

School of Mathematical Sciences

G14TNS Theoretical Neuroscience

Problem sheet 7

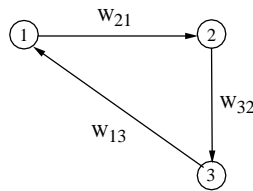
1. Consider the two neuron system

$$\dot{x}_i = -x_i + \sum_{j=1}^2 W_{ij} \tanh(\beta x_j)$$

for $i = 1, 2$. Determine how the stability of the fixed point at the origin, $x_i = 0$, depends upon β by calculating the eigenvalues of the weight matrix in each of the following examples

$$(a) \mathbf{W} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (b) \mathbf{W} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

2. Consider a ring of 3 neurons as shown below. Suppose that the state x_i of the i th neuron



satisfies the following equation

$$\tau \frac{dx_i}{dt} = F_i(x) \equiv -x_i + W_{i,i-1} f(x_{i-1}), \quad f(x) = \tanh \beta x$$

where $W_{10} \equiv W_{13}$ and $x_0 \equiv x_3$.

(a) Let $A_{ij} = \partial F_i / \partial x_j(0)$ be the Jacobian matrix at the origin. Show that the eigenvalues of \mathbf{A} satisfy the characteristic equation

$$(1 + \lambda)^3 = \Gamma, \quad \Gamma = W_{13} W_{32} W_{21} \beta^3$$

(b) Show that if the number of negative (inhibitory) weights is even then the eigenvalues can be written as

$$\lambda_k = -1 + \Gamma^{1/3} e^{i2k\pi/3}, \quad k = 1, 2, 3$$

Hence, show that the origin is stable if $\Gamma < 1$.

(c) Show that if the number of negative (inhibitory) weights is odd then the eigenvalues can be written as

$$\lambda_k = -1 + |\Gamma|^{1/3} e^{i(2k+1)\pi/3}, \quad k = 1, 2, 3$$

Hence, show that the origin is stable provided that $|\Gamma| < 1/\cos(\pi/3)$. Establish that there exists a pair of pure imaginary eigenvalues when $|\Gamma|^{1/3} = 1/\cos(\pi/3)$ (signalling the onset of a supercritical Hopf bifurcation) and show that close to the bifurcation point the frequency of limit cycle oscillations is approximately $\omega \approx \tau^{-1} \tan(\pi/3)$.

3. Consider a one-dimensional network of neurons with the neuron at position x at time t having a somatic potential $u(x, t)$. Let the output firing rate of a neuron be $f(u) = \tanh \beta u$. Let $w(x - y)$ be the connection from neuron y to x . The network then evolves according to the integro-differential equation

$$\frac{1}{\alpha} \frac{\partial u(x, t)}{\partial t} = -u(x, t) + g \int_{-\infty}^{\infty} w(x - y) f(u(y, t)) dy$$

- (a) Obtain the linearised equations of motion about the homogeneous solution and show that it admits solutions of the form $e^{\lambda t + ipx}$, where

$$\lambda/\alpha + 1 = g\beta\tilde{w}(p), \quad \tilde{w}(p) = \int_{-\infty}^{\infty} e^{-ipx} w(x) dx$$

- (b) Let

$$w(x) = e^{-a|x|} - Ae^{-b|x|}, \quad 0 < A < 1, \quad 0 < b < a$$

Sketch $w(x)$. Show that its Fourier transform is given by

$$\tilde{w}(p) = 2 \left[\frac{a}{p^2 + a^2} - A \frac{b}{p^2 + b^2} \right]$$

- (c) Show that $\tilde{w}(p)$ has two maxima and one minima when $A > (b/a)^3$ and sketch the resulting function. Indicate on the figure how one can determine the critical bifurcation parameter g_c for a Turing instability

4. Consider the network of Q 3. with $w(x)$ now given by a difference of two Gaussians:

$$w(x) = w_+(x) - w_-(x), \quad w_{\pm}(x) = A_{\pm} \frac{1}{\sqrt{2\pi\sigma_{\pm}^2}} \exp\left(-\frac{x^2}{2\sigma_{\pm}^2}\right)$$

- (a) Show that $w(x)$ has the form of a mexican hat function provided that

$$\frac{A_+}{\sigma_+} > \frac{A_-}{\sigma_-}, \quad \sigma_- > \sigma_+$$

- (b) Determine the Fourier transform of $w(x)$ and show that the system can undergo a Turing instability along similar lines to Q 3.
(c) Repeat your analysis for the mexican hat shape

$$w(x) = (1 - |x|)e^{-|x|}$$

5. Time independent solutions of the network in Q 3. satisfy

$$u(x) = g \int_{-\infty}^{\infty} w(x - y) f(u(y)) dy$$

Consider the case that $f(u) = \Theta(u - h)$ (where Θ is the heaviside step function).

- (a) Show that *bump* solutions of the form $u(x) > h$ for $x_1 < x < x_2$ and $u(x) < h$ otherwise are given by

$$u(x) = g \int_{x_1}^{x_2} w(x - y) dy$$

(b) Show that when $w(x)$ has the mexican hat shape $w(x) = (1 - |x|)e^{-|x|}$ then

$$u(x) = g \begin{cases} q(x - x_1) - q(x - x_2) & x > x_2 \\ q(x_2 - x) + q(x - x_1) & x_1 \leq x \leq x_2 \\ q(x_2 - x) - q(x_1 - x) & x < x_1 \end{cases}$$

where $q(x) = xe^{-x}$. Sketch this solution.

(c) Show that the width of the bump $\Delta = (x_2 - x_1)$ is determined by

$$\Delta e^{-\Delta} = h/g$$

and hence that there are no bump solutions if $h/g > e^{-1}$.