School of Mathematical Sciences

G14TNS Theoretical Neuroscience

Problem sheet 5

1. Consider the function

$$a(t) = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^{-1} [e^{-\alpha t} - e^{-\beta t}], \qquad t > 0$$

   (a) Show that $a(t)$ is normalised.

   (b) Show that in the limit $\alpha \to \beta$, $a(t)$ approaches the alpha function $\eta(t) = \alpha^2 t e^{-\alpha t}$

   (c) Show that the alpha function has a maximum at $t = \alpha^{-1}$ and sketch $\eta(t)$. Comment on the limit $\alpha \to \infty$.

   (d) Show

$$\sum_{m=0}^{\infty} \eta(t + m\Delta) = \frac{\alpha^2 e^{-\alpha t}}{(1 - e^{-\alpha\Delta})} \left[t + \frac{\Delta e^{-\alpha\Delta}}{1 - e^{-\alpha\Delta}} \right], \qquad t \in [0, \Delta).$$

   (e) Show that $X(t) = \sum_m \eta(t - T_m)$ may be regarded as the solution to the system of ODES

$$\frac{1}{\alpha}\dot{X} = Y - X, \qquad \frac{1}{\alpha}\dot{Y} = -Y + \sum_m \delta(t - T_m)$$

2. Consider the integrate-and-fire network

$$\dot{u}_i = -u_i + I + \epsilon \sum_j W_{ij} \sum_m \eta(t - T_m(j))$$

with reset to zero whenever $u_i(t) = 1$.

   (a) Use the transformation (solution of the uncoupled system)

$$\theta_i(t) = \frac{1}{\Delta} \int_0^{u_i(t)} \frac{dV}{I - V}, \qquad \Delta = \int_0^1 \frac{dV}{I - V}$$

   to show that the dynamics may be re-formulated as

$$\dot{\theta}_i = \frac{1}{\Delta} + \epsilon \sum_j W_{ij} R(\theta_j) \sum_m \eta(t - T_m(j)), \qquad R(\theta) = \frac{e^{\theta\Delta}}{\Delta}$$

   (b) After averaging show that the fixed point equations have the form

$$\epsilon \sum_j W_{ij} H(\theta_j - \theta_i) = 0$$

   (c) Construct the exact phase-locked solution to the system by writing $T_m(j) = (m - \theta_j)T$ and integrating the equations of motion. Compare your results to those obtained using the averaged equations of motion.

3. Consider two neurons with relative phase ϕ satisfying

$$\dot{\phi} = \epsilon[H(-\phi) - H(\phi)], \qquad H(\phi) = \int_0^1 R(\theta - \phi)P(\theta)d\theta$$

where $R(\theta + 1) = R(\theta)$.

(a) Show that if $P(\theta) = \sum_{m=0}^{\infty} \eta((\theta + m)\Delta)$ then

$$H(\phi) = \int_0^{\infty} R(\theta - \phi)\eta(\theta\Delta)d\theta$$

(b) Calculate $H(\phi)$ when $\eta(t) = \alpha^2 te^{-\alpha t}$ and $R(\theta) = 1 + \cos 2\pi\theta$. Determine the equilibrium solutions and their stability.

4. Consider a ring of phase oscillators described by

$$\dot{\theta}_k = \frac{1}{\Delta} + \epsilon \sum_{j=1}^{N} W_j H(\theta_{k+j} - \theta_k)$$

(a) Find the conditions which guarantee solutions of the form $\theta_k = \Omega t + k\beta$. Describe this form of solution or *splay state* in words.

(b) Show that stability of the splay state depends on the elements of the Jacobian:

$$\widehat{\mathcal{H}}_{kj} = \mathcal{H}_{kj} - \delta_{kj} \sum_l \mathcal{H}_{kl}, \qquad \mathcal{H}_{jk} = \epsilon W_{j-k} H'((j-k)\beta)$$

Use translation invariance to show that the eigenvectors of $\widehat{\mathcal{H}}$ are

$$\theta_k(t; p) = e^{\lambda(p)t + 2\pi i k p/N}, \quad \lambda(p) = \epsilon \sum_{k=1}^{N} W_k H' \left[e^{2\pi i p k/N} - 1\right], \qquad p = 0, \ldots, N-1$$

(c) State why the generic bifurcation of a splay state for an odd number of oscillators is a simple Hopf bifurcation.

5. Consider the continuum phase oscillator model with space-dependent delays

$$\frac{\partial \theta(x,t)}{\partial t} = \frac{1}{\Delta} + \epsilon \int_{-\infty}^{\infty} W(y) H(\theta(x+y,t) - \theta(x,t) - |y|/v)dy$$

(a) Find the frequency of solutions defined by $\theta(x,t) = \Omega t + \beta x$.

(b) Show that these solutions are stable if $\text{Re}\lambda(p) < 0$, where

$$\lambda(p) = \epsilon \int_{-\infty}^{\infty} W(y) H'(\beta y - |y|/v)[e^{ipy} - 1]dy$$

(c) For the case that $H(\phi) = \sin 2\pi\phi$ and $W(y) = W(|y|)$ show that

$$\text{Re}\lambda(p) = \pi\epsilon[\Lambda(p, 2\pi\beta_+) + \Lambda(-p, 2\pi\beta_-)]$$

where

$$\Lambda(p, \beta) = W_c(p+\beta) + W_c(p-\beta) - 2W_c(\beta), \qquad W_c(p) = \int_0^{\infty} W(y)\cos(py)dy$$

and $\beta_\pm = \pm\beta - 1/(v)$.

(d) Show that for $W(y) = e^{-|y|}$ and $v \to \infty$ the synchronous solution is stable for $\epsilon > 0$.

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