-a}\right)$$

Define:

$$J_k(x|u) := \mathbb{E}_n^x\left[e^{iuX_{n+1}}(\Delta W_n)^k\right]$$

Then, via integration by parts and recurrence:

$$J_k(x|u) = \frac{iu\bar{s}\Delta t_n}{1-2iu\bar{\kappa}\Delta t_n} J_{k-1}(x|u) + 1_{k\geq 1}\frac{(k-1)\Delta t_n}{1-2iu\bar{\kappa}\Delta t_n} J_{k-2}(x|u)$$

With this recursion, we find expressions for

$\mathbb{E}_n^x[h(t_{n+1}, X_{n+1})]$, $\mathbb{E}_n^x[h(t_{n+1}, X_{n+1})\Delta W_n]$ and $\mathbb{E}_n^x[h(t_{n+1}, X_{n+1})(\Delta W_n)^2]$

Coefficient Recovery and Derivatives via DCT

$$\to h(t_{n+1}, x) = {\sum_{k=0}^{\infty}}' \mathcal{H}(t_{n+1})_k \cos\left(\frac{k\pi(x-a)}{b-a}\right)$$

$$\mathcal{H}(t_{n+1}) = \frac{2}{a-b}\int_a^b h(t_{n+1}, x)\cos\left(\frac{k\pi(x-a)}{b-a}\right)\mathrm{d}x;$$

Given a truncated integration range $[a, b]$ and grid:

$$x_l = a + \left(l + \tfrac{1}{2}\right)\frac{b-a}{K}, \quad l = 0, \ldots, K-1,$$

approximate COS coefficients $\mathcal{H}_k$ of $h(\cdot)$ by a Discrete Cosine Transform:

$$\mathcal{H}_k(t_{n+1}) \approx \frac{2}{K}\sum_{l=0}^{K-1} h(t_{n+1}, x_l)\cos\left(\frac{k\pi(2l+1)}{2K}\right)$$

which is a DCT approximation of type 2.

BCOS Backward Algorithm

→ With the terminal condition, the backward recursion is implementable!

1. Initialise: COS coefficients of payoff $g(x)$ at $t = T$.
2. Loop $n = N - 1 \to 0$
   1. Evaluate $\mathbb{E}[Y], \mathbb{E}[Y\Delta W], \mathbb{E}[f], \mathbb{E}[f\Delta W]$ via COS.
   2. Update Z_n, then Y_n using θ-scheme.
   3. Recover new COS coefficients with DCT.
3. Output (Y_0, Z_0).

Runs in highly efficiently in Python and MATLAB.

⇒ Balint's code is open souce!

See: https://github.com/balintnegyesi/coupled-BCOS

Computational Complexity of the BCOS Method

The computational cost of the BCOS approximation is given by:

$$\mathcal{O}\left(N\left(K+K^2+PK+K\log K\right)\right),$$

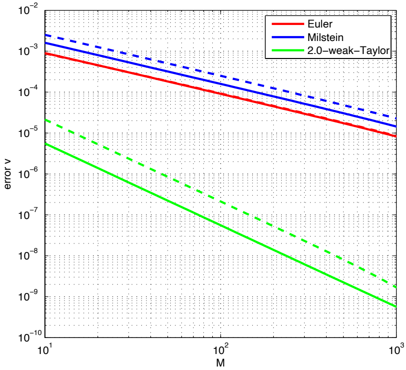
where:

- K: number of COS coefficients used in the spectral expansion;
- $\mathcal{O}(K^2)$: cost for coefficient recovery of the backward component;
- $\mathcal{O}(K\log K)$: cost of computing the COS coefficients via a Discrete Cosine Transform (DCT);
- $\mathcal{O}(PK)$: cost from applying P Picard iterations per time step;
- the outer factor $\mathcal{O}(N)$: accounts for the total number of time steps over the time grid.

Despite higher-order terms, the method remains efficient due to small N typically required, and the availability of fast transforms (DCT).

Example 1 – Bermudan Put (CEV), vol-term X^γ

- Decoupled BSDE!
- CEV parameter $\gamma = 0.2$ and 0.8, 10 exercise dates.
- 2.0-WT error $\leq 10^{-5}$ with $M{=}20$ timesteps. Euler requires ≈ 900.
- Observe 2nd-order slope for both Y_0 and Z_0.



Example 2 – Fully coupled linear–quadratic regulator

Stochastic optimal control

$$\mu(t,x,y,z) = \left(A - B\frac{R_{xu}}{R_u}\right)x + \frac{B^2}{R_u}y + \frac{BD}{R_u}z + \beta,$$

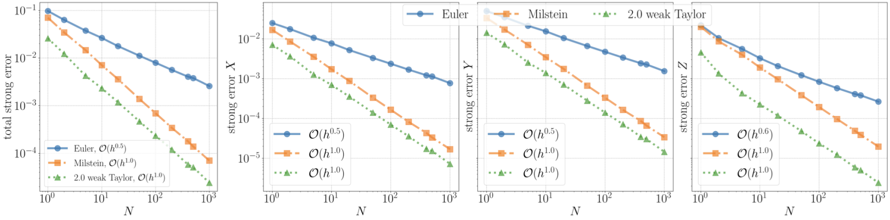
$$\sigma(t,x,y,z) = \left(C - D\frac{R_{xu}}{R_u}\right)x + \frac{DB}{R_u}y + \frac{D^2}{R_u}z + \Sigma, \quad g(x) = -Gx,$$

$$f(t,x,y,z) = \left(A - \frac{BR_{xu}}{R_u}\right)y + \left(C - \frac{DR_{xu}}{R_u}\right)z - \left(R_x - \frac{R_{xu}}{R_u}\right)x.$$

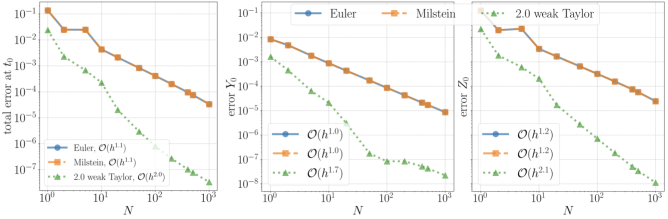
Coupling enters both drift and diffusion via optimal control u_t.

$$A=-1,\ B=0.1,\ \beta=0, C=1, D=0.01,\ \Sigma=0.05,$$
$$R_x=2, R_{xu}=0, R_u=2, G=2.$$

Closed-form solution from Riccati ODE available.



(a) strong convergence



(b) weak convergence at t_0

Euler vs. Milstein

Number of time steps N, cosine terms K.

$N \setminus K$	strong error X			strong error Y			strong error Z		
	128	512	1024	128	512	1024	128	512	1024
10	7.7e−3	7.7e−3	7.7e−3	1.5e−2	1.5e−2	1.5e−2	3.3e−3	3.3e−3	3.3e−3
100	2.4e−3	2.4e−3	2.4e−3	4.7e−3	4.7e−3	4.7e−3	8.3e−4	8.3e−4	8.3e−4
400	1.2e−3	1.2e−3	1.2e−3	2.4e−3	2.4e−3	2.4e−3	4.1e−4	4.1e−4	4.1e−4
1000	7.7e−4	7.7e−4	7.7e−4	1.5e−3	1.5e−3	1.5e−3	2.7e−4	2.7e−4	2.7e−4

(a) Euler with (7)

$N \setminus K$	strong error X			strong error Y			strong error Z		
	128	512	1024	128	512	1024	128	512	1024
10	1.7e−3	1.7e−3	1.7e−3	3.5e−3	3.5e−3	3.5e−3	1.9e−3	1.9e−3	1.9e−3
100	1.7e−4	1.7e−4	1.7e−4	3.3e−4	3.3e−4	3.3e−4	1.9e−4	1.9e−4	1.9e−4
400	4.3e−5	4.3e−5	4.3e−5	8.7e−5	8.7e−5	8.7e−5	4.9e−5	4.9e−5	4.9e−5
1000	1.7e−5	1.7e−5	1.7e−5	3.4e−5	3.4e−5	3.4e−5	2.0e−5	2.0e−5	2.0e−5

(b) Milstein with (8)

Conclusions & Outlook

⇒ Discrete characteristic function + COS gives a **fast, 2nd-order** BSDE solver for scalar FBSDEs.

- The challenge is the multi-D generalization!

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