# School of Mathematical Sciences

## G14TNS Theoretical Neuroscience

# Problem sheet 2

1. Consider the planar system

$$\mu \dot{v} = f(v, w), \qquad \dot{w} = g(v, w)$$

Show that a Hopf bifurcation occurs when

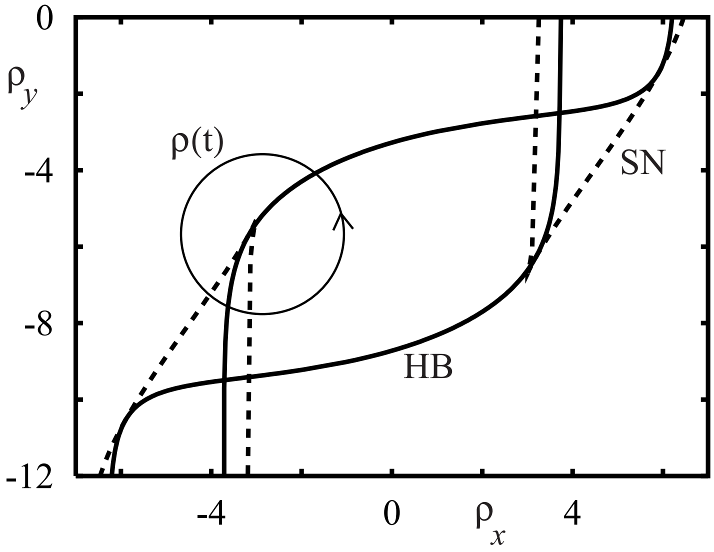
$$f_v(\overline{v}, \overline{w}) = -\mu g_w(\overline{v}, \overline{w}), \qquad f_v(\overline{v}, \overline{w}) g_w(\overline{v}, \overline{w}) - g_v(\overline{v}, \overline{w}) f_w(\overline{v}, \overline{w}) > 0$$

where $(\overline{v}, \overline{w})$ is the fixed point.

2. Consider the Wilson-Cowan equations

$$\begin{aligned} \dot{x} &= -x + f(\rho_x + ax - by), \qquad f(z) = \frac{1}{1 + e^{-z}} \\ \dot{y} &= -y + f(\rho_y + cx - dy) \end{aligned}$$

   (a) Determine the conditions for a Hopf bifurcation (HB).
   (b) Determine the conditions for a saddle-node bifurcation (SN).
   (c) Plot the above curves in the $(\rho_x, \rho_y)$ plane for $a = b = c = 10$ and $d = -2$.
   (d) Describe the type of hybrid bursting behaviour that is found on traversing the orbit $\rho(t)$ shown in the figure.



3. Consider the FitzHugh-Nagumo model

$$\dot{v} = v(a - v)(v - 1) - w + I, \qquad \dot{w} = bv - \gamma w$$

where $0 < a < 1$ and $b, \gamma, I > 0$.

(a) Show that there is a confined set for the system.

(b) Determine the stationary points of the $v$ nullcline in terms of $a$ and $I$.

(c) Construct a piecewise linear approximation of the cubic nullcline with identical stationary points to those found in (b).

(d) Using the approximation in (c) determine the conditions under which the steady state is stable and excitable.

(e) Find the conditions on the parameters and $I_{1,2}$ so that for $I \in (I_1, I_2)$ the steady state is linearly unstable. State why a limit cycle solution is expected to exist.

(f) Obtain the period of the small amplitude limit cycle when $I$ is just greater than $I_1$.

4. Consider the van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

for the special case that $\mu \gg 1$ (strongly nonlinear singular limit).

(a) Obtain the equivalent planar representation

$$\begin{aligned} \dot{x} &= w - \mu F(x), \qquad F(x) = \frac{x^3}{3} - x \\ \dot{w} &= -x \end{aligned}$$

(b) Sketch the nullclines and a typical periodic orbit. Explain your reasoning.

(c) Show that to first order in $\mu$ the period of oscillation is given by

$$\mu[3 - 2\ln 2]$$

5. Consider the canonical phase model

$$\dot{\theta} = (1 - \cos\theta) + (1 + \cos\theta)p, \qquad \theta \in S^1$$

(a) Describe the dynamics for the cases i) $p > 1$, ii) $p = 1$, iii) $0 < p < 1$, iv) $p = 0$, v) $p < 0$.

(b) Show that for $p > 0$ the orbit is given by

$$\theta(t) = 2\tan^{-1}(\sqrt{p}\tan\sqrt{p}t)$$

(c) Find the period of oscillation.

6. Find the conditions on $I$ such that the integrate-and-fire model

$$\dot{v} = -\frac{v}{\tau} + I$$

with threshold $h$ and reset value $0$ can oscillate periodically. Obtain the period of oscillation in closed form.

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