

School of Mathematical Sciences

G14TNS Theoretical Neuroscience

Problem sheet 3

1. Consider the following dynamical system with a stable limit cycle at $r = 1$:

$$\dot{r} = (1 - r)r, \quad \dot{\theta} = 1 + \epsilon(1 - r)$$

- (a) Plot \dot{r} as a function of r .
(b) Draw a phase portrait.
(c) Show that the equation for an isochron of latency angle Φ is

$$\Phi = \theta - \epsilon \ln r$$

and sketch some typical isochrons.

2. Consider the following dynamical system with a stable limit cycle at $r = 1$:

$$\dot{r} = 5(1 - r)(r - 0.5)r, \quad \dot{\theta} = 1 + \epsilon(1 - r)$$

- (a) Plot \dot{r} as a function of r .
(b) Draw a phase portrait.
(c) Show that for $r > 0.5$ the equation for an isochron of latency angle Φ is

$$\Phi = \theta - \frac{2\epsilon}{5} \ln \left(2 - \frac{1}{r} \right)$$

What happens when $r < 0.5$. Sketch some typical isochrons.

3. Consider the leaky-integrator system

$$\tau \dot{x} = -x + I(t)$$

- (a) Show that the solution can be written in the form

$$x(t) = e^{-t/\tau} x(0) + \frac{1}{\tau} \int_0^t e^{-(t-s)} I(s) ds$$

- (b) Show that for $I(t) = I_0 \cos \omega t$ the solution is

$$x(t) = A e^{-t/\tau} + B \cos(\omega t + \phi)$$

where

$$B = \frac{I_0}{\sqrt{1 + \omega^2 \tau^2}}, \quad \tan \phi = -\omega \tau$$

Determine the constant A in terms of the initial condition $x(0) = x_0$.

4. An integrate-and-fire neuron evolves according to the equation

$$\dot{x} = \exp(-\epsilon x) \equiv f(x)$$

with x reset to zero every time x reaches a threshold $h = 1$.

- (a) Determine the period of oscillations T .
 (b) By performing the change of variables

$$\theta(t) = \frac{1}{T} \int_0^x \frac{dy}{f(y)}$$

show that the solution can be written in the form

$$x(\theta) = \frac{1}{\epsilon} \ln[1 + (e^\epsilon - 1)\theta]$$

where $\theta(t) = \theta_0 + t/T$ for some constant phase θ_0 .

5. Consider an integrate-and-fire neuron evolving according to

$$\dot{x} = I[1 + B \cos t], \quad T_n < t < T_{n+1}$$

for constant I and B , with $|B| < 1$. Here T_n is the time at which the neuron reaches a given threshold h for the n th occasion and fires. After firing x is reset to zero.

- (a) Derive the following iterative equation for T_n :

$$T_{n+1} + B \sin T_{n+1} = T_n + B \sin T_n + \frac{h}{I}$$

- (b) Introduce the function $F(x) = x + B \sin x$. Show that F is monotonically increasing (and hence invertible) when $|B| < 1$. Sketch the function $F(x)$ for $0 < B < 1$ and $B > 1$.
 (c) Introduce the phase variable $\phi_n = F(T_n)$ and rewrite (a) in the form

$$\phi_{n+1} = \phi_n + \frac{h}{I}$$

Describe the dynamics when i) h/I is a rational number, ii) when h/I is an irrational number.

6. Consider a periodically forced integrate-and-fire neuron

$$\dot{v} = -\frac{v}{\tau} + A(t), \quad T_n < t < T_{n+1}$$

where $A(t) = A(t + \Delta)$:

$$A(t) = I + \epsilon \begin{cases} +1 & 0 \leq t < \Delta/2 \\ -1 & \Delta/2 < t < \Delta \end{cases}$$

- (a) Show that for $\epsilon = 0$, $p:q$ mode-locking occurs when

$$I\tau = \frac{1}{1 - \exp(-q\Delta/(p\tau))}$$

Comment on the probability of observing a mode-locked state for $I\tau > 1$.

- (b) For non-zero ϵ , show that tangent bifurcations of the 1:1 mode-locked state are defined by

$$\pm\epsilon = -I + \frac{1/\tau}{1 - e^{-\Delta/\tau}}$$

- (c) Show that a non-smooth bifurcation of type (b) is defined by the two equations

$$\begin{aligned} \tau(I + \epsilon)(1 - e^{-\Delta(1/2-\phi)/\tau}) &= 1 \\ \bar{v}e^{-\phi\Delta/\tau} + \tau(I + \epsilon)(1 - e^{-\phi\Delta/\tau}) &= 1, \quad 0 < \phi < 1/2 \end{aligned}$$

where

$$\bar{v} = e^{-\Delta/2\tau} + \tau(I - \epsilon)(1 - e^{-\Delta/2\tau})$$

- (d) Plot the full Arnol'd tongue structure for the 1:1 state. Compare your analysis with a simulation of the XPP ODE file:

```
par I=2,tau=1,epsilon=1,Delta=2
par s=1
```

```
global 1 {v-1}{v=0}
global 1 {phi-Delta}{phi=0;s=1}
global 1 {phi-Delta/2}{s=-1}
```

```
v(0)=0
phi(0)=0
```

```
v'=-v/tau+I+epsilon*s
phi'=1
```

```
aux A=I+epsilon*s
```

```
done
```