2 Neural Systems

2.1 Models of the synapse and dendrite

The synapse

At a synapse presynaptic firing results in the release of neurotransmitters that causes a change in the membrane conductance of the postsynaptic neuron. This postsynaptic current may be written

$$I_s = g_s s(V_s - V)$$

where V is the voltage of the postsynaptic neuron, V_s is the membrane reversal potential and g_s is a constant. The variable s corresponds to the probability that a synaptic receptor channel is in an open conducting state. This probability depends on the presence and concentration of neurotransmitter released by the presynaptic neuron. The sign of V_s relative to the resting potential (assumed to be zero) determines whether the synapse is excitatory ($V_s > 0$) or inhibitory ($V_s < 0$). For simplicity we shall assume that a neuron spends most of its time close to rest such that $V_s - V \approx V_s$, with V_s absorbed into g_s.

The effect of some synapses can be described with a function that fits the shape of the postsynaptic response due to the arrival of action potential at the pre-synaptic release site. A postsynaptic potential (PSP) $s(t)$ would then be given by

$$s(t) = \eta(t - T), \qquad t \geq T$$

where T is the arrival time of a pre-synaptic action potential and $\eta(t)$ fits the shape of a realistic PSP. A common (normalised) choice for $\eta(t)$ is a difference of exponentials:

$$\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^{-1} [e^{-\alpha t} - e^{-\beta t}]$$

or the alpha function :

$$\alpha^2 t e^{-\alpha t}$$

The PSP arising from a train of action potentials is given by

$$s(t) = \sum_m \eta(t - T_m)$$

The dendrite

A nerve fibre consists of a long thin, electrically conducting core surrounded by a thin membrane whose resistance to transmembrane current flow is much greater than that of either the internal core or the surrounding medium. Injected current can travel long distances along the dendritic core before a significant fraction leaks out across the highly resistive cell membrane. Conservation of electric current in an infinitesimal cylindrical element of nerve fibre yields a second–order linear partial differential equation known as the *cable equation*. Let $V(x, t)$ denote the membrane potential at position x along a uniform cable at time t measured relative to the resting potential

of the membrane. Let τ be the cell membrane time constant, D the diffusion constant, then the basic uniform (infinite) cable equation is

$$\frac{\partial V(x,t)}{\partial t} = -\frac{V(x,t)}{\tau} + D\frac{\partial^2 V(x,t)}{\partial x^2} + I(x,t), \quad x \in (-\infty,\infty), \quad t \geq 0$$

where we include the source term $I(x,t)$ corresponding to external input injected into the cable. Diffusion along the dendritic tree generates an effective spatio-temporal distribution of delays as expressed by the associated Green's function of the cable equation.

In response to a unit impulse at x' at $t = 0$ and taking $V(x,0) = 0$ the dendritic potential behaves as $V(x,t) = G(x-x',t)$, where

$$G(x,t) = \frac{1}{\sqrt{4\pi D t}} e^{-t/\tau} e^{-x^2/(4Dt)}$$

The Green's function $G(x,t)$determines the linear response to an instantaneous injection of unit current at a given point on the tree. Using the linearity of the cable equation one may write the general solution as

$$V(x,t) = \int_0^t dt' \int_{-\infty}^{\infty} dx' G(x-x',t-t') I(x',t') + \int_{-\infty}^{\infty} dx' G(x-x',t) V(x',0)$$

Assuming the soma is at $x = 0$, $V(x,0) = 0$ and the synaptic input is a train of spikes at $x = x'$, $I(x,t) = \delta(x-x') \sum_m \delta(t-T_m)$ we have that

$$s(t) = V(0,t) = \sum_m \eta(t-T_m), \qquad \eta(t) = G(x',t)$$

Green's function for the infinite cable equation

$$G_t = -\frac{G}{\tau} + DG_{xx}, \qquad G(x,0) = \delta(x)$$

Introduce Fourier transform

$$G(k,t) = \int_{-\infty}^{\infty} dk e^{-ikx} G(x,t) dx, \qquad G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ikx} G(k,t) dk$$

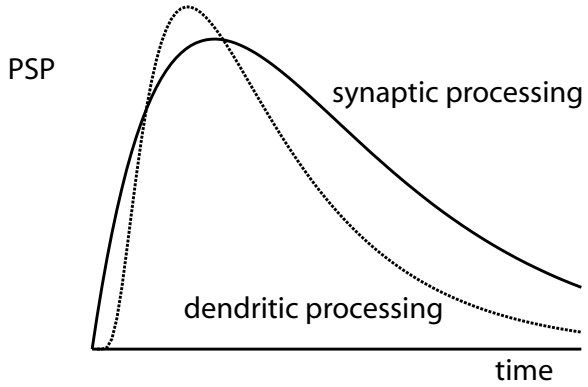
then

$$G_t(k,t) = -\epsilon(k), \qquad G(k,0) = 1, \quad \epsilon(k) = \frac{1}{\tau} + Dk^2$$

with solution $G(k,t) = G(k,0)\exp(-\epsilon(k)t)$. Inverse transforming gives

$$\begin{aligned} G(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} e^{-\epsilon(k)t} = e^{-t/\tau} e^{-x^2/4Dt} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-Dt[k+ix/2Dt]^2} dk \\ &= \frac{1}{\sqrt{4\pi D t}} e^{-t/\tau} e^{-x^2/(4Dt)} \end{aligned}$$

where we complete the square and use the fact that $\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}$.



Distributed delays arising from synaptic and dendritic processing have similar shapes.

2.2 Return maps for pulse-coupled oscillators

Here we consider a simple model for synchronous firing of biological oscillators based upon two pulse coupled integrate-and-fire oscillators. In this case case synaptic coupling is fast, and $\eta(t) \to \delta(t)$. The (oscillatory) dynamics for the network is given by

$$\dot{v}_i = -v_i + I, \qquad I > 1, \qquad 0 \le v_i \le 1, \qquad i = 1, 2$$

When $v_i = 0$ the ith oscillator *fires* and v_i is reset to zero. Pulsatile coupling simply acts to *pull* a freely evolving oscillator up by an amount ϵ, or pulls them to firing, whichever is less. That is,

$$v_i(t) = 1 \Rightarrow \; v_j(t^+) = \min(1, v_j(t) + \epsilon)$$

In the absence of coupling the solution there is a natural phase variable given by

$$\phi = \frac{t}{\Delta} \mod 1 \equiv g(v) = \frac{1}{\Delta}\int_0^v \frac{du}{-u + I} = \frac{1}{\Delta} \ln\left(\frac{I}{I - v}\right), \qquad \Delta = \ln\left(\frac{I}{I-1}\right)$$

so that we may write the solution in the form $v = g^{-1}(\phi) \equiv f(\phi)$, where $f : [0,1] \to [0,1]$ is a smooth , monotonic increasing and concave down ($f' > 0$ and $f'' < 0$) function given explicitly by :

$$f(\phi) = I(1 - e^{-\phi\Delta})$$

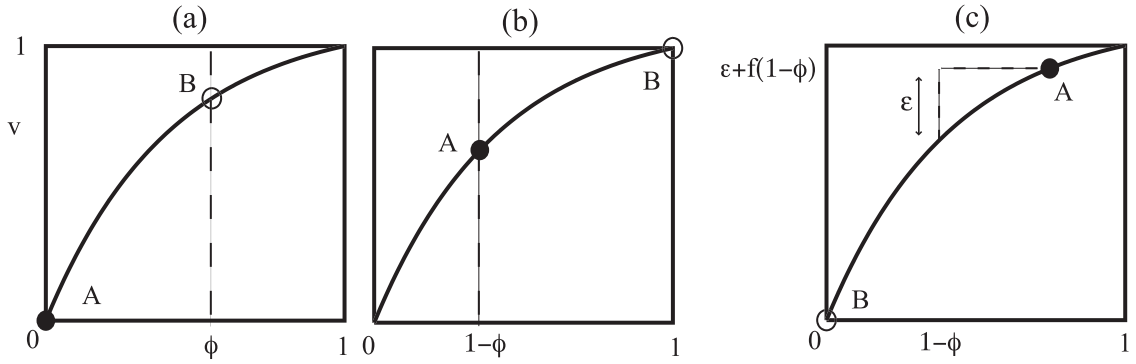
The two neuron system can then be viewed as two points moving along the curve $v = f(\phi)$. Interactions then correspond to *phase-shifts*. Between firing the two neurons move with the same constant velocity $\dot{\phi} = 1/\Delta$. To prove that the system always becomes synchronised we calculate the so-called return map.

The return map is defined as follows. Call the two oscillators A and B, and strobe the system at the instant after A has fired (so its phase is zero). Let ϕ denote the phase of B. The *return map* $R(\phi)$ is defined to be the phase of B immediately after the next firing of A. After a time $1 - \phi$, B reaches threshold and A moves to a value of $v_A = f(1 - \phi)$. An instant later B fires and v_A jumps to $\epsilon + f(1 - \phi)$ or 1, whichever is less. If $v_A = 1$ then the oscillators have synchronised. We shall assume $v_A = \epsilon + f(1-\phi) < 1$. The corresponding phase of A is $g(\epsilon + f(1-\phi))$. The *firing map* h is defined by

$$h(\phi) = g(\epsilon + f(1 - \phi))$$

Thus, after one firing, the system has moved from an initial state $(\phi_A, \phi_B) = (0, \phi)$ to a current state $(\phi_A, \phi_B) = (h(\phi), 0)$, which is essentially the same as when it started with the oscillators interchanged and ϕ replaced by $h(\phi)$. To obtain the return map we follow the system for one more iteration of h:

$$R(\phi) = h(h(\phi))$$



A system of two oscillators governed by $v = f(\phi)$, and interacting by the pulse-coupling rule. (a) The state of the system immediately after oscillator A has fired. (b) The state of the system just before oscillator B reaches the firing threshold. The phase difference between the oscillators is the same as in (a). (c) The state of the system just after B has fired. B has jumped back to zero, and the state of A is now $\min(1, \epsilon + f(1-\phi))$.

The assumption $\epsilon + f(1-\phi) < 1$ is satisfied for $\epsilon \in [0,1)$ and $\phi \in (\delta, 1)$, where $\delta = 1 - g(1-\epsilon)$. Thus the domain of h is more correctly the subinterval $(\delta, 1)$. Similarly the domain of R is the subinterval $(\delta, h^{-1}(\delta))$. This interval is nonempty because $\delta < h^{-1}(\delta)$ for $\epsilon < 1$. We will now show that there is a unique fixed point for R and it is a repeller.

We first show that $h'(\phi) < -1$ and $R'(\phi) > 1$ for all ϕ:
It suffices to show that $h'(\phi) < -1$, since $R'(\phi) = h'(h(\phi))h'(\phi)$. Now

$$h'(\phi) = -g'(\epsilon + f(1-\phi))f'(1-\phi), \quad \text{and} \quad f'(1-\phi) = 1/[g'(f(1-\phi))]$$

since $f = g^{-1}$. Hence,

$$h'(\phi) = -\frac{g'(\epsilon + f(1-\phi))}{g'(f(1-\phi))}$$

Now $g'' > 0$ (concavity) and $\epsilon > 0$ so $g'(\epsilon + x) > g'(x)$. Using the fact that $g' > 0$ gives $h' < -1$.

We then establish that there exists a unique fixed point for R in $(\delta, h^{-1}(\delta))$, and it is a repeller: To prove existence, it suffices to find a fixed point for h since $R = h \circ h$. The fixed point for h is given by

$$F(\phi) \equiv \phi - h(\phi) = 0, \quad \text{with} \quad F(\delta) < 0,\ F(h^{-1}(\delta)) > 0$$

Hence, $F'(\phi) = 1 - h'(\phi) > 2 > 0$. Hence h has a unique fixed point $\overline{\phi}$. Since $R(\overline{\phi}) = \overline{\phi}$ and $R'(\phi) > 1$ we have

$$\begin{aligned} R(\phi) &> \phi \qquad \text{if } \phi > \overline{\phi} \\ R(\phi) &< \phi \qquad \text{if } \phi < \overline{\phi} \end{aligned}$$

showing that $\overline{\phi}$ is a repeller. Hence, from any initial data the system is driven toward $\phi = 0$ or $\phi = 1$. Whenever one of these states is reached the system becomes synchronised and will remain so for all time.

It may also be proved rigorously that a system of N globally coupled IF oscillators almost always synchronize in the presence of instantaneous excitatory interactions.

2.3 Networks of interacting phase oscillators

Theorem: Phase equations for oscillatory networks

Consider a family of weakly connected systems

$$\dot{X}_i = F_i(X_i) + \epsilon G(X), \qquad X \in \mathbb{R}^n, \qquad i = 1, \ldots, n$$

such that each equation in the uncoupled system ($\epsilon = 0$) has an exponentially orbitally stable limit cycle $\gamma_i \subset \mathbb{R}^m$ having natural frequency $\Omega_i \neq 0$. Then the oscillatory weakly connected system can be reduced to a phase model of the form

$$\dot{\theta}_i = \Omega_i + \epsilon g_i(\theta_1, \ldots, \theta_n), \qquad \theta_i \in \mathbb{S}^1, \quad i = 1, \ldots, n$$

defined on the n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$. ie there is an open neighbourhood W of $M = \gamma_1 \times \ldots \times \gamma_n \subset \mathbb{R}^{mn}$ and a continuous function $h : W \to \mathbb{T}^n$ that maps solutions of the full model to those of the phase model. ■

The proof of this theorem is based around the fact that the direct product of hyperbolic limit cycles $M = \gamma_1 \times \ldots \times \gamma_n$ is a normally hyperbolic invariant manifold (Limit cycles are hyperbolic if Floquet multipliers are not on the unit circle). The invariant manifold theorem guarantees the persistence of a manifold M_ϵ, ϵ close to M. The restriction of the dynamics to M (non-zero ϵ) then has something to say about the dynamics on M_ϵ for small enough ϵ.

Since γ_i is homeomorphic to $\mathbb{S}^1$, we can parameterise it using the phase variable $\theta_i \in \mathbb{S}^1$, $\dot{\theta}_i = \Omega_i$:

$$\Gamma_i : \mathbb{S}^1 \to \gamma_i, \qquad \Gamma_i(\theta_i(t)) = x_i(t) \in \gamma_i, \quad t \in [0, 2\pi/\Omega_i]$$

Then we have

$$\dot{x}_i = \frac{d\Gamma_i(\theta_i(t))}{dt} = \Gamma_i'(\theta_i)\dot{\theta}_i = \Gamma_i'(\theta_i)\Omega_i = F_i(\Gamma_i(\theta_i(t)))$$

for all t. Therefore

$$\Gamma_i'(\theta_i) = \frac{F_i(\Gamma_i(\theta_i))}{\Omega_i}$$

Applying this transformation to the full system (when $\epsilon \neq 0$) gives

$$\Gamma_i'(\theta_i)\dot{\theta}_i = F_i(\Gamma_i(\theta_i)) + \epsilon G_i(\Gamma(\theta))$$

Multiplying both sides by

$$R_i(\theta_i) = \frac{\Omega_i F_i(\Gamma_i(\theta_i))^\top}{|F_i(\Gamma_i(\theta_i)|^2}$$

gives

$$\boxed{\dot{\theta}_i = \Omega_i + \epsilon R_i(\theta_i) G_i(\Gamma(\theta))}$$

To illustrate the analysis of such systems we consider the following restricted *phase-difference* example

$$\dot{\theta}_1 = \omega_1 + K_1 \sin(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \omega_2 + K_2 \sin(\theta_1 - \theta_2)$$

In the uncoupled state ($K_1 = K_2 = 0$) we have $\theta_1(t) = \theta_1(0) + \omega_1 t$ and $\theta_2(t) = \theta_2(0) + \omega_2 t$ such that $d\theta_2/d\theta_1 = \omega_2/\omega_1$. If the slope is rational, $\omega_2/\omega_1 = p/q$, $p, q \in \mathbb{Z}$, then all trajectories lie on closed orbits of the torus (with coords (θ_1, θ_2)). For irrational slopes the flow

is quasiperiodic. Each trajectory is dense on the torus (ie comes arbitrarily close to any given point). Introducing $\phi = \theta_1 - \theta_2$ the coupled system takes the form

$$\dot{\phi} = \omega_1 - \omega_2 - (K_1 + K_2)\sin(\phi)$$

There are two fixed points if $|\omega_1 - \omega_2| < K_1 + K_2$, defined by $\sin\phi^* = (\omega_1 - \omega_2)/(K_1 + K_2)$, and a saddle-node (tangent) bifurcation occurs when $|\omega_1 - \omega_2| = K_1 + K_2$. In this case $\dot{\phi} = 0$ so that $\dot{\theta}_1 = \dot{\theta}_2 = \text{constant} = \omega^*$, where

$$\omega^* = \omega_2 + K_2 \sin\phi^* = \frac{K_1\omega_2 + K_2\omega_1}{K_1 + K_2}$$

We may regard ω^* as a co-operative frequency that is an emergent property of the coupled system. When no-cooperative frequency can be established the two oscillators cannot phase-lock (although they may still frequency lock).

Consider now G_i to describe the synaptic input to neuron i in a network of N neurons with connections specified by W_{ij}:

$$G_i(t) = \sum_{j=1}^{N} W_{ij} s_j(t), \qquad s_j(t) = \sum_m \eta(t - T_m(j))$$

where neuron j fires at times $T_m(j)$. In the case that all oscillators are identical all PRCs and natural frequencies are the same, so that $R_i = R$, and $\Omega_i = \Omega = 1/\Delta$. We may then shift to rotating coordinates $\theta_i = \psi_i + \Omega t$. In the absence of any coupling the phase variable ψ_i is constant and all oscillators fire with their natural frequency Ω. For weak coupling, each oscillator still approximately fires at the unperturbed rate but now $\psi_i(t)$ slowly drifts. To first order in ϵ, we can take the firing-times to be $T_m(j) = (m - \psi_j)\Delta$ to obtain the phase equation

$$\boxed{\dot{\psi}_i = \epsilon \sum_j W_{ij} R(\psi_i + \Omega t) P(\psi_j + \Omega t)}$$

where $P(\psi + 1) = P(\psi)$ for all ψ and

$$P(\psi) = \sum_{m \in \mathbb{Z}} \eta((\psi + m)\Delta), \qquad 0 \le \psi < 1, \qquad \eta(t) = 0,\ t < 0$$

Averaging

Further simplification can be obtained in the weak coupling regime by averaging over the natural period Δ. This leads to a phase equation of the form

$$\dot{\psi}_i = \epsilon \sum_j W_{ij} \frac{1}{\Delta} \int_{-\psi_j\Delta}^{-\psi_j\Delta + \Delta} R(\psi_i + \Omega t) P(\psi_j + \Omega t) dt = \epsilon \sum_j W_{ij} H(\psi_j - \psi_i)$$

(choosing the limits of integration in this fashion avoids integrating through discontinuities in $P(\psi_j + \Omega t)$), where

$$H(\psi) = \int_0^1 R(\theta - \psi) P(\theta) d\theta$$

which may be interpreted as a *phase interaction* function. When describing a piece of cortex or a central pattern generator circuit with a set of oscillators, the biological realism of the model typically resides in the phase interaction function.

Note that certain caution has to be exercised in applying averaging theory. In general, one can only establish that a solution of the unaveraged equations is ϵ-close to a corresponding solution of the averaged system for times of $\mathcal{O}(\epsilon^{-1})$. No such problem arises in the case of hyperbolic fixed points corresponding to phase-locked states.

Stability of phase locked states

We define a (1:1) phase-locked solution to be of the form $\theta_i(t) = \phi_i + t/T$, where ϕ_i is a constant phase and T is the collective period of the coupled oscillators. Substitution into the averaged system gives

$$\frac{1}{T} = \frac{1}{\Delta} + \epsilon \sum_j W_{ij} H(\phi_j - \phi_i)$$

After choosing some reference oscillator, these N equations determine the collective period T and $N-1$ relative phases with the latter independent of ϵ. In order to analyze the local stability of a phase-locked solution $\Phi = (\phi_1, ..., \phi_N)$, we linearise the system by setting $\theta_i(t) = \phi_i + \Omega t + \widetilde{\theta}_i(t)$ and expanding to first-order in $\widetilde{\theta}_i$:

$$\frac{d\widetilde{\theta}_i}{dt} = \epsilon \sum_j \widehat{\mathcal{H}}_{ij}(\Phi)\widetilde{\theta}_j$$

where

$$\widehat{\mathcal{H}}_{ij}(\Phi) = W_{ij} H'(\phi_j - \phi_i) - \delta_{i,j} \sum_k W_{ik} H'(\phi_k - \phi_i)$$

and $H'(\phi) = dH(\phi)/d\phi$. One of the eigenvalues of the Jacobian $\widehat{\mathcal{H}}$ is always zero, and the corresponding eigenvector points in the direction of the flow, that is $(1, 1, \ldots, 1)$. The phase-locked solution will be stable provided that all other eigenvalues have a negative real part..

2.4 Rhythm generation in biological central pattern generators

Inhibitory synchrony

In the particular case of two oscillators

$$\dot{\theta}_1 = \Omega + \epsilon H(\theta_2 - \theta_1)$$
$$\dot{\theta}_2 = \Omega + \epsilon H(\theta_1 - \theta_2)$$

a phase-locked solution satisfies $\theta_2(t) - \theta_1(t) = \phi$ where the constant phase difference ϕ is a zero of the function $K(\phi) = \epsilon[H(-\phi) - H(\phi)]$. A given phase-locked state is then stable provided that $K'(\phi) < 0$. Note that by symmetry both the in-phase ($\phi = 0$) and anti-phase ($\phi = 1/2$) states are guaranteed to exist. For synaptic coupling

$$H(\phi) = \int_0^1 R(\psi - \phi) \sum_{m \in \mathbb{Z}} \eta((\psi + m)\Delta) d\psi = \int_0^\infty R(\psi - \phi)\eta(\psi\Delta) d\psi$$

Choosing $R(\psi) = -\sin 2\pi\psi$ (which fits the numerically obtained PRC of the Hodgkin-Huxley model) and $\eta(t) = \alpha^2 t e^{-\alpha t}$ gives

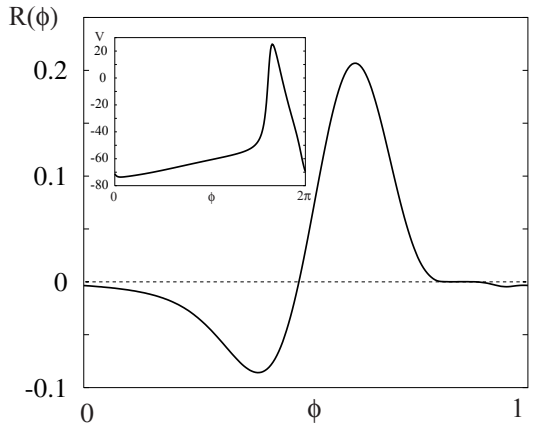
$$H(\phi) = \alpha^2 \frac{d}{d\alpha} \int_0^\infty \frac{e^{2\pi i(\psi - \phi)}}{2i} e^{-\alpha\psi\Delta} d\psi + cc \equiv z + z^*$$

Now

$$\frac{d}{d\alpha} \int_0^\infty e^{\psi(2\pi i - \alpha\Delta)} d\psi = \frac{d}{d\alpha} \frac{1}{\alpha\Delta - 2\pi i} = -\frac{\Delta}{(\alpha\Delta - 2\pi i)^2}$$

and so

$$H(\phi) = 2Re(z) = \frac{[1 - (2\pi/\alpha\Delta)^2]\sin(2\pi\phi) - 4\pi/\alpha\Delta \cos(2\pi\phi)}{\Delta[1 + (2\pi/\alpha\Delta)^2]^2}$$



Numerically obtained PRC of the Hodgkin-Huxley model.

giving

$$K(\phi) = -\frac{2\epsilon[(1-(2\pi/\alpha\Delta))^2]\sin(2\pi\phi)}{\Delta[1+(2\pi/\alpha\Delta)^2]^2}$$

As long as $2\pi/\alpha\Delta \neq 1$ there are equilibrium states at $\phi = 0$ and $\phi = 1/2$ as expected. The stability of the synchronous solution is governed by the sign of $K'(0)$:

$$\text{sgn } K'(0) = \text{sgn } \left\{-\epsilon[(1-(2\pi/\alpha\Delta))^2]\right\}$$

For inhibitory coupling ($\epsilon < 0$) synchronisation will occur if $K'(0) < 0$, ie if

$$\frac{1}{\alpha} > \frac{\Delta}{2\pi}$$

Thus inhibitory synchronization can arise when the synapse is *slow* ($\alpha \to 0$). It is also a simple matter to show that the anti-synchronous solution ($\phi = 1/2$) is stable for a sufficiently *fast* synapse ($\alpha \to \infty$).

Swimming lamprey

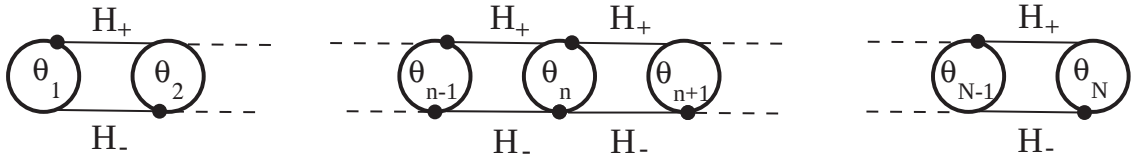
The analysis of phase-locking provides a basis for understanding many motor control systems. In fact phase-locked oscillators form the basis for virtually all rhythmic motor behaviour: breathing, swimming, running, chewing, ...

The lamprey is an eel-like vertebrate which swims by generating travelling waves of neural activity that pass down its spinal cord. The spinal cord contains about 100 segments, each of which is a simple half-center neural circuit capable of generating alternating contraction and relaxation of the body muscles on either side of body during swimming.

We consider a simple model of N phase-oscillators arranged on a chain with nearest-neighbour interactions to model the known physiology of the lamprey spinal cord. A travelling wave is recognised as a phase-locked state with a constant phase difference between adjacent segments. The intersegmental phase differences are defined as $\phi_i = \theta_{i+1} - \theta_i$. If $\phi_i < 0$ then the wave travels from head to tail whilst for $\phi_i > 0$ the wave travels from the tail to the head.

For a chain we set $W_{ij} = \delta_{i-1,j}W_- + \delta_{i+1,j}W_+$ to obtain

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + W_+ H(\theta_2 - \theta_1) \\
\dot{\theta}_i &= \omega_i + W_+ H(\theta_{i+1} - \theta_i) + W_- H(\theta_{i-1} - \theta_i), \qquad i = 2, \ldots, N-1 \\
\dot{\theta}_N &= \omega_N + W_- H(\theta_{N-1} - \theta_N)
\end{aligned}$$



A chain of N phase oscillators $\phi_i = \theta_{i+1} - \theta_i$ with $H_\pm(\phi) = W_\pm H(\phi)$.

Pairwise subtraction and substitution of $\phi_i = \theta_{i+1} - \theta_i$ leads to an $N-1$ dimensional system for the phase differences

$$\dot{\phi}_i = \Delta\omega_i + W_+[H(\phi_{i+1}) - H(\phi_i)] + W_-[H(-\phi_i) - H(-\phi_{i-1})], \qquad i = 1 \ldots N-1$$

with boundary conditions

$$H(-\phi_0) = 0 = H(\phi_{N+1})$$

where $\Delta\omega_i = \omega_{i+1} - \omega_i$. There are at least two different mechanisms that can generate travelling wave solutions.

The first is based on the presence of a gradient of frequencies along the chain, that is, $\Delta\omega_i$ has the same sign for all i, with the wave propagating from the high frequency region to the low frequency region. This can be established explicitly in the case of an isotropic, odd interaction function, $W_\pm = 1$ and $H(\phi) = -H(-\phi)$, where we have

$$\dot{\phi}_i = \Delta\omega_i + H(\phi_{i+1}) + H(\phi_{i-1}) - 2H(\phi_i)$$

The fixed points $\Phi = (\phi_1, \ldots, \phi_N)$ satisfy the matrix equation

$$\mathbf{H}(\Phi) = -\mathbf{A}^{-1}\mathbf{D}$$

where $\mathbf{H}(\Phi) = (H(\phi_1), \ldots, H(\phi_N))^\top$, $\mathbf{D} = (\Delta\omega_1, \ldots, \Delta\omega_N)^\top$, and $\mathbf{A}$ is a tridiagonal matrix with elements $A_{ii} = -2$, $A_{i,i+1} = A_{1+1,i} = 1$. For the sake of illustration suppose that $H(\phi) = \sin(\phi + \sigma)$, for some synaptic communication delay σ. Then a solution Φ will exist if every component of $\mathbf{A}^{-1}\mathbf{D}$ lies between ± 1. Let $a_0 = \max\{|(\mathbf{A}^{-1}\mathbf{D})_i|\}$. If $a_0 < 1$ then for each $i = 1, \ldots, N$ there are two distinct solutions $\phi_i^\pm$ in the interval $[0, 1]$ with $H'(\phi_i^-) > 0$ and $H'(\phi_i^+) < 0$. In other words, there are 2^N phase-locked solutions. Linearising about a phase-locked solution and exploiting the structure of the matrix $\mathbf{A}$, it can be proven that only the solution $\Phi^- = (\phi_1^-, \ldots, \phi_N^-)$ is stable. Assuming that the frequency gradient is monotonic, this solution corresponds to a stable travelling wave. When the gradient becomes too steep to allow phase-locking (i.e. $a_0 > 1$), two or more pools of oscillators (frequency plateaus) tend to form that oscillate at different frequencies. Waves produced by a frequency gradient do not have a constant speed or, equivalently, constant phase lags along the chain.

Constant speed waves in a chain of identical oscillators can be generated by considering phase-locked solutions defined by $\phi_i = \phi$ for all i, with a collective period of oscillation Ω determined using $\dot{\theta}_1 = \Omega$ to give

$$\Omega = \omega_1 + W_+ H(\phi_1)$$

The steady state equations are then

$$\begin{aligned} \Delta\omega_1 + W_+ H(-\phi) &= 0 \\ \Delta\omega_i &= 0, \qquad i = 2, \ldots, N-2 \\ \Delta\omega_{N-1} - W_- H(\phi) &= 0 \end{aligned}$$

Thus, a travelling wave solution requires that all frequencies must be the same except at the ends of the chain. One travelling solution is given by $\Delta\omega_{N-1} = 0$ with

$$\Delta\omega_1 = -W_- H(-\phi), \qquad H(\phi) = 0$$

For the choice $H(\phi) = \sin(\phi + \sigma)$ we have that $\phi = -\sigma$ and $\Delta\omega_1 = -W_- \sin(2\sigma)$. If $2\sigma < \pi$ then $\Delta\omega_1 = \omega_2 - \omega_1 < 0$ and ω_1 must be larger than ω_2 and hence all the remaining ω_i for a forward travelling wave to exist. Backward swimming can be generated by setting $\omega_1 = 0$ and solving in a similar fashion.

2.5 Travelling waves in neural systems

Many spatially extended dynamical systems support travelling waves. For example, action potentials on axons, spiral waves of electrical activity on hearts, flame fronts in forest fires, etc. It is quite often possible to calculate the properties of such waves by constructing an associated homoclinic or heteroclinic connection.

Global connections

Consider a continuous-time dynamical system defined by

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n$$

with equilibria $x^{(0)}, x^{(1)}, x^{(2)}, \ldots$
An orbit Γ starting at a point $x \in \mathbb{R}^n$ is called *homoclinic* to the equilibrium point $x^{(0)}$ if $\phi(x,t) \to x^{(0)}$ as $t \to \pm\infty$.
An orbit Γ starting at a point $x \in \mathbb{R}^n$ is called *heteroclinic* to the equilibrium points $x^{(1)}$ and $x^{(2)}$ if $\phi(x,t) \to x^{(1)}$ as $t \to -\infty$ and $\phi(x,t) \to x^{(2)}$ as $t \to +\infty$

Let us consider the simple example of a spatially extended bi-stable system of the form

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v)$$

where $f(v) = v(v-1)(\alpha - v)$. By a travelling wave we mean solutions of the form

$$v(x,t) = V(x + ct) = V(\xi)$$

for some undetermined *speed* c. $\xi = x + ct$ is called the travelling wave variable. When written as a function of ξ the wave appears stationary. Substitution into the bistable equation yields a second order ODE

$$V_{\xi\xi} - cV_\xi + f(V) = 0$$

Equivalently we have

$$V_\xi = W$$
$$W_\xi = cW - f(V)$$

The fixed points are $(V,W) = (0,0)$ (saddle) and $(V,W) = (1,0)$ (saddle). To find travelling fronts we look for a heteroclinic trajectory that connect these equilibria. The equilibria at $V = \alpha$ is either a node or a spiral (since the eigenvalues of the linearisation have the same sign). Our

goal is to choose a value of c such that the trajectory that leaves the saddle $V = 0$ at $\xi = -\infty$ can be made to connect with the saddle $V = 1$ at $\xi = +\infty$. This is known as *shooting*. Since we expect $V_\xi \to 0$ as $\xi \to \pm\infty$ we try solutions of the form

$$W = BV(V-1)$$

implying that $W_V = B(2V-1)$. From the first order form we may take the ratio of W_ξ and V_ξ to obtain

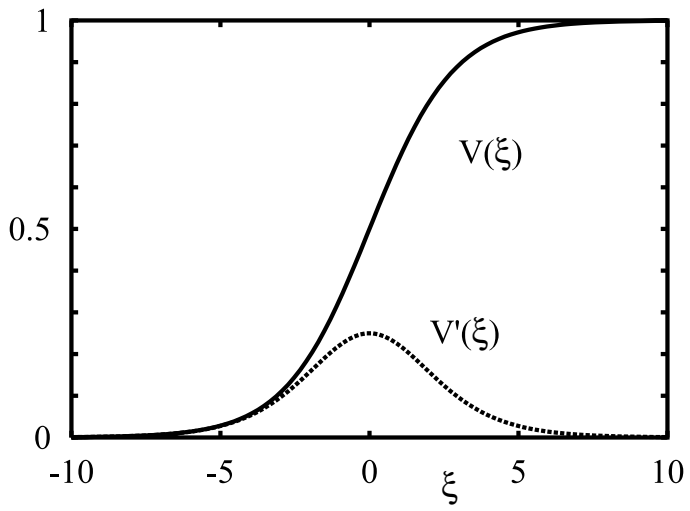
$$W_V = c - \frac{\alpha - V}{B}$$

where we use $f(V) = W(\alpha - V)/B$. Comparing the two forms for W_V and equating powers of V yields $B = 1/\sqrt{2}$ and

$$c = \frac{2\alpha - 1}{\sqrt{2}}$$

Finally solving $V_\xi = W$ we have an exact form of the solution given by

$$v(x,t) = \frac{1}{1 + \exp[-(x+ct)/\sqrt{2}]}$$



The spatially extended FitzHugh-Nagumo model

The following system of PDEs is the FitzHugh-Nagumo caricature of the Hodgkin-Huxley equations modelling the nerve impulse propagation along an axon:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) - u$$
$$\frac{\partial u}{\partial t} = \beta v$$

where $v(x,t)$ represents the membrane potential and $u(x,t)$ is a phenomenological *recovery* variable; $f(v) = v(\alpha - v)(v-1)$, $1 > \alpha > 0$, $\beta > 0$, $-\infty < x < \infty$, $t > 0$. Travelling waves are solutions of the form

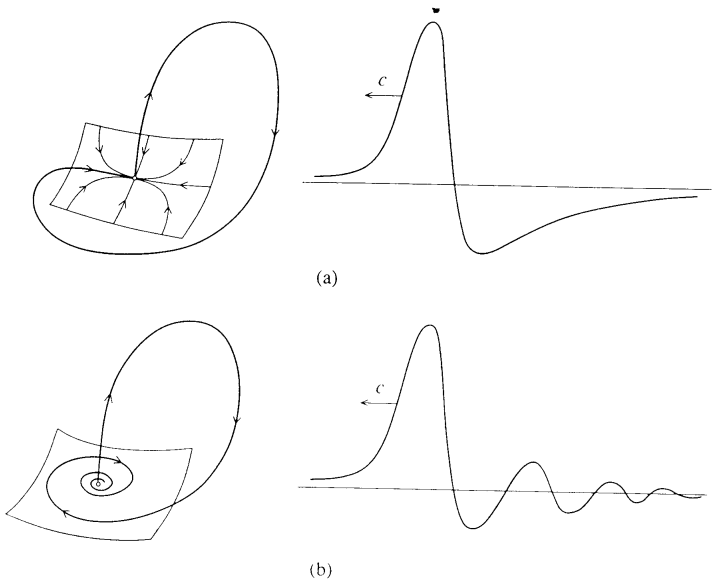
$$v(x,t) = V(\xi), \qquad u(x,t) = U(\xi), \qquad \xi = x + ct$$

for some unknown c. They satisfy the following first order system of ODEs

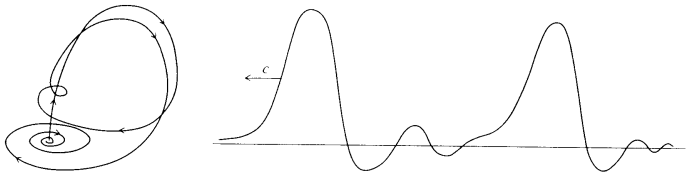
$$\begin{aligned}\dot{V} &= W \\ \dot{W} &= cW - f(V) + U \\ \dot{U} &= \frac{\beta}{c}V\end{aligned}$$

where the dot denotes differentiation with respect to ξ. Any bounded orbit corresponds to a travelling wave such that $c = c(\alpha, \beta)$.

For all $c > 0$ the wave system has a unique equilibrium at $(V, W, U) = (0, 0, 0)$ with one positive eigenvalue λ_1 and two eigenvalues $\lambda_{2,3}$ with negative real parts. (To show this first verify this assuming the eigenvalues are real. Then show that the characteristic equation cannot have roots on the imaginary axis, and finally, use the continuous dependence of the eigenvalues on the parameters). The equilibrium can either be a saddle or a saddle-focus with a 1D unstable and a 2D stable manifold. The transition between saddle and saddle-focus is caused by the presence of a double negative eigenvalue.



Impulses with (a) monotone and (b) oscillating tails.



A double pulse.

The piecewise linear FitzHugh-Nagumo model

Consider the FitzHugh-Nagumo system in the form

$$\epsilon \frac{\partial v}{\partial t} = \epsilon^2 \frac{\partial^2 v}{\partial x^2} + f(v, w)$$
$$\frac{\partial w}{\partial t} = g(v, w)$$

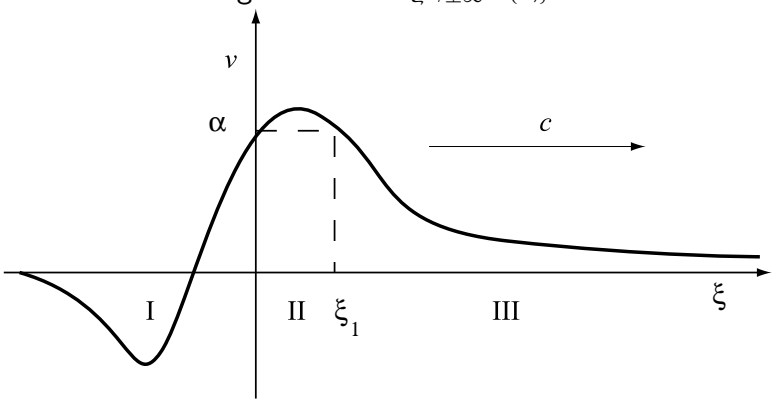
so that travelling waves are specified by the ODEs:

$$\epsilon^2 v_{\xi\xi} + c\epsilon v_\xi + f(v, w) = 0$$
$$c w_\xi + g(v, w) = 0$$

where $\xi = x - ct$. Analysis of the wave and its speed is possible for the piecewise linear dynamics

$$f(v, w) = \Theta(v - \alpha) - v - w, \qquad g(v, w) = v$$

where $\Theta(x)$ is a step function such that $\Theta(x) = 1$ if $x \geq 0$ and is zero otherwise. Consider solutions of the form shown in the figure with $\lim_{\xi \to \pm\infty} v(\xi) = 0$.



Regions I, II and III are defined respectively by $\xi < 0$, $0 < \xi < \xi_1$ and $\xi_1 < \xi$, and in the travelling coordinate frame $v(0) = v(\xi_1) = \alpha$. Note that ξ_1 and c are undetermined. In regions I and III look for solutions of the form $v = A\exp(\lambda\xi)$ and $w = B\exp(\lambda\xi)$. This gives the characteristic equation

$$p(\lambda) = \epsilon^2\lambda^3 + c\epsilon\lambda^2 - \lambda + 1/c \equiv \epsilon^2(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

Since $p(0) > 0$ then at least one root (say λ_1) is real and negative. If the remaining two roots $\lambda_{2,3}$ are real they are positive. (All this can be seen from a plot of $p(\lambda)$). In region II $f(v, w) = 1 - v - w$ so that

$$\epsilon^2 v_{\xi\xi} + c\epsilon v_\xi - v - w = -1$$
$$c w_\xi + v = 0$$

The inhomogeneous solution solution is $(v, w) = (0, 1)$. We now look for general solutions of the form *inhomogeneous solution* + *homogeneous solution* which decay at $\pm\infty$. A solution with this form is

$$w(\xi) = \begin{cases} Ae^{\lambda_1 \xi} & \xi \geq \xi_1 \\ 1 + \sum_{i=1}^{3} B_i e^{\lambda_i \xi} & 0 \leq \xi \leq \xi_1 \\ \sum_{i=2}^{3} C_i e^{\lambda_i \xi} & \xi \leq 0 \end{cases}$$

where $v = -cw_\xi$. Note that although the solution and its first derivative are required to be continuous there must be a jump in $v_{\xi\xi}$ (due to the nature of the nonlinearity $f(v, w)$) at $\xi = 0$ and $\xi = \xi_1$ where $(v = \alpha)$. In fact

$$v_{\xi\xi}\big|_{0^-}^{0^+} = -1 \qquad \text{and} \qquad v_{\xi\xi}\big|_{\xi_1^-}^{\xi_1^+} = +1$$

Continuity of the solution and its first derivative gives

$$\begin{aligned} v(0^+) &= v(0^-) = \alpha & \qquad v(\xi_1^+) &= v(\xi_1^-) = \alpha \\ v_\xi(0^+) &= v_\xi(0^-) & \qquad v_\xi(\xi_1^+) &= v_\xi(\xi_1^-) \\ v_{\xi\xi}(0^+) &= v_{\xi\xi}(0^-) - 1 & \qquad v_{\xi\xi}(\xi_1^+) &= v_{\xi\xi}(\xi_1^-) + 1 \end{aligned}$$

At $\xi = 0$, and using $v = -cw_\xi$ gives

$$\begin{aligned} \lambda_1 B_1 + \lambda_2(B_2 - C_2) + \lambda_3(B_3 - C_3) &= 0 \\ \lambda_1^2 B_1 + \lambda_2^2(B_2 - C_2) + \lambda_3^2(B_3 - C_3) &= 0 \\ \lambda_1^3 B_1 + \lambda_2^3(B_2 - C_2) + \lambda_3^3(B_3 - C_3) &= 1/c \end{aligned}$$

or $\Lambda(B_1, B_2 - C_2, B_3 - C_3) = (0, 0, 1/c)$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix}$$

which may be solved using Cramer's rule. Now

$$\begin{aligned} \det\Lambda &= \lambda_1\lambda_2\lambda_3 \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = \lambda_1\lambda_2\lambda_3 \begin{vmatrix} 1 & 0 & 0 \\ \lambda_1 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 \\ \lambda_1^2 & \lambda_2^2 - \lambda_1^2 & \lambda_3^2 - \lambda_1^2 \end{vmatrix} \\ &= \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \begin{vmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 1 \\ \lambda_1^2 & \lambda_2 + \lambda_1 & \lambda_3 + \lambda_1 \end{vmatrix} = \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \end{aligned}$$

Introduce

$$\Lambda_1 = \begin{vmatrix} 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_2^2 & \lambda_3^2 \\ 1/c & \lambda_2^3 & \lambda_3^3 \end{vmatrix} = \lambda_2\lambda_3(\lambda_3 - \lambda_2)/c, \qquad \Lambda_2 = \lambda_1\lambda_3(\lambda_1 - \lambda_3)/c, \qquad \Lambda_3 = \lambda_1\lambda_2(\lambda_2 - \lambda_1)/c$$

Remembering that $\epsilon^2 p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, then

$$p'(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) + (\lambda - \lambda_1)(\lambda - \lambda_3) + (\lambda - \lambda_2)(\lambda - \lambda_3)$$

so that

$$\epsilon^2 p'(\lambda_1) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3), \qquad \epsilon^2 p'(\lambda_2) = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3), \qquad \epsilon^2 p'(\lambda_3) = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$$

Finally we may write the solution as

$$\boxed{\begin{bmatrix} B_1 \\ B_2 - C_2 \\ B_3 - C_3 \end{bmatrix} = \frac{1}{\det\Lambda} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{bmatrix} = \frac{1}{c\epsilon^2} \begin{bmatrix} \frac{1}{\lambda_1 p'(\lambda_1)} \\ \frac{1}{\lambda_2 p'(\lambda_2)} \\ \frac{1}{\lambda_3 p'(\lambda_3)} \end{bmatrix}}$$

$\boxed{\xi = 0}$

At $\xi = \xi_1$

$$\begin{aligned} \lambda_1(A - B_1)e^{\lambda\xi_1} - \lambda_2 B_2 e^{\lambda_2\xi_1} - \lambda_3 B_3 e^{\lambda_3\xi_1} &= 0 \\ \lambda_1^2(A - B_1)e^{\lambda\xi_1} - \lambda_2^2 B_2 e^{\lambda_2\xi_1} - \lambda_3^2 B_3 e^{\lambda_3\xi_1} &= 0 \\ \lambda_1^3(A - B_1)e^{\lambda\xi_1} - \lambda_2^3 B_2 e^{\lambda_2\xi_1} - \lambda_3^3 B_3 e^{\lambda_3\xi_1} &= -1/c \end{aligned}$$

A similar piece of algebra shows that

$$\boxed{\begin{bmatrix} (A - B_1)e^{\lambda_1 \xi} \\ B_2 e^{\lambda_2 \xi} \\ B_3 e^{\lambda_3 \xi} \end{bmatrix} = \frac{1}{c\epsilon^2} \begin{bmatrix} -\frac{1}{\lambda_1 p'(\lambda_1)} \\ \frac{1}{\lambda_2 p'(\lambda_2)} \\ \frac{1}{\lambda_3 p'(\lambda_3)} \end{bmatrix}}$$

$$\boxed{\xi = \xi_1}$$

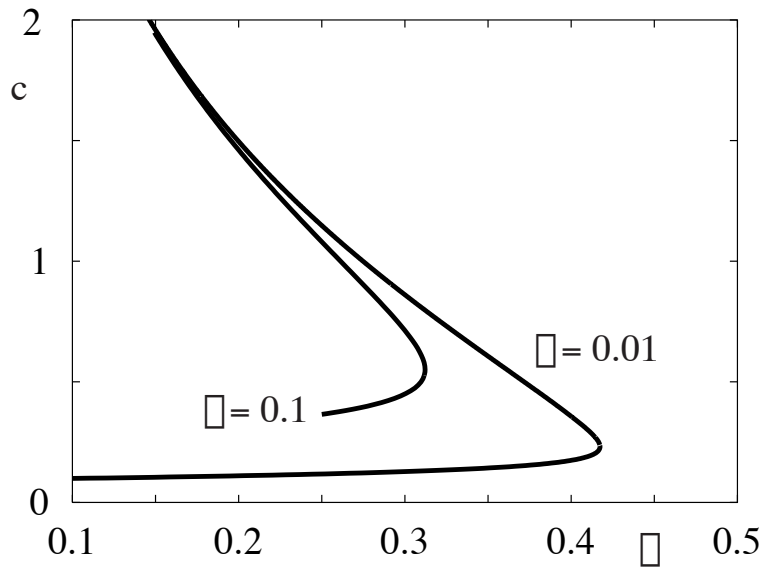
Together with the equations $v(0) = \alpha$ and $v(\xi_1) = \alpha$ we have eight equations in the six unknowns $A, B_1, B_2, B_3, C_1, C_2, \xi_1, c$. We shall now eliminate all but ξ and c to obtain $(v, \xi_1) = (v(\alpha), \xi_1(\alpha))$. From $v(0) = \alpha$ and $v(\xi_1) = \alpha$ we have

$$\alpha = -c(\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3)$$
$$\alpha = -cA\lambda_1 e^{\lambda_1 \xi_1}$$

Using $\boxed{\xi = 0}$ and $\boxed{\xi = \xi_1}$ we may then obtain the two equations

$$e^{\lambda_1 \xi_1} + \epsilon^2 p'(\lambda_1)\alpha - 1 = 0$$
$$\frac{e^{-\lambda_2 \xi_1}}{p'(\lambda_2)} + \frac{e^{-\lambda_3 \xi_1}}{p'(\lambda_3)} + \frac{1}{p'(\lambda_1)} + \epsilon^2 \alpha = 0$$

which can be solved numerically.



Speed of the travelling pulse solution in the piecewise linear FitzHugh-Nagumo model. A linear stability analysis shows that the fast wave is stable and the slow wave unstable.

2.6 Firing rate models, population models and mean field descriptions for large networks

Firing rate models

In many continuum models for the propagation of electrical activity in neural tissue it is assumed that the synaptic input current is a function of the pre-synaptic firing rate function. To see how this might arise consider a one-dimensional continuum of spiking single neurons with synaptic input at position x given by

$$\begin{aligned} u(x,t) &= g\int_{-\infty}^{\infty} w(x-y)\sum_m \eta(t-T_m(y))dy \\ &= g\int_{-\infty}^{\infty} w(y)\int_0^{\infty}\eta(s)\sum_{m\in\mathbb{Z}}\delta(s-t+T_m(x-y))dyds \end{aligned}$$

This models the effect of an idealized action potential (delta-Dirac function) arriving at a synapse and initiating a postsynaptic current $\eta(t-T_m)$ at time T_m. The convolution over space takes into account the connectivity pattern of the synapses between neurons. For simplicity it is also often assumed to be isotropic and homogeneous. If the synaptic response is *slow* then neurons feel an effectively constant input over some long time-scale. In this case it is natural to replace the spike train $\sum_m \delta(t-T_m(x))$ with a (smooth) firing rate function of synaptic activity $f(u(x,t))$. The function $f(u)$ is the firing rate response of the single neuron model to constant input. For a given firing rate function of synaptic current the synaptic dynamics is therefore determined by the integral equation

$$\boxed{u(x,t) = g\int_{-\infty}^{\infty} w(y)\int_0^{\infty}\eta(s)f(u(x-y,t-s)dyds}$$

There are a number of firing rate functions that may be derived from biophysical models. For example type I cortical neurons have firing rates of the form $f\sim\sqrt{u-h}$ (near a saddle-node on a limit cycle bifurcation at $u=h$). Typically, as the (constant) current, u is increased, most cortical neurons switch from a resting constant potential to an active mode. In the active mode, either trains of spikes are generated or burst of spikes. A common choice for the firing rate function is the sigmoid

$$f(u) = \frac{1}{1+\exp(-\beta(u-h))}$$

which satisfies the Ricatti equation $df/du = \beta f(1-f)$ and saturates to one for large u. This functional form, with threshold h and steepness parameter β, is not derived from a biophysical model, rather it is seen as a reasonable fit to experimental data.

Population models and mean field descriptions for large networks

Many cortical networks seem to be adequately described by *Wilson-Cowan* equations of the form

$$\begin{aligned} \tau\frac{\partial E}{\partial t} &= -E + (1-kE)f_E\left(W_{EE}*E - W_{IE}*I + P\right) \\ \tau\frac{\partial I}{\partial t} &= -I + (1-kI)f_I\left(W_{EI}*E - W_{II}*I + Q\right) \end{aligned}$$

where

$$(w*u)(x,t) = \int_{-\infty}^{\infty} w(x-y)u(y,t)dy$$

Here $E(x,t)$ (for excitatory) and $I(x,t)$ (for inhibitory) are the mean firing rates of neurons at positions x and $P(x)$ and $Q(x)$ are the external inputs to the network. The terms $(1-kE(x,t))$ and $(1-kE(x,t))$ model refractoriness by decreasing the firing rates $f_{I,E}$ within a period k after high activity. The four connectivity functions $W_{EE}(x), W_{EI}(x), W_{IE}(x), W_{II}(x)$ represent the spatial spread of synaptic connections in the network. A simple lattice version of these equations (dropping refractory terms) might be written

$$\tau\dot{Y}_i = -Y_i + \ f(\sum_{j=1}^{N} W_{ij}Y_j), \qquad i = 1,\dots,N$$

Using the so-called Σ exchange transformation $\mathbf{X} = \mathbf{WY}$ we obtain

$$\tau\dot{X}_i = -X_i + \sum_{j=1}^{N} W_{ij}f(X_j) + I_i, \qquad i = 1,\dots,N$$

where we include an external drive $\mathbf{I}$. For the case that $W_{ij} = W_{ji}$ (symmetric $\mathbf{W}$) and $W_{ii} = 0$ (no self-interactions) and f is monotonically increasing we obtain the so-called *Hopfield model*.

For the Hopfield model one can guarantee that almost all trajectories converge to stable fixed point due to the existence of an energy function $H(\mathbf{X})$ such that $dH/dt \leq 0$ with the equality only holding at a fixed point where $\dot{X}_i = 0$ for all $i = 1,\dots,N$. The energy function is given by

$$H(\mathbf{X}) = -\frac{1}{2}\sum_{i,j=1}^{N} W_{i,j}V_iV_j + \sum_{i=1}^{N}\int_0^{V_i} f^{-1}(Y)dY - \sum_{i=1}^{N} I_iV_i$$

where $V_i = f(X_i)$.

Proof: Differentiating with respect to time gives

$$\frac{dH}{dt} = -\sum_{i=1}^{N}\left(\sum_{j=1}^{N} W_{ij}V_j - X_i + I_i\right)\frac{dV_i}{dt} = -\tau\sum_{i=1}^{N}\frac{dX_i}{dt}\frac{dV_i}{dt} = -\tau\sum_{i=1}^{N}\left(\frac{dX_i}{dt}\right)^2 f'(X_i)$$

Since f is a monotonically increasing function, $f'(X_i) \geq 0$ for all X_i. Moreover, $(\dot{X}_i)^2 \geq 0$, so $\dot{H} \leq 0$ with equality only holding at the fixed point where $\dot{X}_i = 0$. Thus each trajectory of a Hopfield network evolves by continuously decreasing the energy function H until a stable fixed point is reached (so that there are no attractors other than fixed points).

The Hopfield net has been used extensively in modelling associative memories where fixed point attractors are treated as *memories* that can be addressed by *cues* in the form of initial network states. When $f(u) = \mathrm{sgn}(u)$ then $X_i = S_i \in \{+1,-1\}$ and

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} W_{ij}S_iS_j$$

which is recognised as the Hamiltonian of a *spin glass*. Tools from statistical mechanics (such as mean field theories and the replica approach) have been used to study such systems in the thermodynamic limit $N \to \infty$.

2.7 Pattern formation in neural systems and drug-induced visual hallucinations

Upon choosing a simple synaptic response such as $\eta(t) = \alpha e^{-\alpha t}$ the integral neural field model can be written as an integro-differential equation

$$\frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} = -u(x,t) + g\int_{-\infty}^{\infty} w(x-y)f(u(y,t))dy$$

[using $\dot{\eta} = -\alpha\eta$ and $\eta(0) = \alpha$]. We now describe how a spatially homogeneous state can become unstable to spatially heterogeneous perturbations, resulting in the formation of periodic patterns. We do this using a *Turing* instability analysis.

Turing instability analysis

One solution of the neural field equation is the spatially uniform resting state $u(x,t) = \overline{u}$ for all x, t, defined by

$$\overline{u} = gf(\overline{u}) \int_{-\infty}^{\infty} w(y)dy$$

We linearise about this state by letting $u(x,t) \to \overline{u} + u(x,t)$ so that $f(u(x,y)) \to f(\overline{u}) + f'(\overline{u})u$ to obtain

$$\frac{1}{\alpha}\frac{\partial u(x,t)}{\partial t} = -u(x,t) + g\beta \int_{-\infty}^{\infty} w(y)u(x-y,t)dy, \qquad \beta = f'(\overline{u})$$

This has solutions of the form $e^{\lambda t}e^{ipx}$, giving

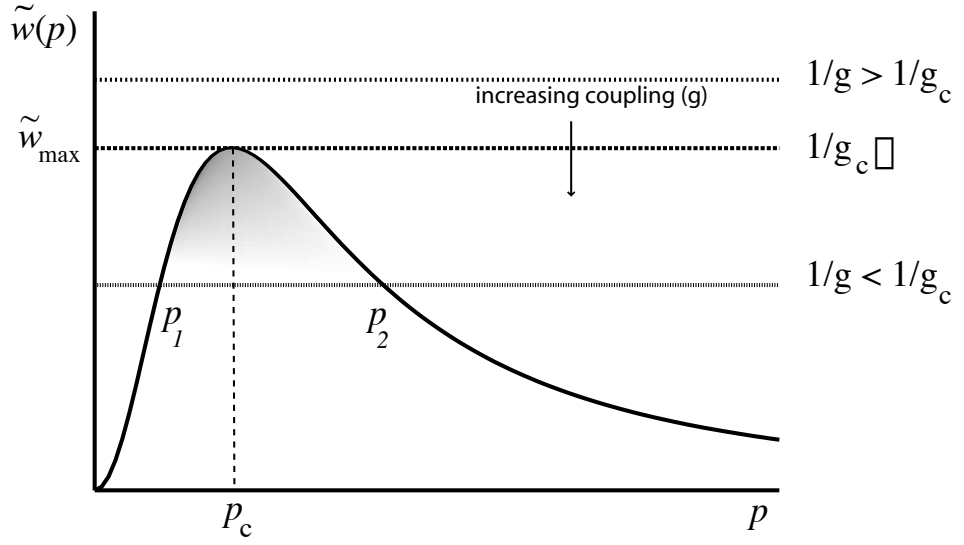
$$\frac{\lambda(p)}{\alpha} = -1 + g\beta\widetilde{w}(p), \qquad \widetilde{w}(p) = \int_{-\infty}^{\infty} w(y)\ e^{-ipy}dy$$

We recognise $\widetilde{w}(p)$ as the Fourier transform of $w(x)$. The uniform steady state is linearly stable if $\mathrm{Re}\lambda(p) < 0$ for all $p \in \mathbb{R}$, $p \neq 0$. If we assume that $w(x) = w(-x)$ then $\widetilde{w}(p)$ is a real even function of p and the stability condition is simply

$$\widetilde{w}(p) < \frac{1}{\beta g}, \qquad \text{for all } \ p \in \mathbb{R}, \quad p \neq 0$$

Now consider the case that $\widetilde{w}(p)$ has a positive maximum $\widetilde{w}_{max}$ at $p = \pm p_c$, that is $\widetilde{w}(p_c) = \widetilde{w}_{max}$ and $\widetilde{w}(p) < \widetilde{w}_{max}$ for all $p \neq p_c$. Let

$$g_c\beta = \frac{1}{\widetilde{w}_{max}}$$

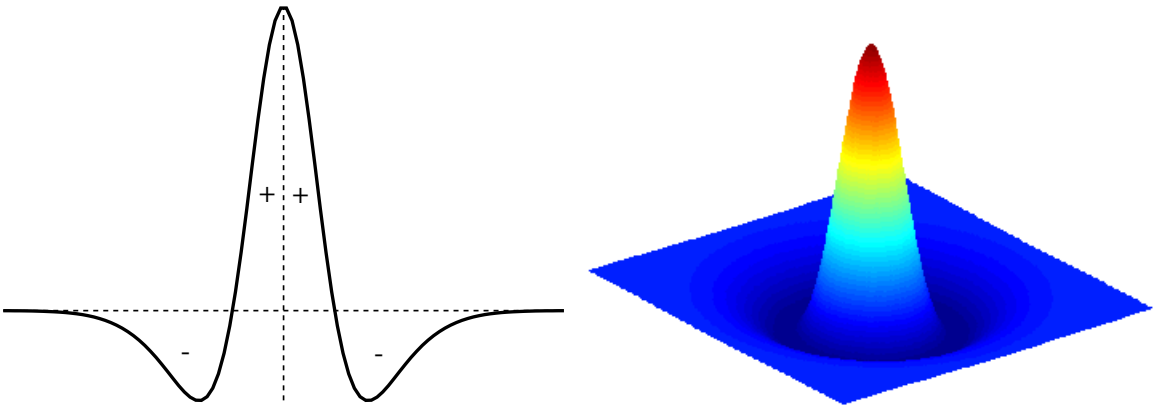


- For $g < g_c$ we have $g\widetilde{w}(p) \leq g\widetilde{w}_{max} < 1$ for all p and the resting state is linearly stable.
- At the critical point $g = g_c$ we have $\beta g_c\widetilde{w}(p_c) = 1$ and $\beta g_c\widetilde{w}(p) < 1$ for all $p \neq p_c$. Hence, $\lambda(p) < 0$ for all $p \neq p_c$, but $\lambda(p_c) = 0$. This signals the point of instability due to excitation of the pattern $e^{\pm ip_c x}$.

- Beyond the bifurcation point, $g > g_c$, $\lambda(p_c) > 0$ and this pattern grows with time. In fact there will typically exist a range of values of $p \in (p_1, p_2)$ for which $\lambda(p) > 0$, signalling a set of growing patterns. As the patterns grow, the linear approximation breaks down and nonlinear terms dominate behaviour.
- The saturating property of $f(u)$ tends to create patterns with finite amplitude, that scale as $\sqrt{g - g_c}$ close to bifurcation and have wavelength $2\pi/p_c$.
- If $p_c = 0$ then we would have a *bulk instability* resulting in the formation of another homogeneous state.

A biologically motivated choice for $w(x)$ is the so-called mexican hat function which represents short-range excitation and long range inhibition. An example of such a function is $w(x) = w_+(x) - w_-(x)$, where

$$w_\pm(x) = \frac{A_\pm}{\sqrt{2\pi\sigma_\pm^2}} e^{-x^2/2\sigma_\pm^2}, \qquad \frac{A_+}{\sigma_+} > \frac{A_-}{\sigma_-}, \qquad \sigma_- > \sigma_+$$



One and two-dimensional Mexican hat functions.

A generalisation of our argument to two dimensions shows that the linearised equations of motion have solutions of the form

$$e^{\lambda t} e^{i\mathbf{p}\cdot\mathbf{r}}, \qquad \frac{\lambda(p)}{\alpha} = -1 + g\beta\widetilde{w}(p), \qquad p = |\mathbf{p}|$$

with $\mathbf{r}, \mathbf{p} \in \mathbb{R}^2$ and

$$\widetilde{w}(p) = \int w(|\mathbf{r}|)\ e^{-i\mathbf{p}\cdot\mathbf{r}} d\mathbf{r}$$

The bifurcation point is defined by $\beta g_c \widetilde{w}(p_c) = 1$, with $\mathbf{p}_c = (p_1, p_2)$ so that near bifurcation we expect spatially heterogeneous solutions of the form

$$\exp i[p_1 x + p_2 y], \qquad p_c^2 = p_1^2 + p_2^2$$

For a given p_c there are an infinite number of choices for p_1 and p_2. It is therefore convenient to restrict attention to doubly periodic solutions that tessellate the plane. These can be expressed in terms of the basic symmetry groups of hexagon, square and rhombus. Solutions can then be constructed from combinations of the basic functions $e^{ip_c \mathbf{R}\cdot\mathbf{r}}$, for appropriate choices of the basis vectors $\mathbf{R}$. If ϕ is the angle between two basis vectors $\mathbf{R}_1$ and $\mathbf{R}_2$, we can distinguish three types of lattice according to the value of ϕ: square lattice ($\phi = \pi/2$), rhombic lattice

$(0 < \phi < \pi/2, \phi = \pi/3)$ and hexagonal $(\phi = \pi/3)$. Hence, all doubly periodic functions may be written as a linear combination of plane waves

$$u(\mathbf{r}) = \sum_j c_j e^{ip_c \mathbf{R}_j \cdot \mathbf{r}}, \qquad |\mathbf{R}_j| = 1$$

For hexagonal lattices we use

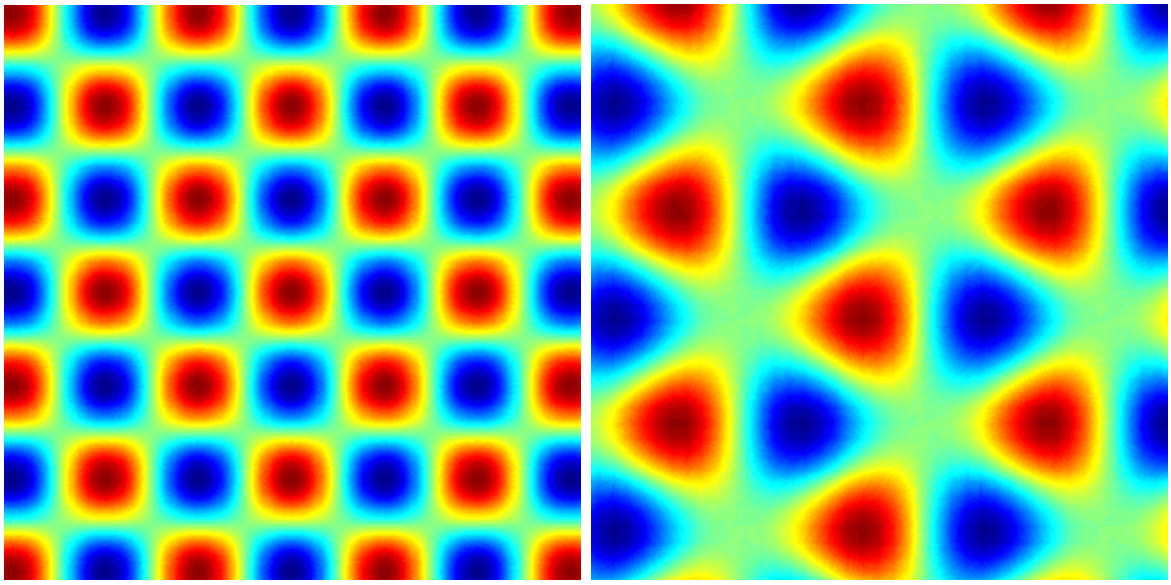
$$\mathbf{R}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{R}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}, \qquad \mathbf{R}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

For square lattices we use

$$\mathbf{R}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{R}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

while the rhombus tessellation uses

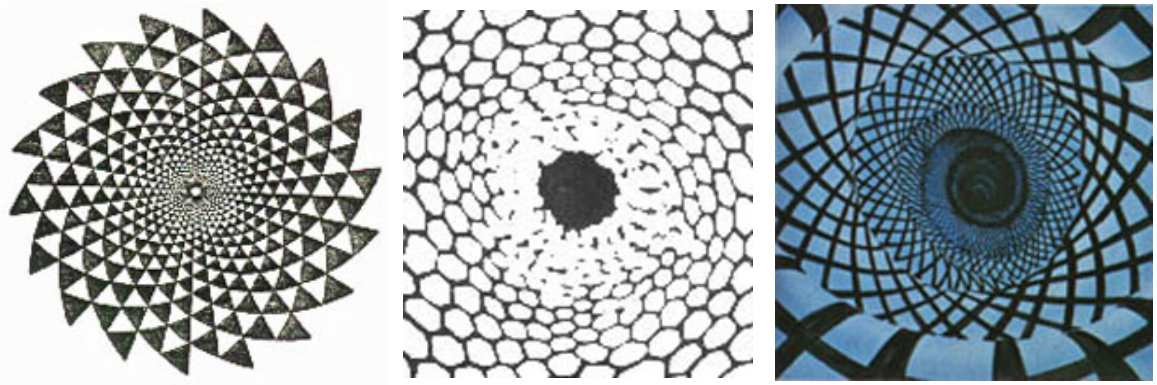
$$\mathbf{R}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{R}_2 = \begin{bmatrix} \cos\eta \\ \sin\eta \end{bmatrix}$$



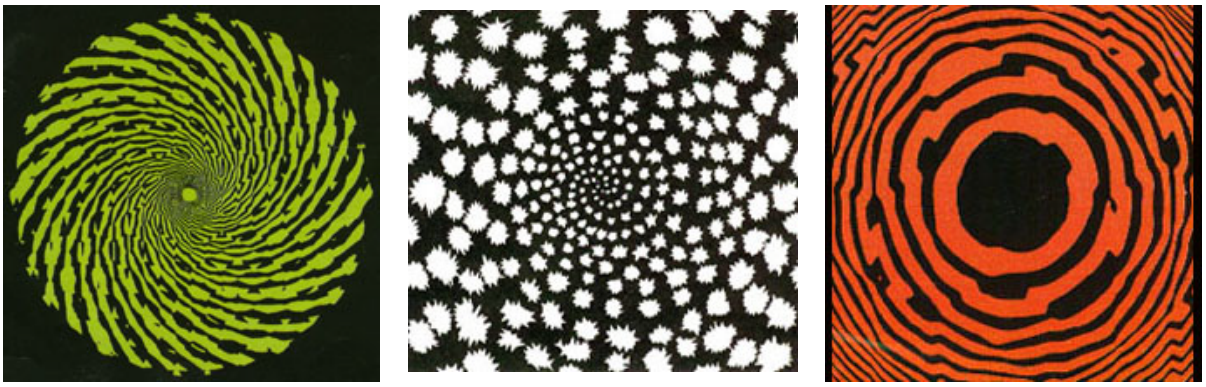
Density plot of doubly periodic square and hexagonal functions.

Visual hallucinations

Geometric visual hallucinations are seen in many situations, for example: after being exposed to flickering lights, after the administration of certain anaesthetics, on waking up or falling asleep, following deep binocular pressure on ones eyeballs, and shortly after the ingesting of drugs such as LSD and Marihuana.



Phosphene hallucination produced by deep binocular pressure on the eyeballs. *Honeycomb* and lattice-tunnel hallucination generated by Marihuana.

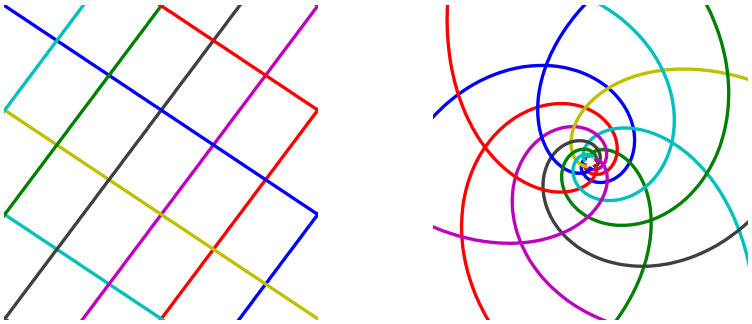


Spiral, spiral tunnel and tunnel hallucinations generated by LSD.

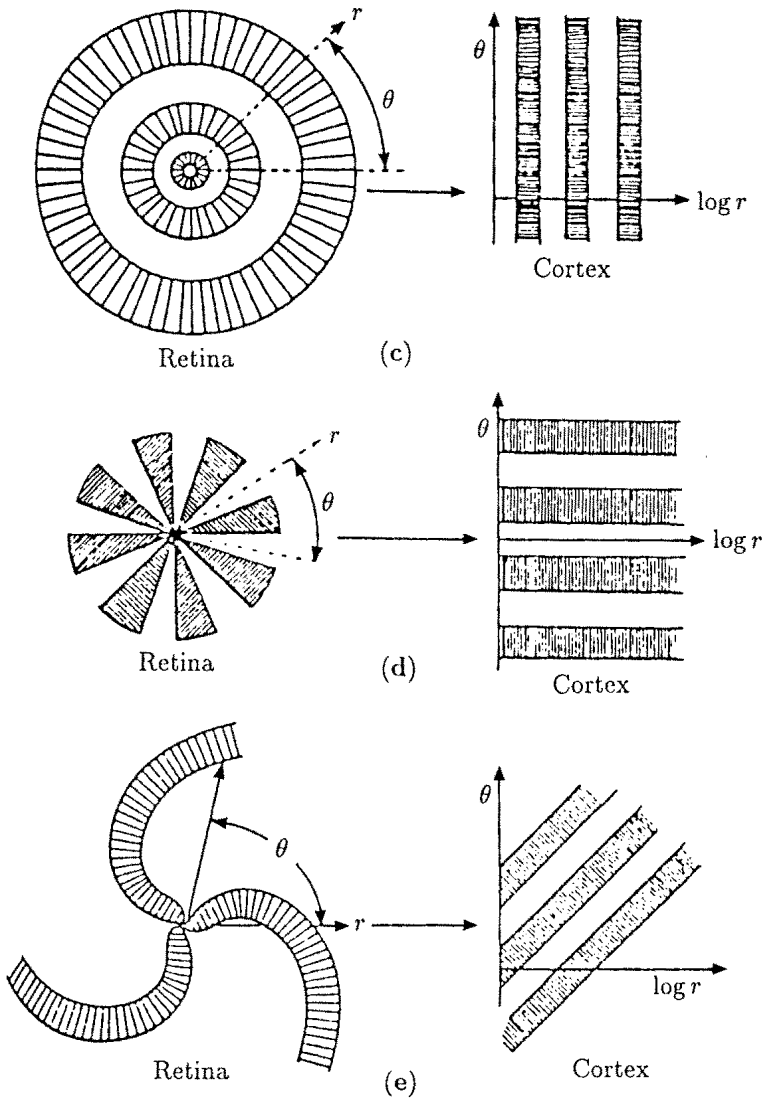
To see how doubly periodic functions can give rise to visual hallucinations we first need to calculate what visual hallucinations look like, not in the standard polar coordinates of the visual field, but in the coordinates of V1 (layer 1 of visual cortex). It is well established that there is a topographic map of the visual field in V1, the retinotopic representation, and that the central region of the visual field has a much bigger representation in V1 than it does in the visual field. The reason for this is partly that there is a nonuniform distribution of retinal ganglion cells, each of which connects to V1 via the lateral geniculate nucleus (LGN). Except very close to the fovea, a point on the retina denoted by the complex coordinate $z = 2\beta r e^{i\theta}$ is mapped onto the point with complex coordinate $w = x + iy$ in the visual cortex according to the complex logarithmic conformal mapping

$$w = \alpha \ln z = \alpha \ln 2\beta r + i\alpha\theta$$

where α and β are constants. The transformation of various geometric shapes is summarised in the next two pictures.

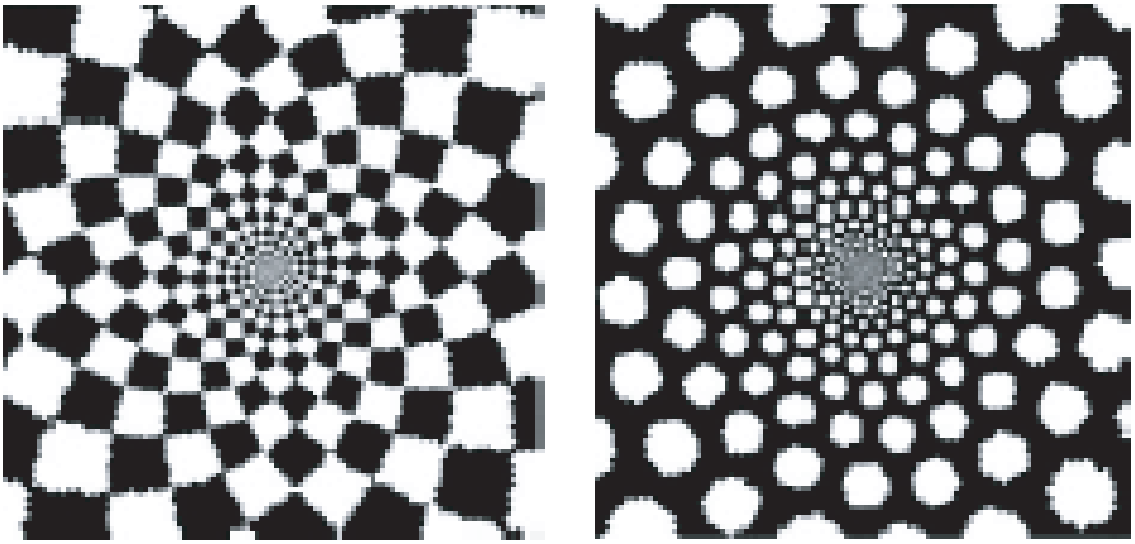


Application of the complex logarithmic map to a regular grid.



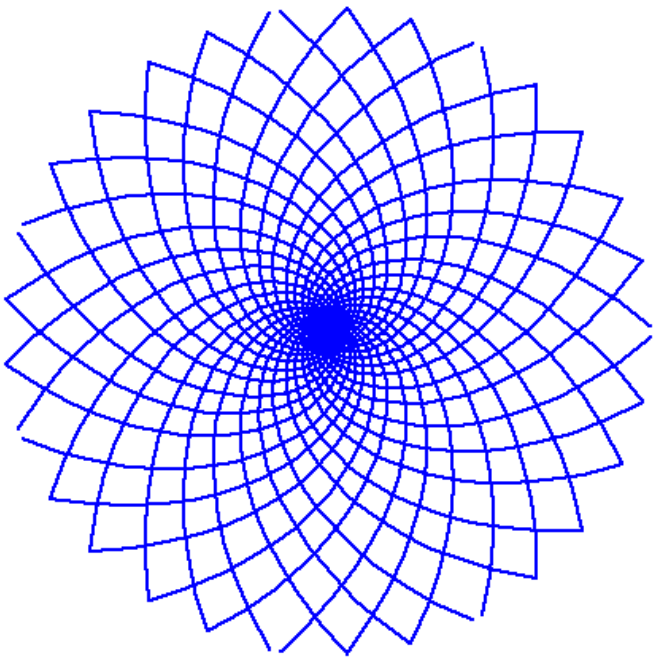
Corresponding patterns under the retino-cortical map.

Hence, if there are cortical mechanisms that give rise to doubly periodic cellular patterns of squares and hexagons or rolls along some constant direction then, by the inverse retino-cortical map, this could underly many common forms of visual hallucination. As we have already seen, such a mechanism exists based upon the variation of a bifurcation parameter that leads to a Turing instability of a homogeneous steady state.



Action of the inverse retinocortical map on square planforms (left) and hexagonal planforms (right).

For your entertainment why not colour the picture!



S Coombes 2008