School of Mathematical Sciences

G14TNS Theoretical Neuroscience

Problem sheet 6

1. Consider a chain of coupled oscillators described by

$$\dot{\theta}_1 = \omega_1 + W_+ \sin(\theta_2 - \theta_1 + \sigma)$$
$$\dot{\theta}_i = \omega_i + W_+ \sin(\theta_{i+1} - \theta_i + \sigma) + W_- \sin(\theta_{i-1} - \theta_i + \sigma), \qquad i = 2, \dots, N-1$$
$$\dot{\theta}_N = \omega_N + W_- \sin(\theta_{N-1} - \theta_N + \sigma)$$

   (a) Determine the conditions that yield a forward travelling wave.

   (b) Show that the Jacobian of the system is

$$\begin{bmatrix} -W_+ - W & W_+ & 0 & \cdots & 0 \\ W & -W_+ - W & W_+ & & \\ 0 & W & -W_+ - W & W_+ & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & W & -W_+ - W \end{bmatrix}$$

   where $W = W_- \cos(2\sigma)$.

   (c) Show that the forward travelling wave is stable if $W_\pm > 0$ and $\sigma < \pi/4$.

2. Consider the PDE for the variable $v(x,t)$:

$$v_t = f(v) + D v_{xx}, \qquad f(v) = -\frac{v}{\tau} + \Theta(v - h)$$

where $\Theta(v)$ is a heavside step function.

   (a) Write down the corresponding ODE system for solutions of the form $v(x,t) = v(\xi)$, where $\xi = x - ct$.

   (b) Choosing an origin such that $v(0) = h$, show that a general solution to this ODE takes the form

$$v(\xi) = \begin{cases} \tau + A \exp \lambda_+(c)\xi & \xi \le 0 \\ B \exp \lambda_-(c)\xi & \xi > 0 \end{cases}$$

   where

$$\lambda_\pm(c) = \frac{-c \pm \sqrt{c^2 + 4D/\tau}}{2D}$$

   (c) Determine the constants A and B by assuming continuity of the solution and its first derivative at $\xi = 0$.

   (d) Show that the speed of the wave is specified by

$$c = \sqrt{D} \frac{\frac{1}{h} - \frac{2}{\tau}}{\sqrt{\frac{1}{h} - \frac{1}{\tau}}}, \qquad h < \tau$$

   and comment on the case when $h > \tau$.

3. Consider the FitzHugh-Nagumo system of PDEs modelling the nerve impulse propagation along an axon:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) - u$$
$$\frac{\partial u}{\partial t} = \beta v$$

where $v(x,t)$ represents the membrane potential and $u(x,t)$ is a phenomenological *recovery* variable; $f(v) = v(\alpha - v)(v-1)$, $1 > \alpha > 0$, $\beta > 0$, $-\infty < x < \infty$, $t > 0$. Travelling waves are solutions of the form

$$v(x,t) = V(\xi), \qquad u(x,t) = U(\xi), \qquad \xi = x + ct$$

for some unknown speed c.

(a) Derive a system of three ODEs for the travelling wave profiles.

(b) Check that the system has a unique equilibrium with one positive eigenvalue and two eigenvalues with negative real parts.

(c) Conclude that the equilibrium can either be a saddle or a saddle-focus with a one-dimensional unstable and a two dimensional stable invariant manifold, and show that for fixed $\beta > 0$ the boundary between these two cases is defined by

$$D = \{(\alpha, c) \ : \ c^4(4\beta - \alpha^2) + 2\alpha c^2(9\beta - 2\alpha^2) + 27\beta^2 = 0\}$$

[Hint: At the boundary the characteristic polynomial has a double root].

(d) Sketch possible profiles of travelling impulses in both regions.

4. Consider the FitzHugh-Nagumo system in the form

$$\epsilon \frac{\partial v}{\partial t} = \epsilon^2 \frac{\partial^2 v}{\partial x^2} + f(v, w)$$
$$\frac{\partial w}{\partial t} = g(v, w)$$

(a) Show that periodic waves of the form $v(x,t) = v(kx - \omega t)$ satisfy the system of ODEs

$$\epsilon^2 k^2 u_\xi = -\omega u - f(v, w)$$
$$v_\xi = u$$
$$w_\xi = -\frac{v}{\omega}$$

where $\xi = kx - \omega t$.

(b) Consider the case that

$$f(v, w) = \Theta(v - \alpha) - v - w, \qquad g(v, w) = v$$

where $\Theta(x)$ is a step function such that $\Theta(x) = 1$ if $x \geq 0$ and is zero otherwise. Discuss how one might construct the dispersion relation $\omega = \omega(k)$, and hence the speed $c = \omega/k$.

S Coombes 2008