# School of Mathematical Sciences

## G14TNS Theoretical Neuroscience

# Problem sheet 4

1. Consider a basic model of an oscillator:

$$\dot{\theta} = f(\theta), \qquad \theta \in [0, 2\pi)$$

where $f(\theta) = f(\theta + 2\pi)$.

   (a) Find the solution and period of oscillation when $f(\theta) = \omega$.

   (b) Discuss the possible bifurcations that can occur when $f(\theta) = \omega - a \sin\theta$ and $a$ is varied. Construct the periodic solution explicitly and show that when $a \approx \omega$ the period of oscillation goes like

$$\frac{\sqrt{2}\pi}{\sqrt{\omega}} \frac{1}{\sqrt{\omega - a}}$$

2. Consider the *firefly* model

$$\dot{\Theta} = \Omega, \quad \dot{\theta} = \omega + A f(\Theta - \theta)$$

where $f$ is now given by the triangle wave function:

$$f(\phi) = \begin{cases} \phi & -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \\ \pi - \phi & \frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2} \end{cases}$$

and $f$ is extended periodically outside the interval $-\pi/2 \leq \phi \leq 3\pi/2$.

   (a) Sketch $f(\phi)$

   (b) Find the range of entrainment

   (c) Assuming that the firefly is phase-locked to the stimulus, find a formula for the phase difference $\phi^*$.

   (d) Find a formula for $T_{\text{drift}}$, the period of an oscillatory solution.

3. Consider the system of two coupled oscillators

$$\dot{\theta}_1 = 1 - \gamma \sin 2\pi(\theta_1 - \theta_2), \quad \dot{\theta}_2 = \omega + \gamma \sin 2\pi(\theta_1 - \theta_2)$$

where $\theta_1, \theta_2 \in S^1 = [0, 1] \mod 1$, $\omega$ is fixed in the range $[0, 1]$ and $\gamma \geq 0$ is a variable parameter.

   (a) Suppose that $2\gamma > (1 - \omega)$. By subtracting the two given equations show that the oscillators become phase-locked for almost all initial conditions.

(b) Suppose that $2\gamma < (1-\omega)$. By adding and subtracting the given pair of equations obtain the exact solutions

$$\theta_1(t) = \frac{\theta}{2} + \frac{1}{2\pi}\left(\pi(1+\omega)t + \arctan u(t)\right)$$

$$\theta_2(t) = \frac{\theta}{2} + \frac{1}{2\pi}\left(\pi(1+\omega)t - \arctan u(t)\right)$$

where

$$u(t) = \frac{1}{1-\omega}\{2\gamma + \sigma \tan(\sigma t/2 + c)\}$$

$$\sigma = \sqrt{(1-\omega)^2 - 4\gamma^2}$$

and where $c$ is defined via

$$\sigma \tan c = -\{2\gamma + (1-\omega)\tan(\pi\theta)\}$$

The initial conditions are $\theta_1(0) = 0, \theta_2(0) = \theta$.

(c) Let $\Sigma \subset S^1 \times S^1$ be the cross-section $\Sigma = \{(\theta_1, \theta_2) | \theta_1 = 0\}$. Determine the corresponding Poincaré map $P : \Sigma \to \Sigma$ using part b).

4. Consider a network of oscillators with the following dynamics:

$$\dot{\phi}_k = \Omega + a \sin\phi_k + \frac{K}{N}\sum_{n=1}^{N} \sin\phi_n$$

The so-called in-phase solution is given by $\phi_n(t) = \phi^*(t)$ for all $n = 1, \ldots, N$, where $\phi^*(t)$ denotes the common waveform. Thus

$$\frac{d\phi^*}{dt} = \Omega + (a+1)\sin\phi^*$$

(a) Find conditions under which the in-phase solution is periodic, and show that all the Floquet multipliers of this solution are equal to unity.
**Hint**: Substitute $\phi_n(t) = \phi^*(t) + \eta_n(t)$ into the network equations and linearise. Then perform a change of variables

$$\mu = \frac{1}{N}\sum_{j=1}^{N} \eta_j, \quad \xi_n = \eta_{n+1} - \eta_n, \quad n = 1, \ldots, N-1$$

(b) By integrating the resulting equations for $\xi_n$ and $\mu$ show that $\xi_n(T) = \xi_n(0)$ and $\mu(T) = \mu(0)$ where $T$ is the period of the periodic solution $\phi^*(t)$.

(c) Use a reversibility argument based on the invariance of the original system under the transformation $t \to -t$, $\phi_n \to \pi - \phi_n$ to prove that the in-phase periodic solution of the linearised system is not attracting, even if the nonlinear terms are kept.

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