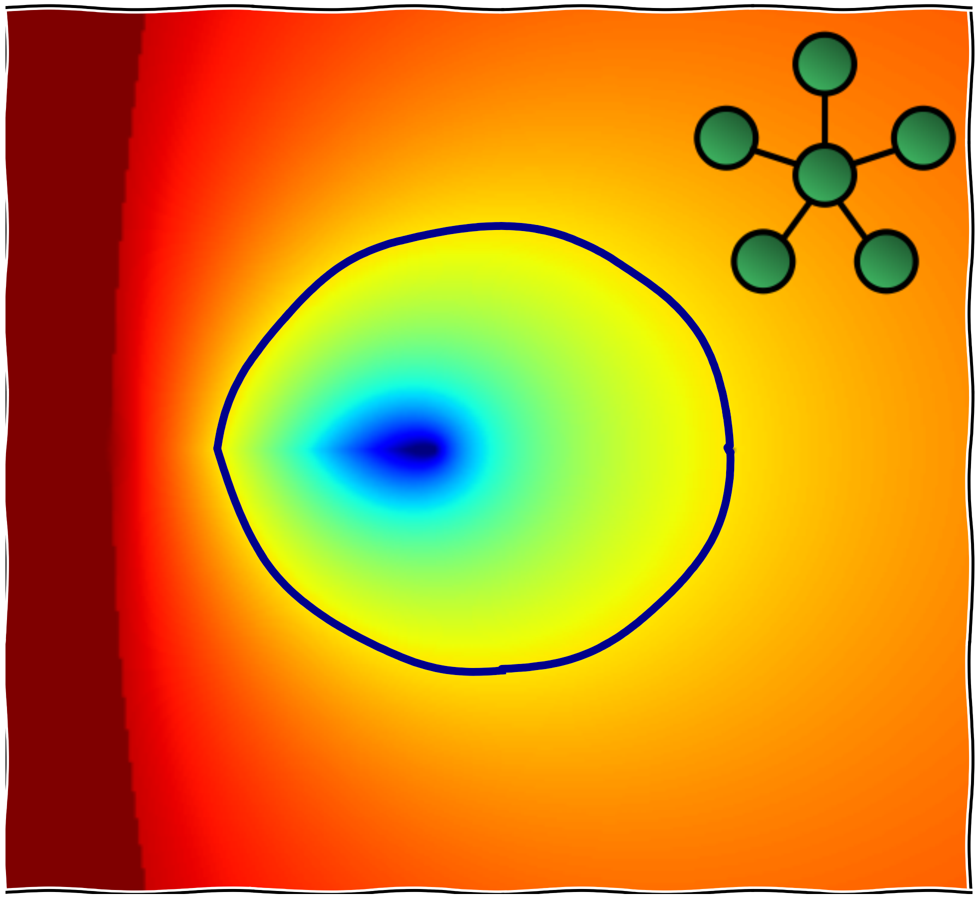
Networks and Synchrony: The Master Stability Function

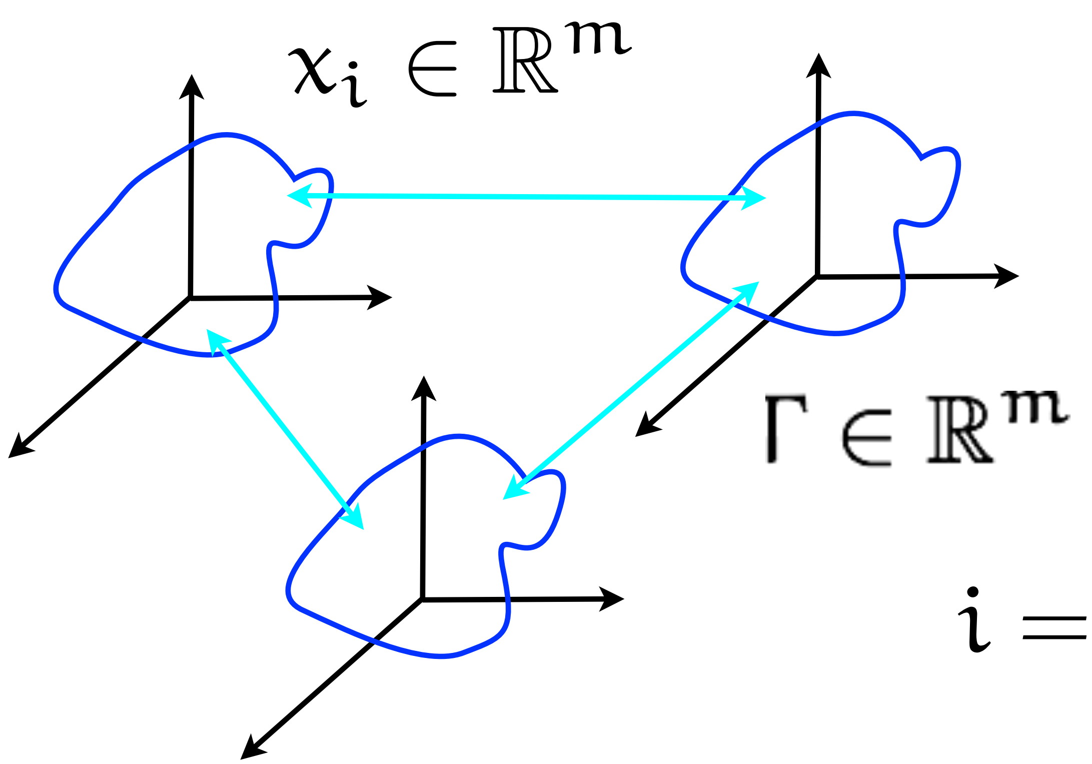


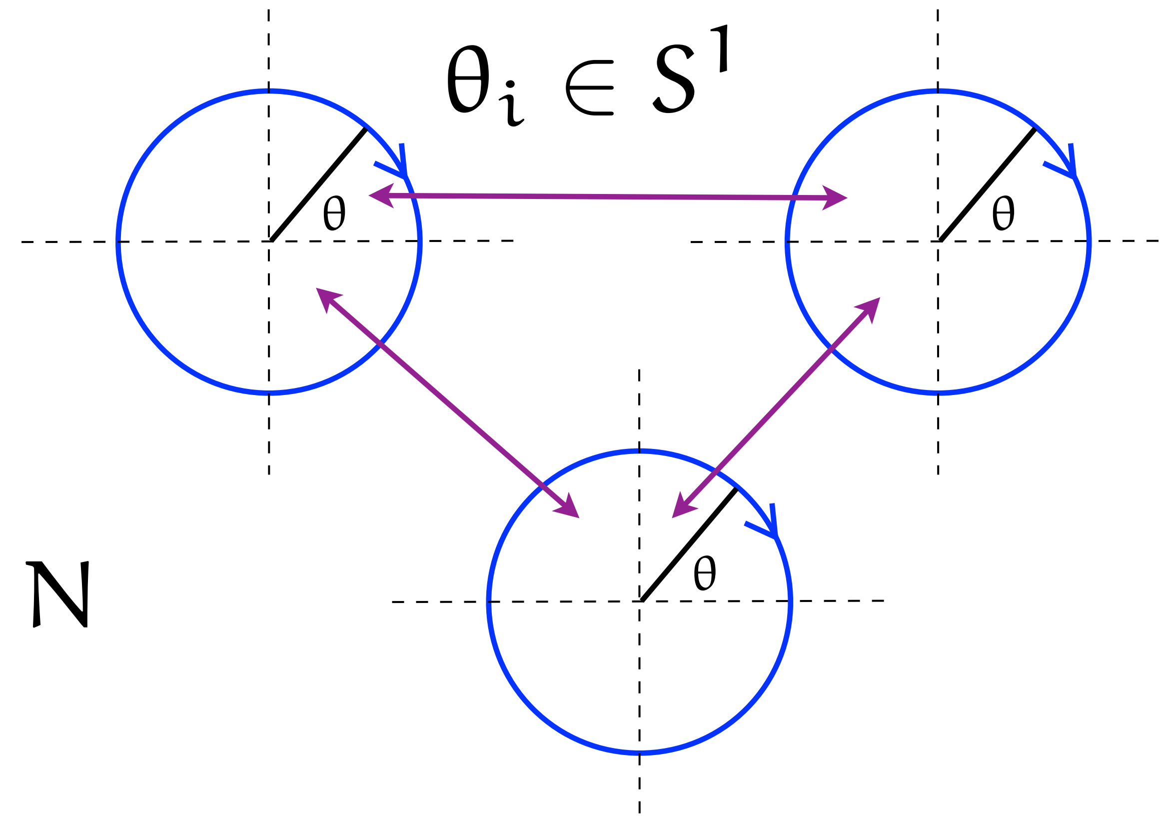
Steve Coombes

CMMB Centre for Mathematical Medicine and Biology

The University of Nottingham

Beyond weak coupling





$$i = 1, \ldots, N$$

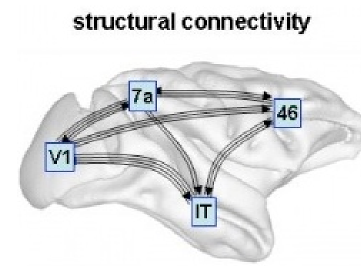
$$\dot{x}_i = f(x_i) + \epsilon \sum_{j=1}^{N} w_{ij} G(x_j)$$

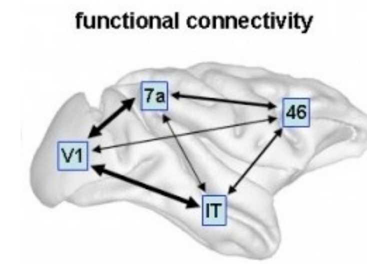
Ask Jeff!

$$\dot{\theta}_i = \omega + \epsilon \sum_{j=1}^{N} w_{ij} H(\theta_j - \theta_i)\rangle$$

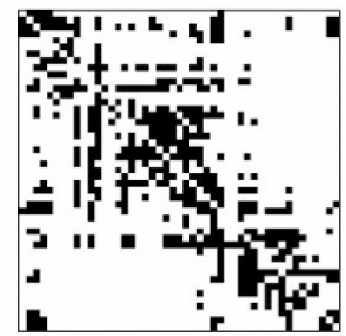
Phase oscillator networks in neuroscience

Biorobotics lab at EPFL
http://biorob.epfl.ch/

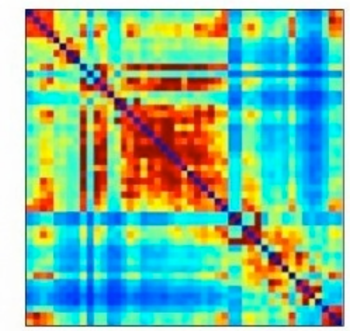




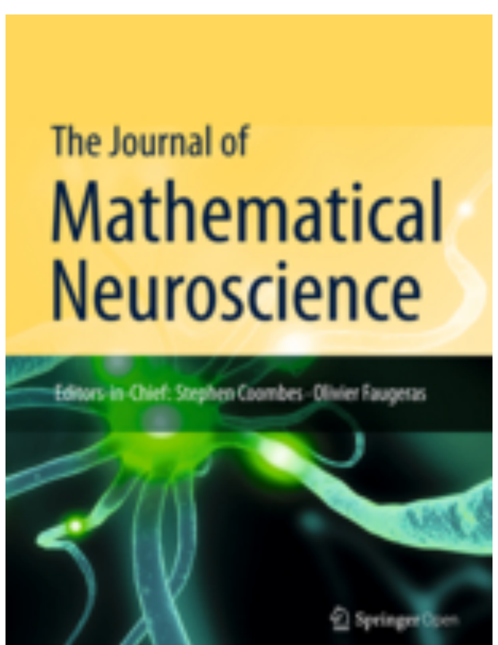
$\dot{\theta}_i = \omega_i$ **+**



$\stackrel{?}{=}$

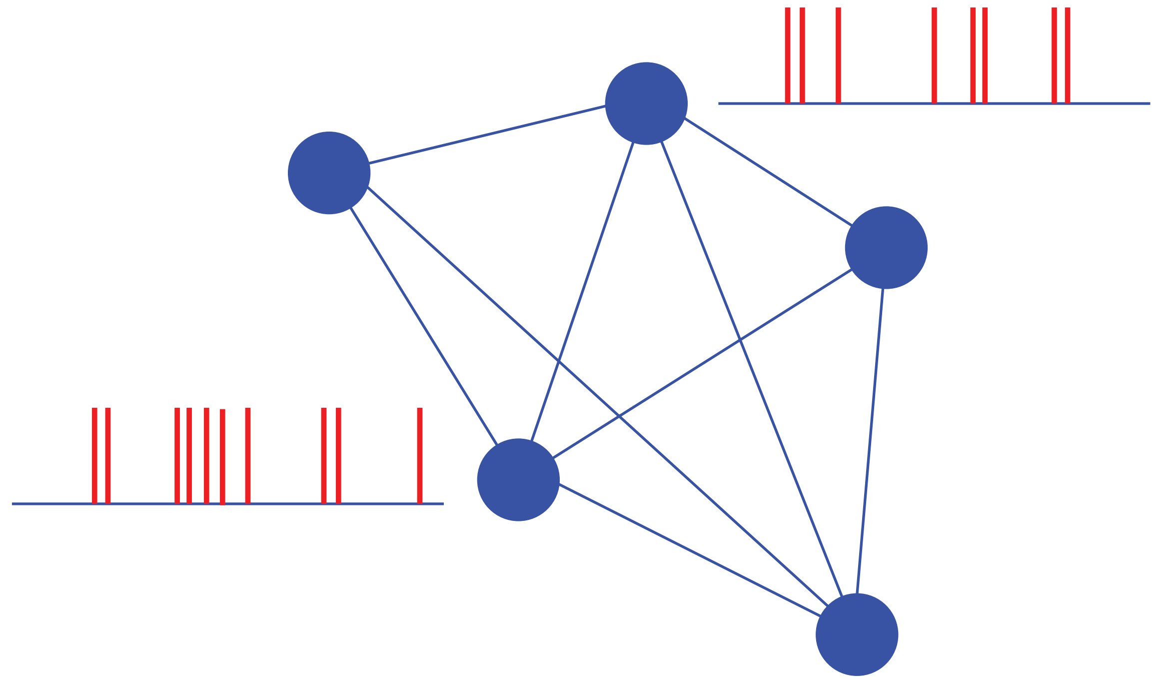


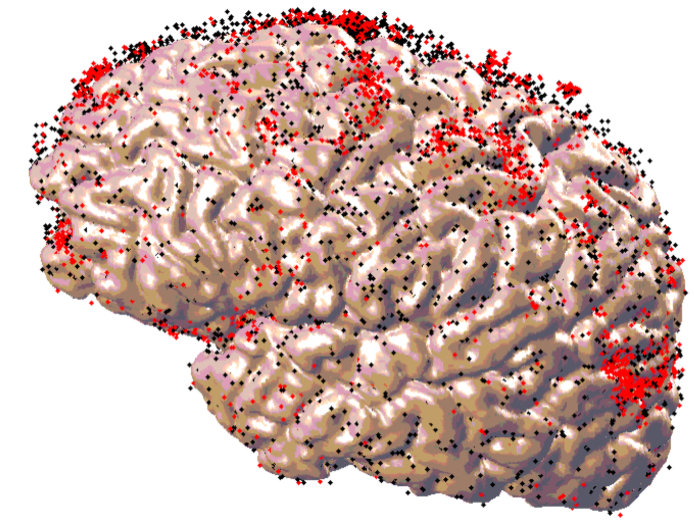
$$\widehat{\mathcal{H}}_{ij}(\Phi) = \epsilon \left[w_{ij} H'(\phi_j - \phi_i) - \delta_{ij} \sum_k w_{ik} H'(\phi_k - \phi_i) \right]$$

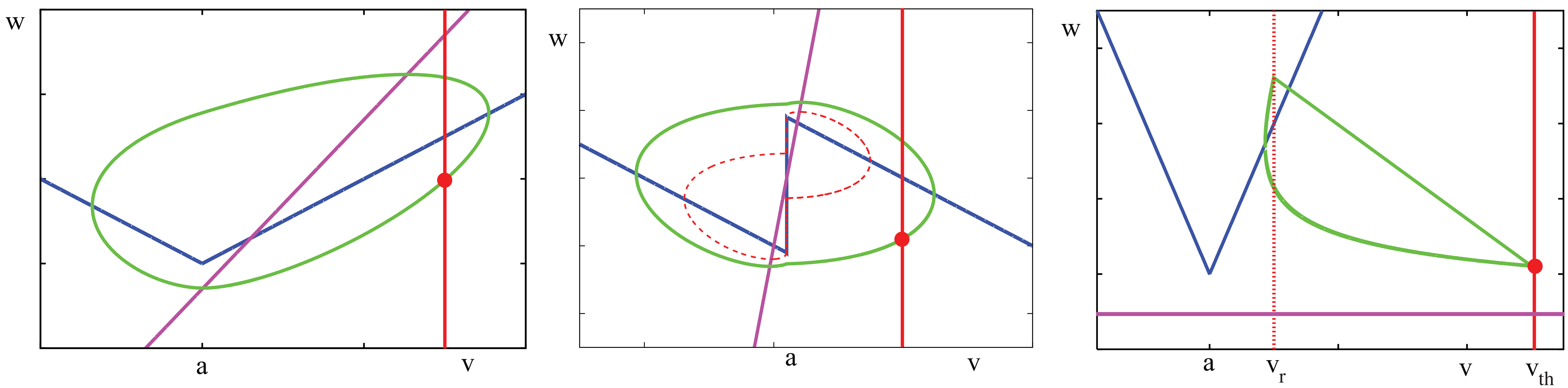


Review
Mathematical Frameworks for Oscillatory Network Dynamics in Neuroscience Ashwin P, Coombes S and Nicks R
The Journal of Mathematical Neuroscience 2016, **6**:2 (6 January 2016)

… strong coupling, event driven interactions, …

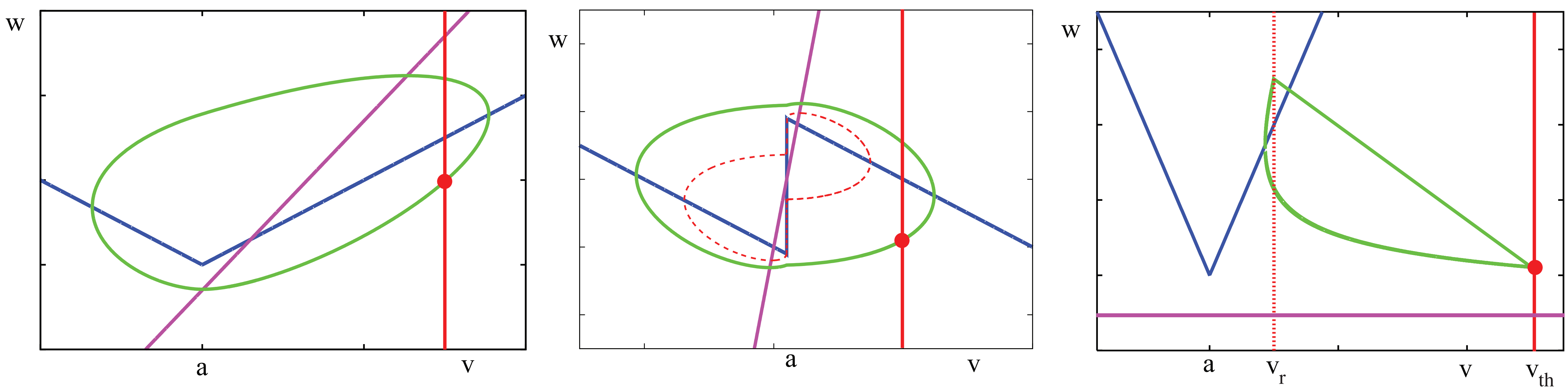






Challenge of studying networks of non smooth and discontinuous ***threshold*** elements.

The approach for Wilson-Cowan applies to other PWL models and networks and no need for weak coupling



S Coombes, M Sayli, R Thul, R Nicks, M A Porter and Y M Lai 2024
Oscillatory networks: Insights from piecewise-linear modelling, SIAM Review, to appear

$$\dot{z} = F(z)$$

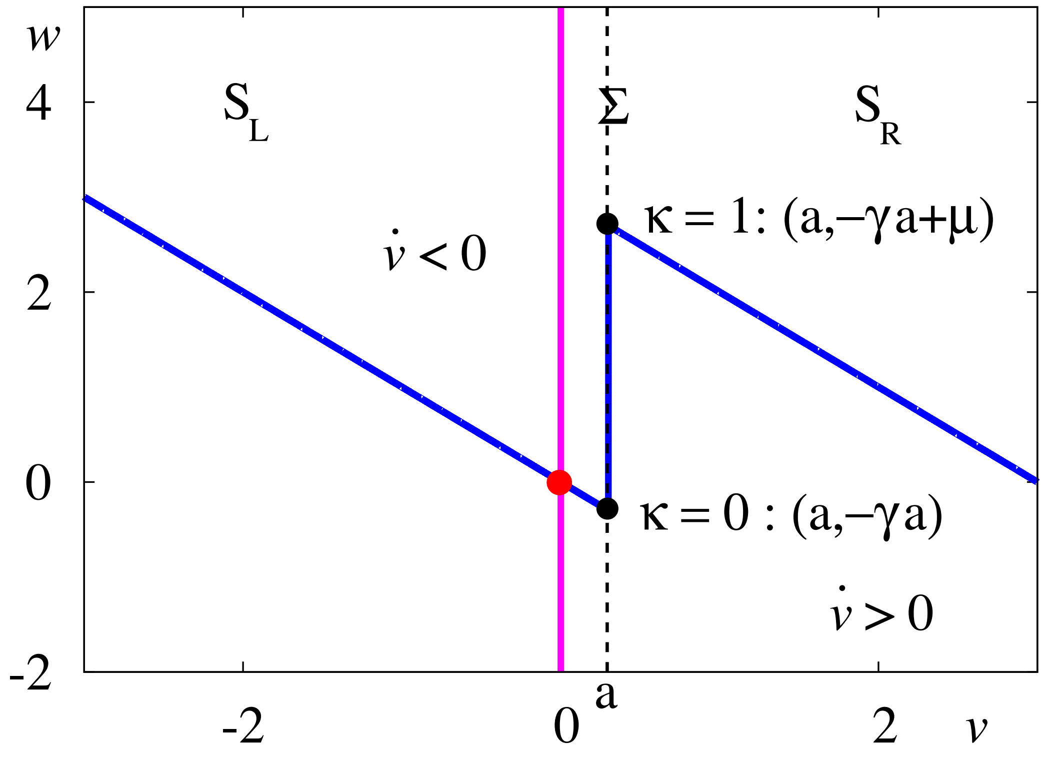
$$F(z) = \begin{cases} F_L \equiv A_L z + c_L & v < a \\ F_R \equiv A_R z + c_R & v > a \end{cases}$$

[for all models]

Matrix exponential solutions

$$z(A, c; t, t_0) = G(A; t - t_0) z(t_0) + K(A; t - t_0) c,$$

$$G(A; t) = e^{At}, \qquad K(A; t) = \int_0^t G(A; s) ds = A^{-1} [G(A; t) - I_2]$$



$$a = v(\Delta_R)$$

$$a = v(\Delta)$$

$$w(\Delta) = w(0)$$

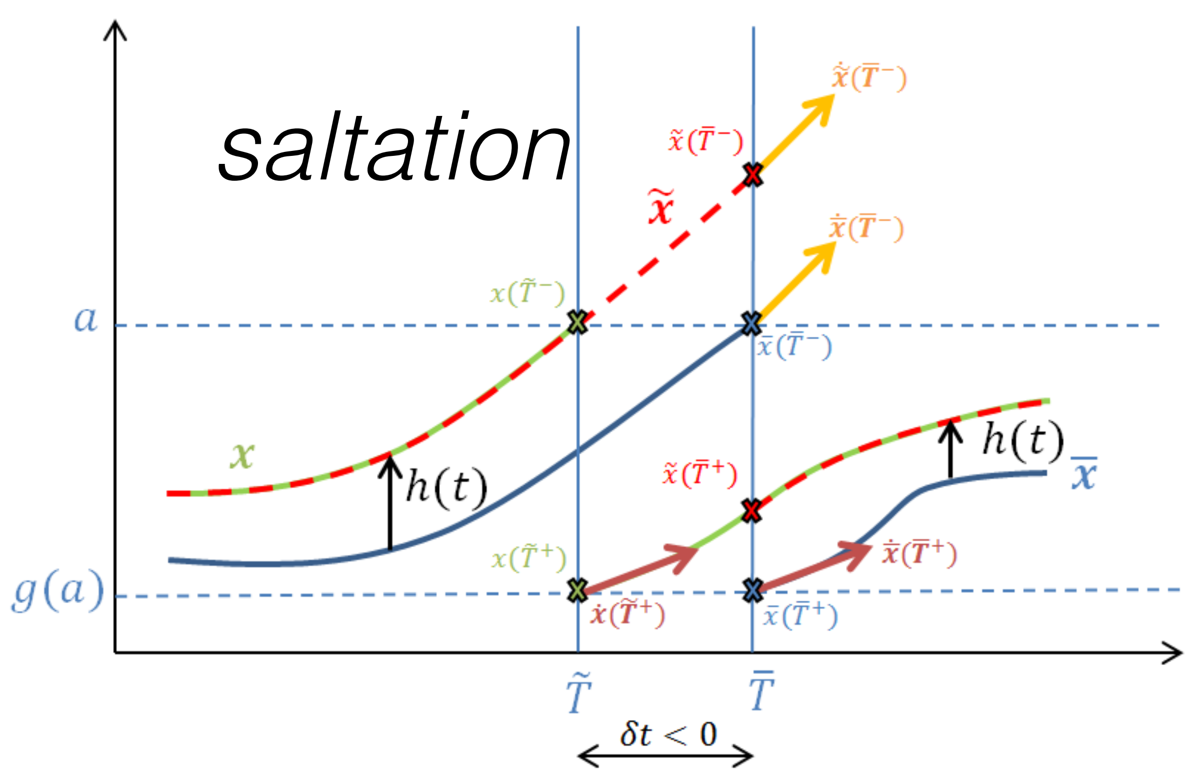
Glueing!

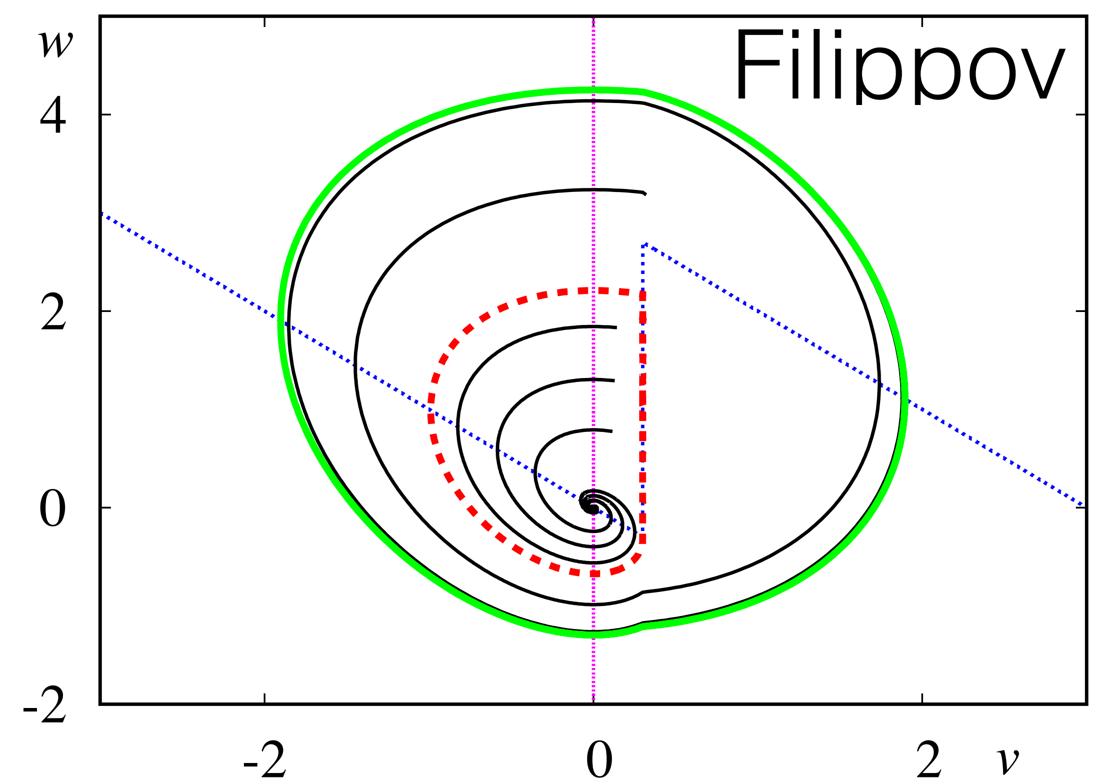
Floquet exponent

Need to be careful when propagating perturbations through switching manifold

$$\sigma_{\text{smooth}} = \frac{1}{\Delta}\int_0^{\Delta} \mathrm{Tr}\ DF(\overline{z}(s))ds$$

$$\sigma = \frac{1}{\Delta}\sum_{\mu\in L,R}\left[\Delta_\mu \,\mathrm{Tr}\, A_\mu + \log\frac{\dot{v}(T_\mu^+)}{\dot{v}(T_\mu^-)}\right]$$





Network synchrony: MSF

L M Pecora and T L Carroll. Master stability functions for synchronized coupled systems. Physical Review Letters, 80:2109–2112, 1998.

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} w_{ij} \left[\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)\right]$$

$$\equiv \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} \mathcal{G}_{ij} \mathbf{H}(\mathbf{x}_j)$$

$$\mathbf{x}_i, \mathbf{F}, \mathbf{H} \in \mathbb{R}^m$$

$$i = 1, \ldots, N$$

Graph Laplacian

$$\mathcal{G}_{ij} = -w_{ij} + \delta_{ij} \sum_k w_{ik}$$

Synchronisation manifold

$$\mathbf{x}_1(t) = \mathbf{x}_2(t) = \ldots = \mathbf{x}_N(t) = \mathbf{s}(t) \qquad \dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})$$

Variational problem $\mathbf{x}_i(t) = \mathbf{s}(t) + \delta\mathbf{x}_i(t)$

$$\frac{d}{dt}\delta\mathbf{x}_i = \mathbf{DF}(\mathbf{s})\delta\mathbf{x}_i - \sigma\mathbf{DH}(\mathbf{s})\sum_{j=1}^{N}\mathcal{G}_{ij}\delta\mathbf{x}_j$$

Nice notation $\mathbf{U} = (\delta\mathbf{x}_1, \ldots, \delta\mathbf{x}_N) \in \mathbb{R}^{N\times m}$

$$\dot{\mathbf{U}} = (I_N \otimes \mathbf{DF}(\mathbf{s}))\,\mathbf{U} - \sigma\,(\mathcal{G} \otimes \mathbf{DH}(\mathbf{s}))\,\mathbf{U}$$

Block diagonalise using

$$\mathcal{G}P = P\Lambda$$

$$\mathbf{V} = (P \otimes I_m)^{-1}\mathbf{U} \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$$

$$\dot{\mathbf{V}} = (I_N \otimes \mathbf{DF}(\mathbf{s}))\,\mathbf{V} - \sigma\,(\Lambda \otimes \mathbf{DH}(\mathbf{s}))\,\mathbf{V}$$

$$A \otimes B = \begin{bmatrix} A_{11}B & \dots & A_{1n_2}B \\ \vdots & \ddots & \vdots \\ A_{n_1 1}B & \dots & A_{n_1 n_2}B \end{bmatrix}$$

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

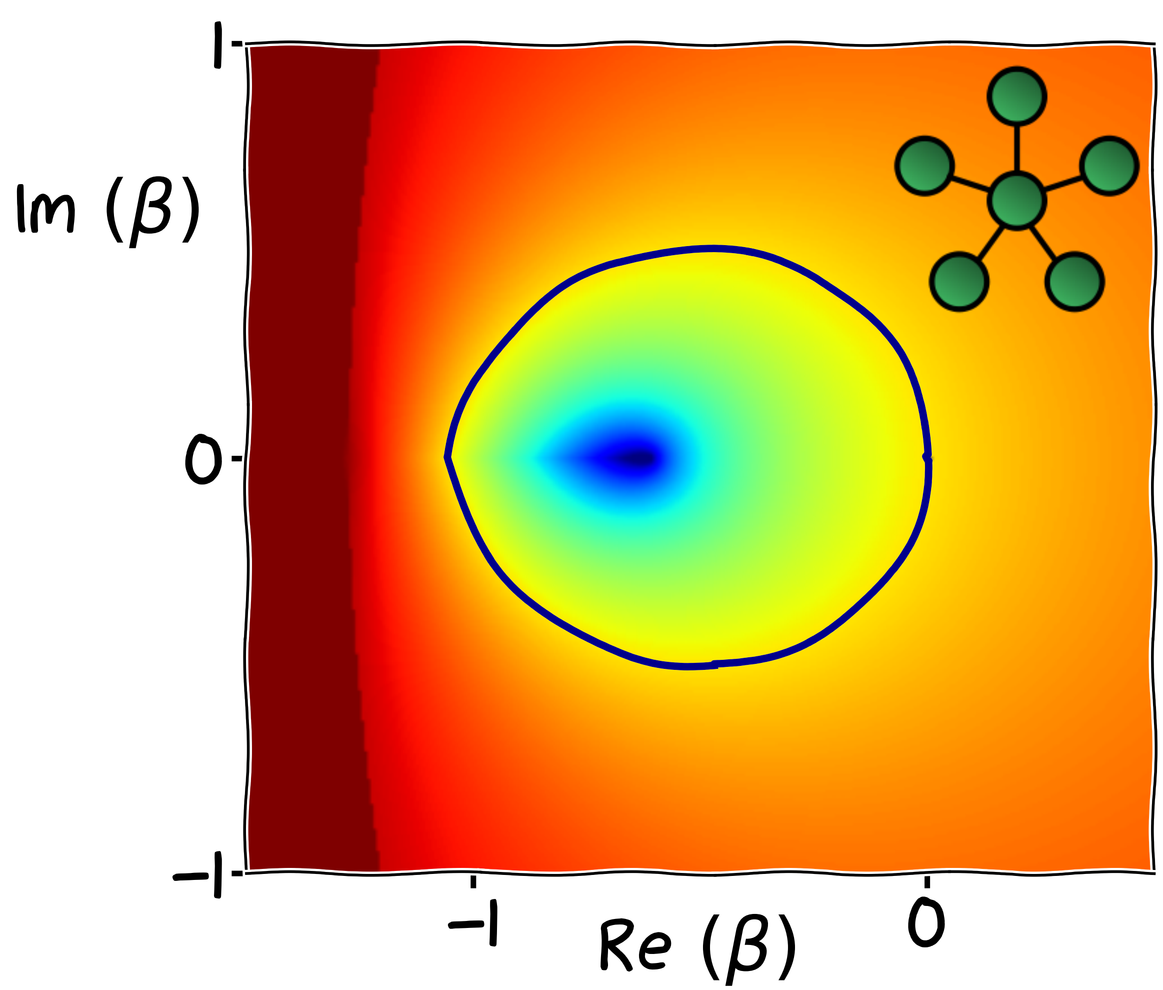
$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

N-block structure with the dynamics in each block, indexed by $l = 1, \ldots, N$:

$$\xi_l \in \mathbb{C}^m$$

$$\dot{\xi}_l = [D\mathbf{F}(\mathbf{s}) - \beta_l D\mathbf{H}(\mathbf{s})]\, \xi_l \qquad \beta_l = \sigma\lambda_l \in \mathbb{C}$$

The **MSF** is defined as the function which maps the complex number β to the greatest Floquet exponent of the variational equation. The synchronous state of the system of coupled oscillators is stable if the MSF is negative at $\beta = \sigma\lambda_l$ where λ_l ranges over the eigenvalues of the matrix $\mathcal{G}$ (excluding $\lambda_1 = 0$).



Saltation also acts blockwise

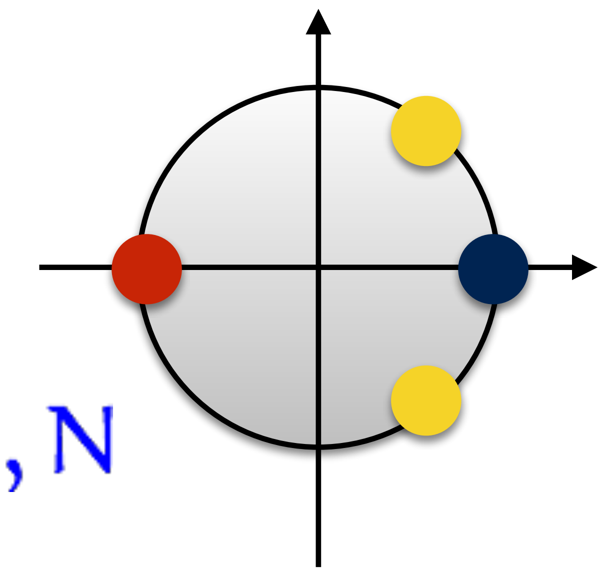
What does it all look like? Planar example:

$$\beta_l = \sigma\lambda_l \in \mathbb{C}$$

$$\dot{\xi} = [\mathbf{DF}(\mathbf{s}) - \beta_l \mathbf{DH}(\mathbf{s})]\,\xi, \qquad \xi \in \mathbb{R}^2$$

eigenvalues of $\mathcal{G}$

… modified Floquet problem.



Non smooth result: $\xi(\Delta) = \Gamma(l)\xi(0)$

$$\Gamma(l) = K_L G_L(l) K_R G_R(l) \in \mathbb{R}^{2\times 2}, \qquad l = 1, \ldots, N$$

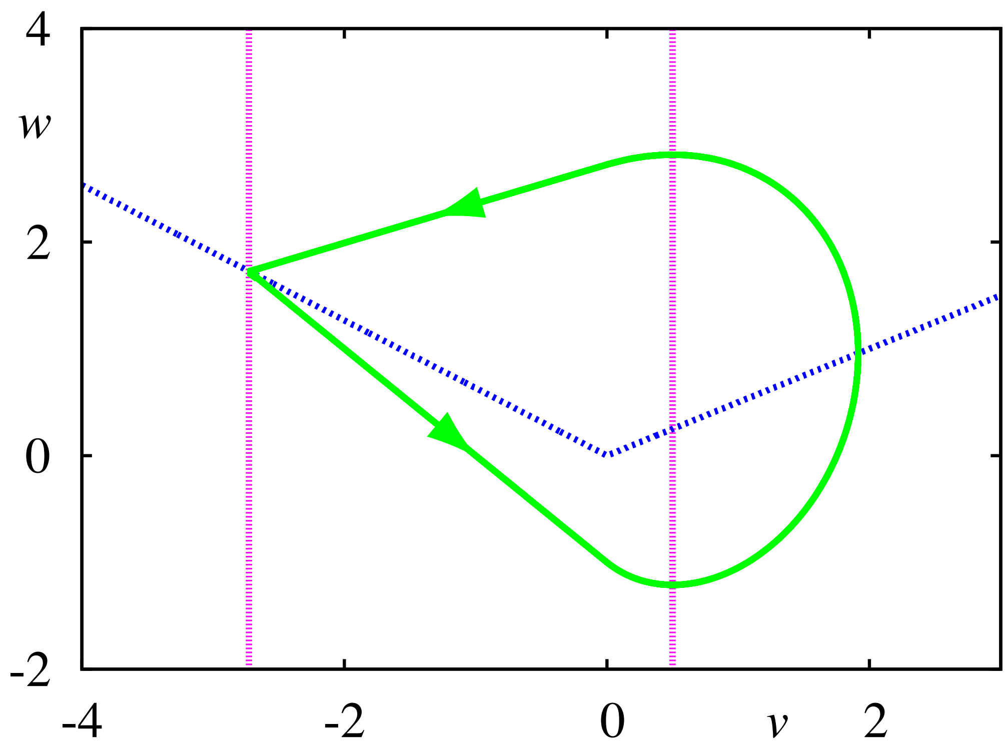
$$G_\mu(l) = G(A_\mu - \beta_l J; \Delta_\mu), \qquad K_\mu = K(T_\mu), \qquad \mu \in \{L, R\}$$

$$K(t) = \begin{bmatrix} \dot{v}(t^+)/\dot{v}(t^-) & 0 \\ (\dot{w}(t^+) - \dot{w}(t^-))/\dot{v}(t^-) & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

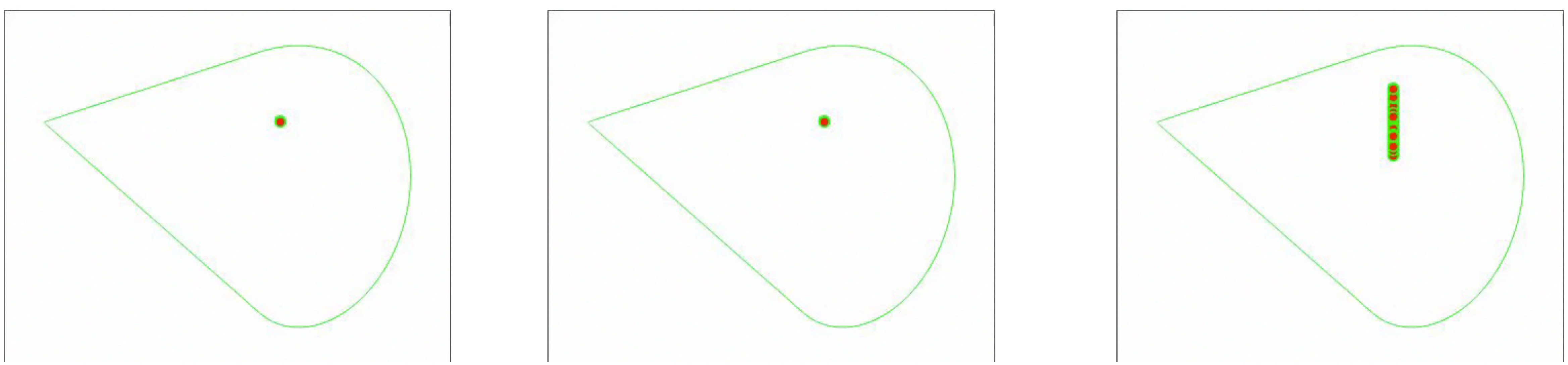
Saltation matrix

[coupling on one variable]

Network of *homoclinic* oscillators



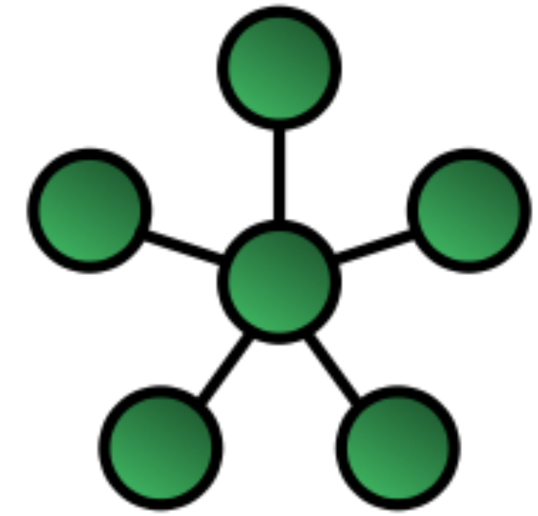
Global linear coupling on “v”



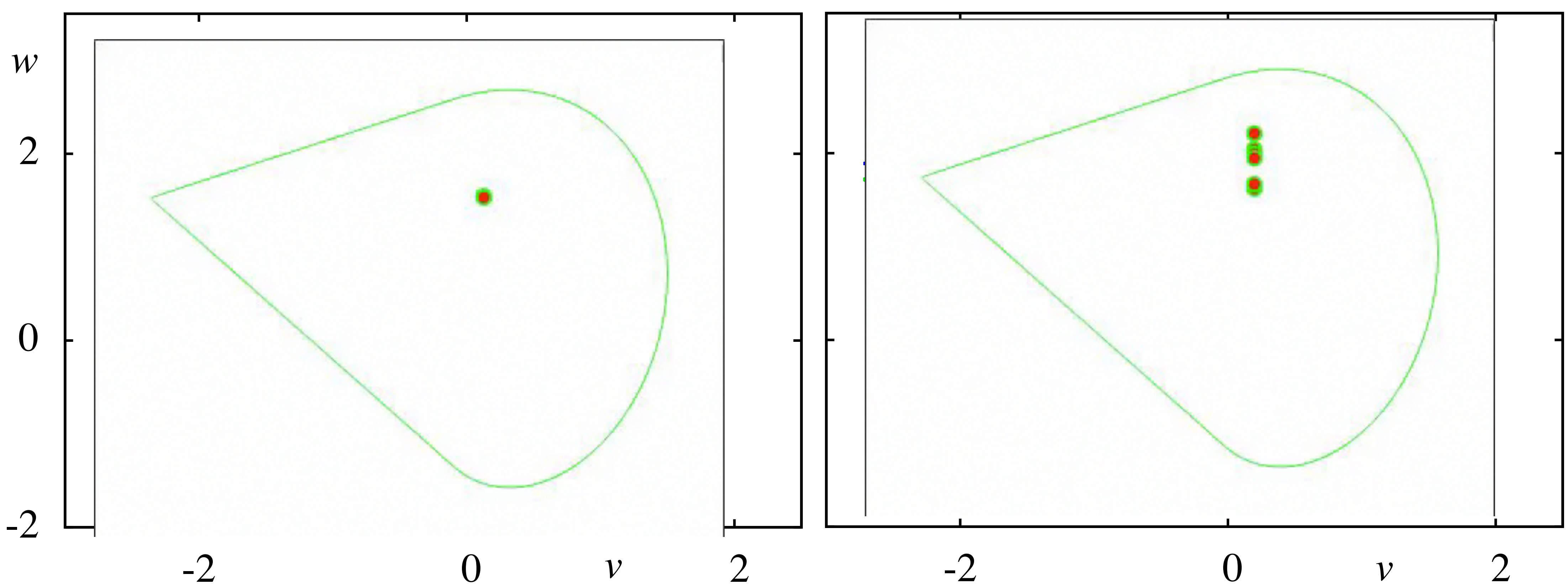
Synchrony unstable for weak coupling and restabilises via an inverse period doubling bifurcation at $\epsilon = \epsilon_{\mathrm{pd}}$ in excellent agreement with simulations (independent of $\mathbf{N}$).

Star Network

$$
w = \begin{bmatrix}
0 & 1/(N-1) & 1/(N-1) & \cdots & 1/(N-1) \\
1 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & & \vdots \\
1 & 0 & 0 & \cdots & 0
\end{bmatrix}
$$



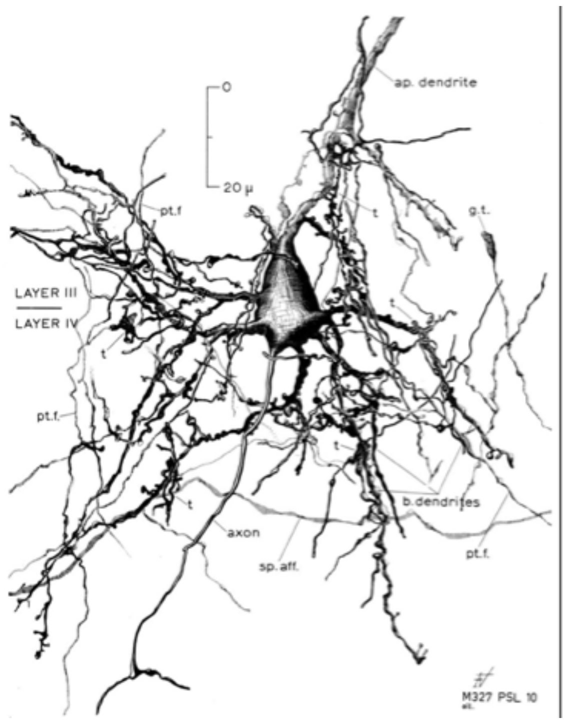
Synchrony is *always* unstable

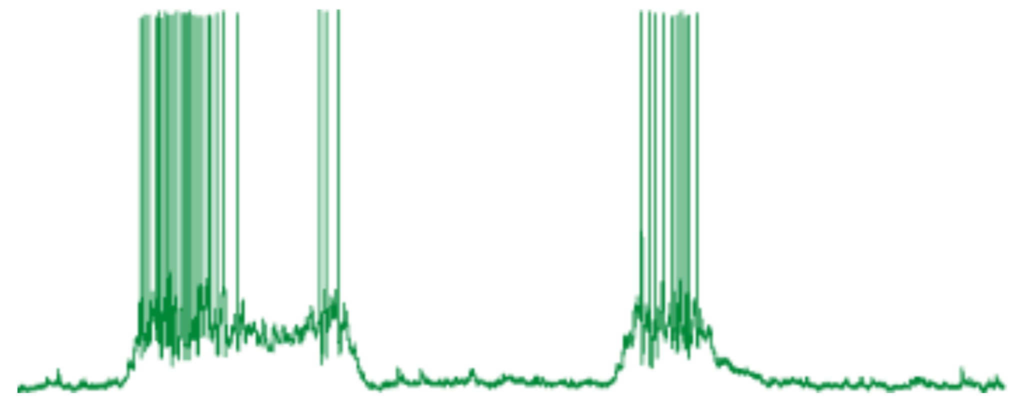


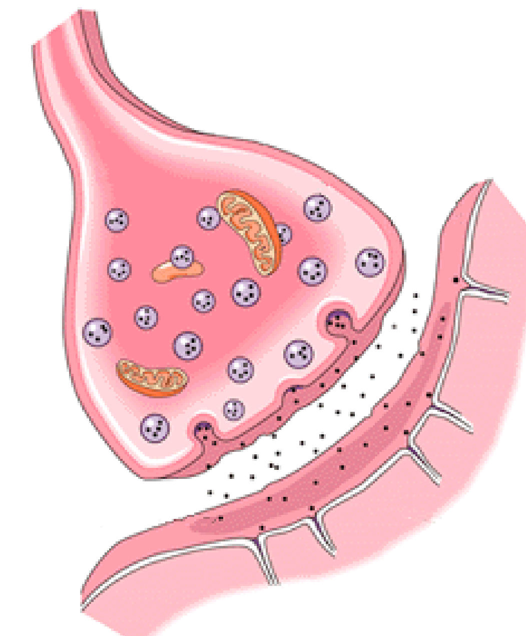
S Coombes and R Thul 2016 Synchrony in networks of coupled nonsmooth dynamical systems, European Journal of Applied Mathematics, Vol 27(6), 904–9

[remote synchronisation]

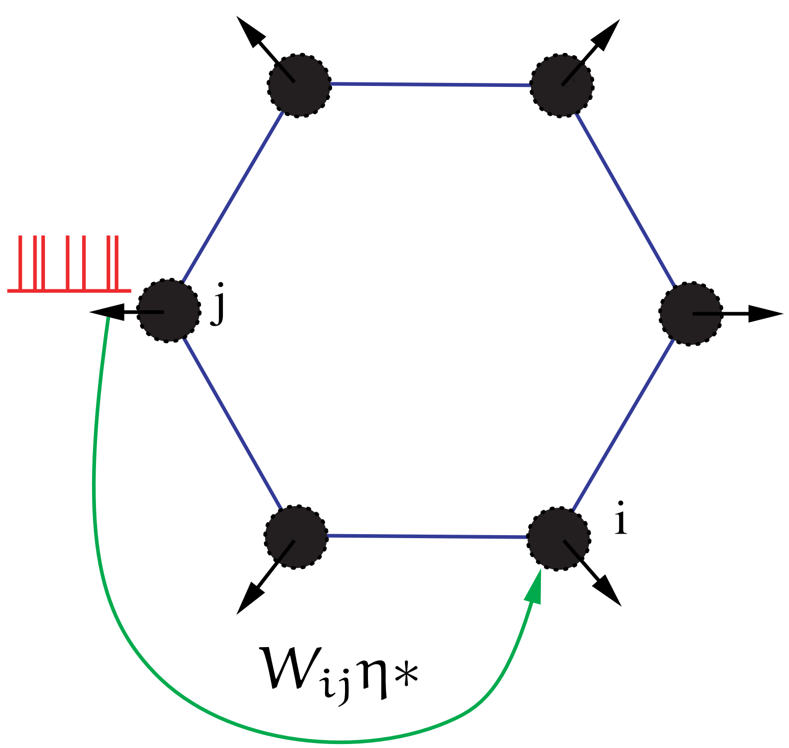
… and now for event driven synaptic coupling

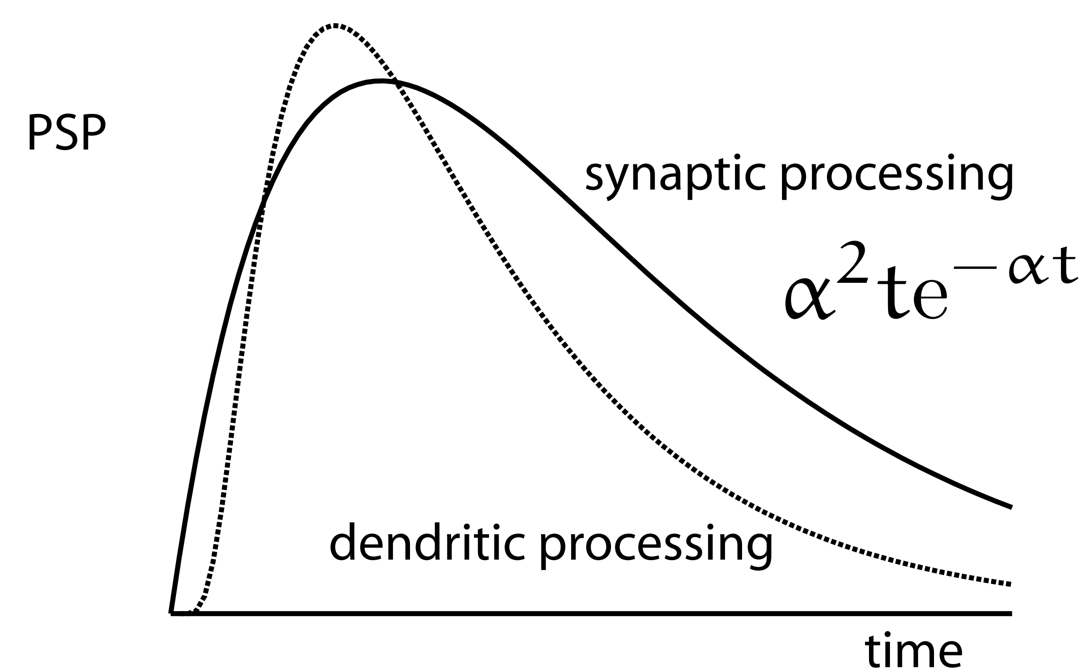






$$I_i(t) = \sigma \sum_{j=1}^{N} W_{ij} \sum_{m \in \mathbb{Z}} \eta(t - T_j^m)$$

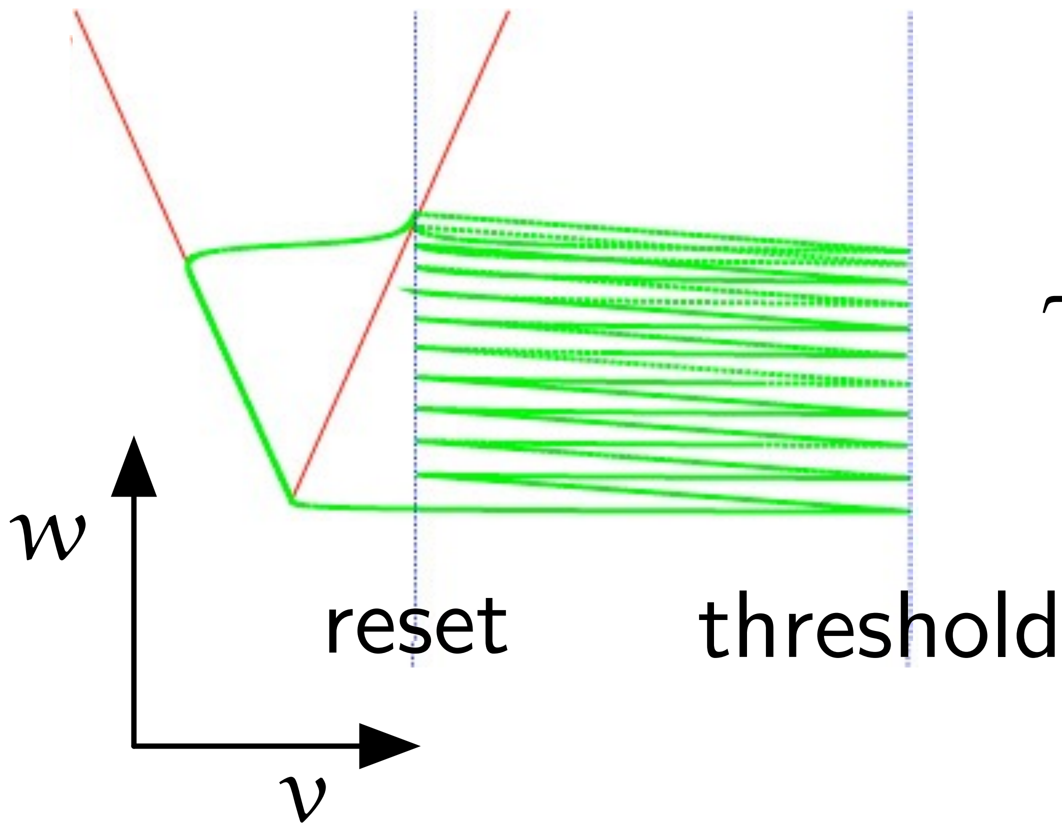


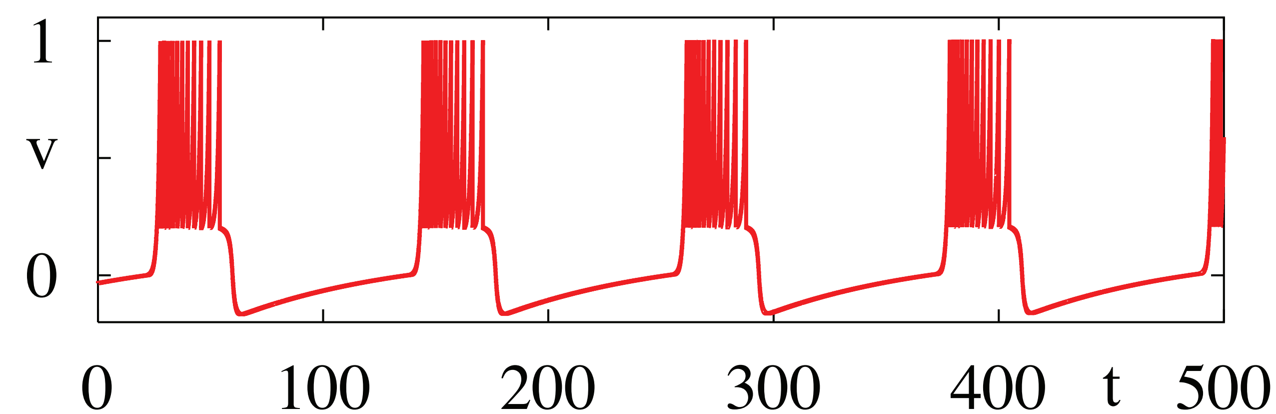


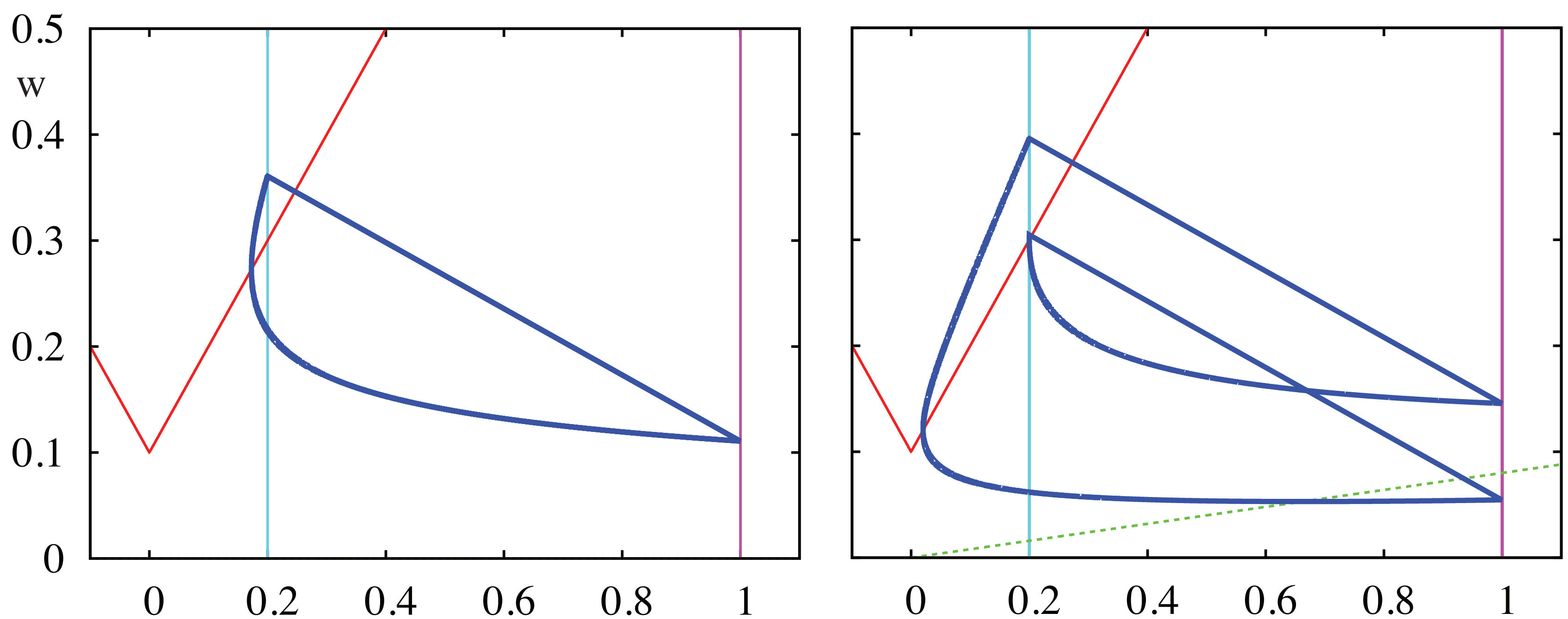
PWL IF

$$\dot{v} = |v| + I - w \qquad v^+ = v_r$$

$$\tau\dot{w} = -w \qquad w^+ = w^- + \kappa/\tau$$







S Coombes, R Thul and K C A Wedgwood 2012 Nonsmooth dynamics in spiking neuron models, Physica D, Vol 241, 2042–2057

Synaptically coupled network

$$\dot{\mathbf{z}}_i = \mathbf{F}(\mathbf{z}_i) + \sigma \sum_{j=1}^{N} W_{ij} \mathbf{H}(\mathbf{z}_j)$$

$$\mathbf{z}_i = (v_i, w_i, s_i, u_i) \qquad \mathbf{H}(\mathbf{z}) = (s, 0, 0, 0)$$

$$\left(1 + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\right) s_i = u_i$$

$$\left(1 + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\right) u_i = \sum_{m\in\mathbb{Z}} \delta(t - T_j^m).$$

$$s_i(t) = \sum_{m\in\mathbb{Z}} \eta(t - T_i^m)$$

$$A_R = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1/\tau & 0 & 0 \\ 0 & 0 & -\alpha & \alpha \\ 0 & 0 & 0 & -\alpha \end{bmatrix}$$

$$\mathbf{z}_i \to \mathbf{g}(\mathbf{z}_i) = (v_r, w_i + \kappa/\tau, s_i, u_i + \alpha)$$

$$h(\mathbf{z}_i; v_{th}) = v_i - v_{th} = 0$$

A pwl system with saltation matrices that describe firing

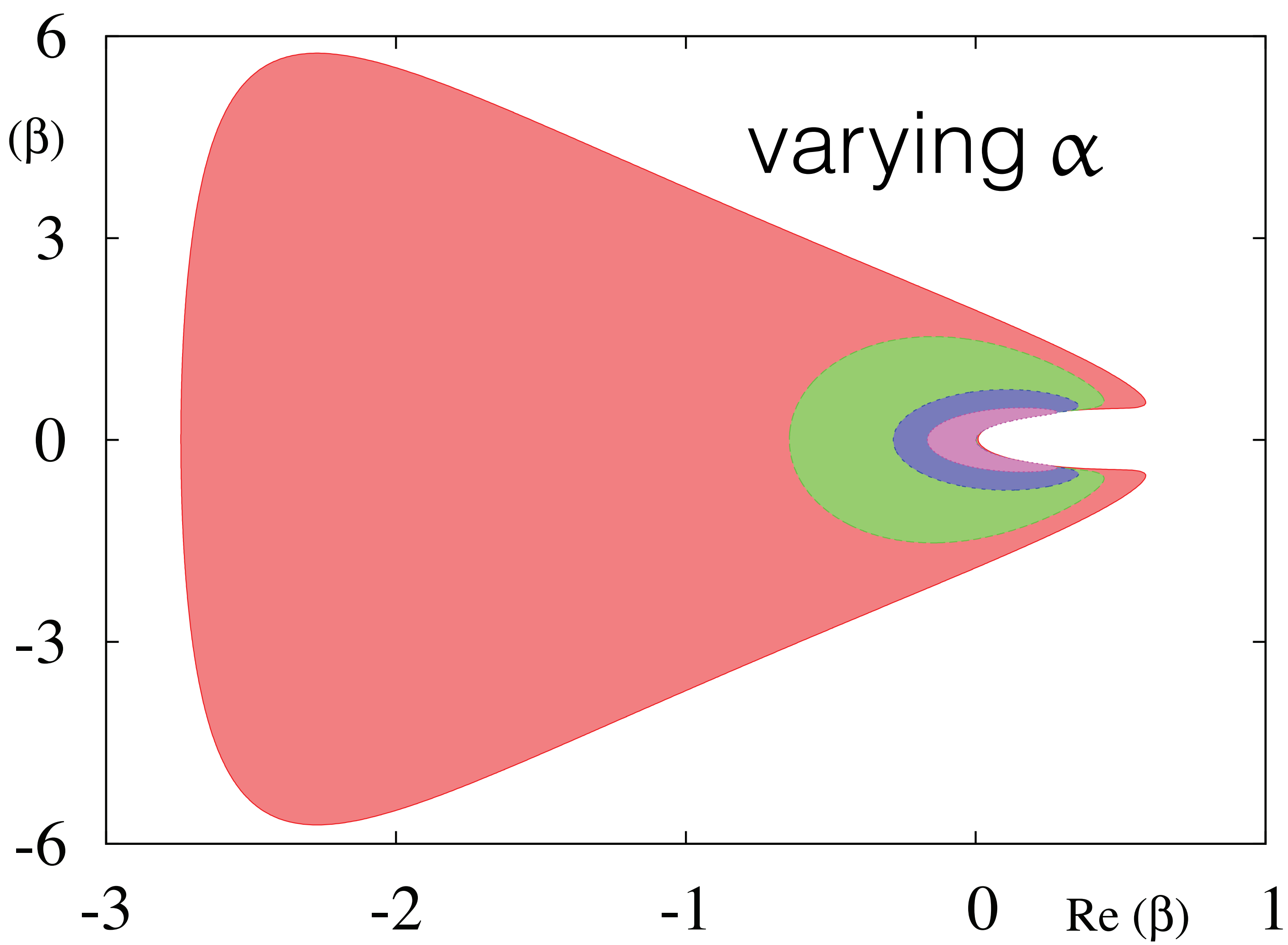
$$K(T) = Dg(\mathbf{z}(T^-)) + \frac{[\dot{\mathbf{z}}(T^+) - Dg(\mathbf{z}(T^-))\dot{\mathbf{z}}(T^-)]\,[\nabla_{\mathbf{z}} h(\mathbf{z}(T^-))]^{\mathsf{T}}}{\nabla_{\mathbf{z}} h(\mathbf{z}(T^-)) \cdot \dot{\mathbf{z}}(T^-)}$$

$$= \begin{bmatrix} \dot{v}(T^+)/\dot{v}(T^-) & 0 & 0 & 0 \\ (\dot{w}(T^+) - \dot{w}(T^-))/\dot{v}(T^-) & 1 & 0 & 0 \\ (\dot{s}(T^+) - \dot{s}(T^-))/\dot{v}(T^-) & 0 & 1 & 0 \\ (\dot{u}(T^+) - \dot{u}(T^-))/\dot{v}(T^-) & 0 & 0 & 1 \end{bmatrix}$$

Balance ensures synchrony $\sum_{j=1}^{N} W_{ij} = 0$

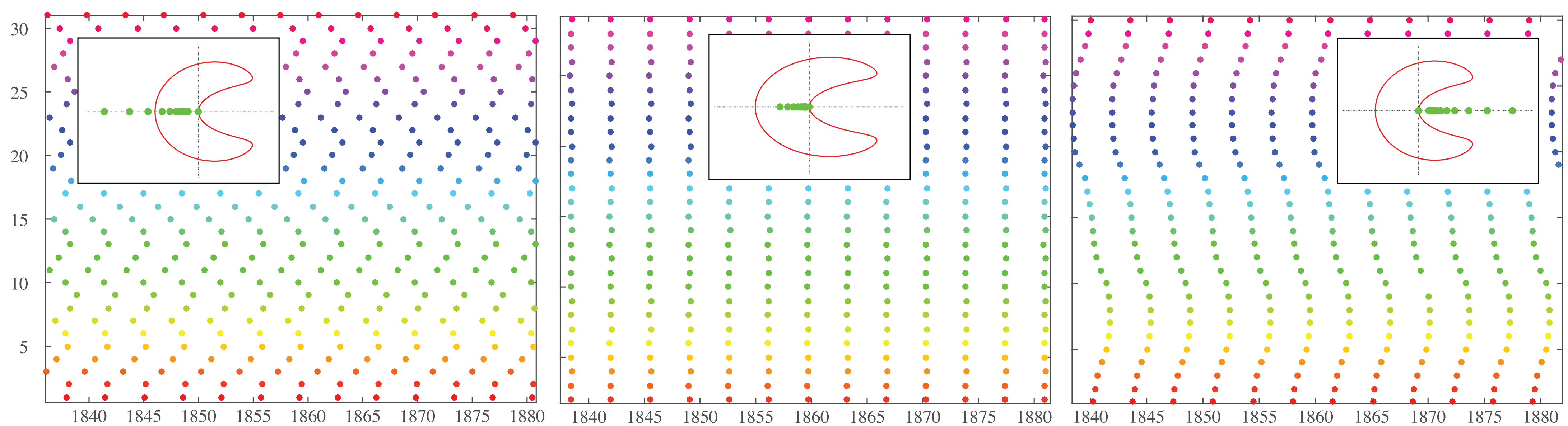
R Nicks, L Chambon and S Coombes 2018 Clusters in nonsmooth oscillator networks, Physical Review E, Vol 97, 032213

MSF: $\Gamma_l = K(\Delta)\exp\{(A_R + \sigma\lambda_l D\mathbf{H})\Delta\}$



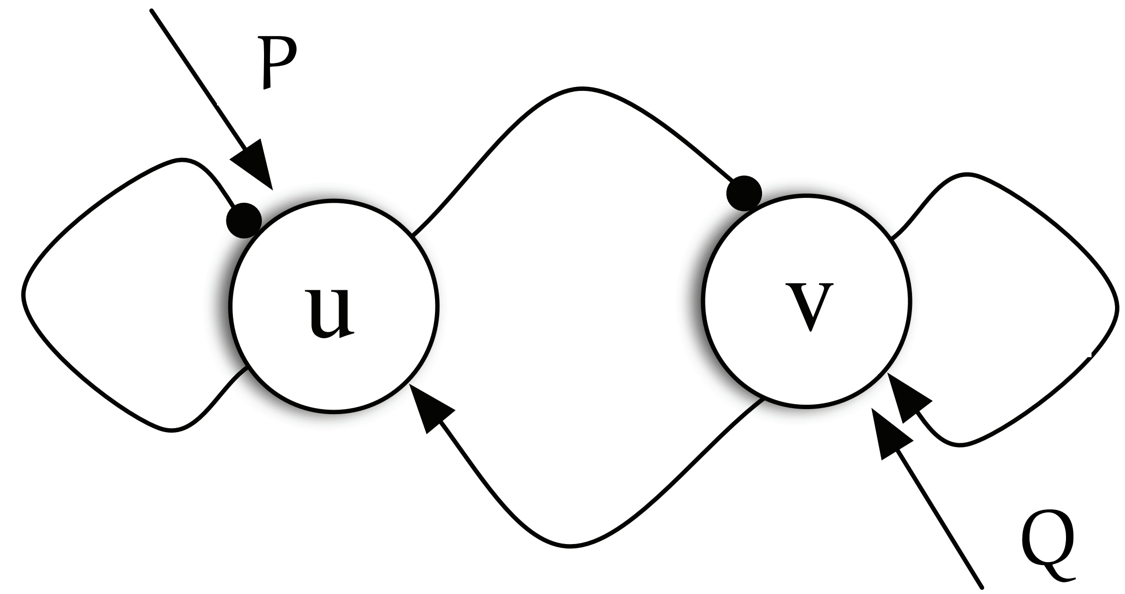
$\beta = \sigma\lambda_l$

Mex-hat
ring network
(N=31)
[circulant]



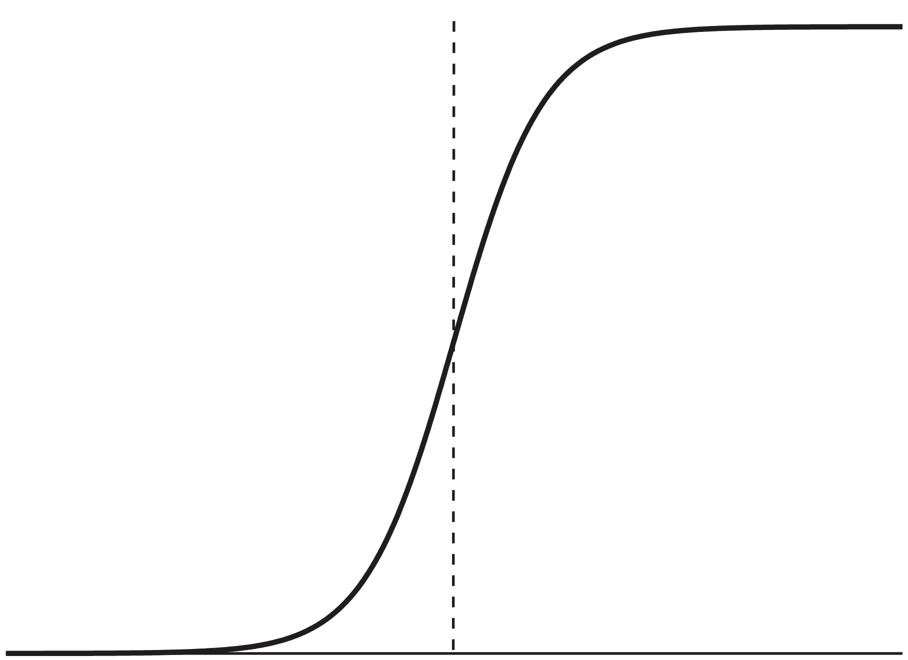
Wilson-Cowan network(s)

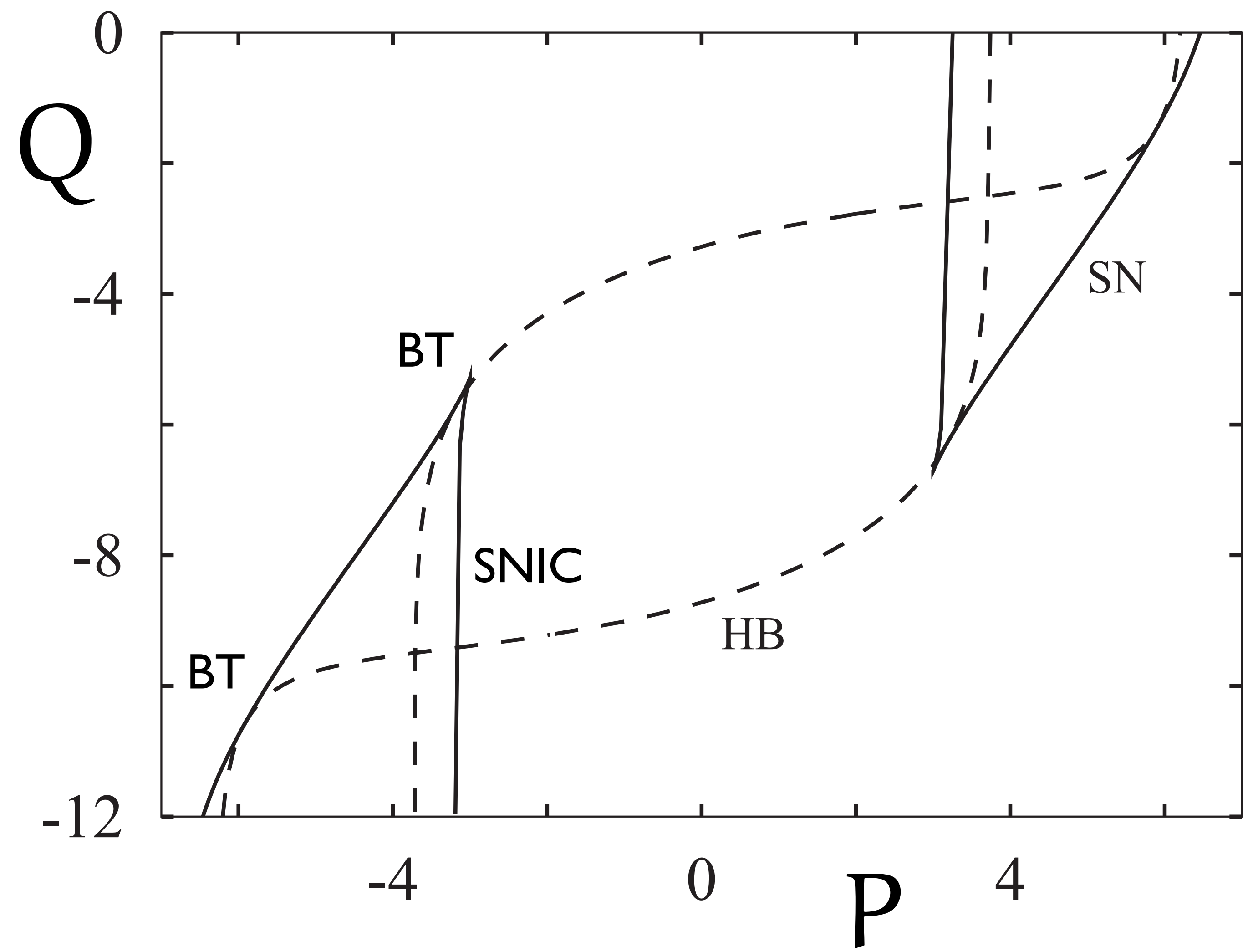
$$\dot{u} = -u + f(I_u + w^{uu}u - w^{vu}v),$$
$$\tau\dot{v} = -v + f(I_v + w^{uv}u - w^{vv}v)$$



Firing rate:

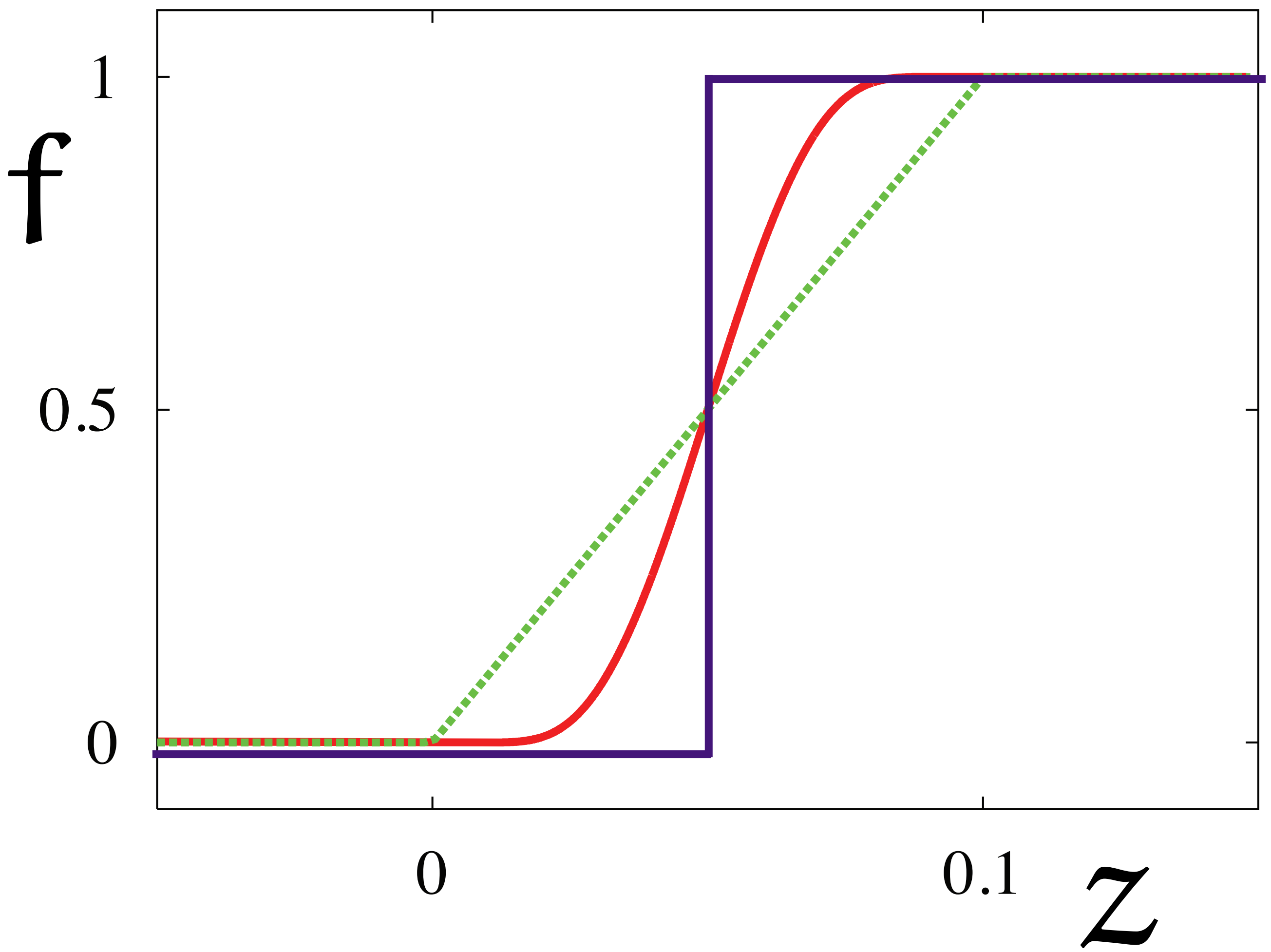
$$f(z) = \frac{1}{1 + e^{-\beta z}}$$





PW-Linear and PW-constant choices

(non-smooth *interactions*)

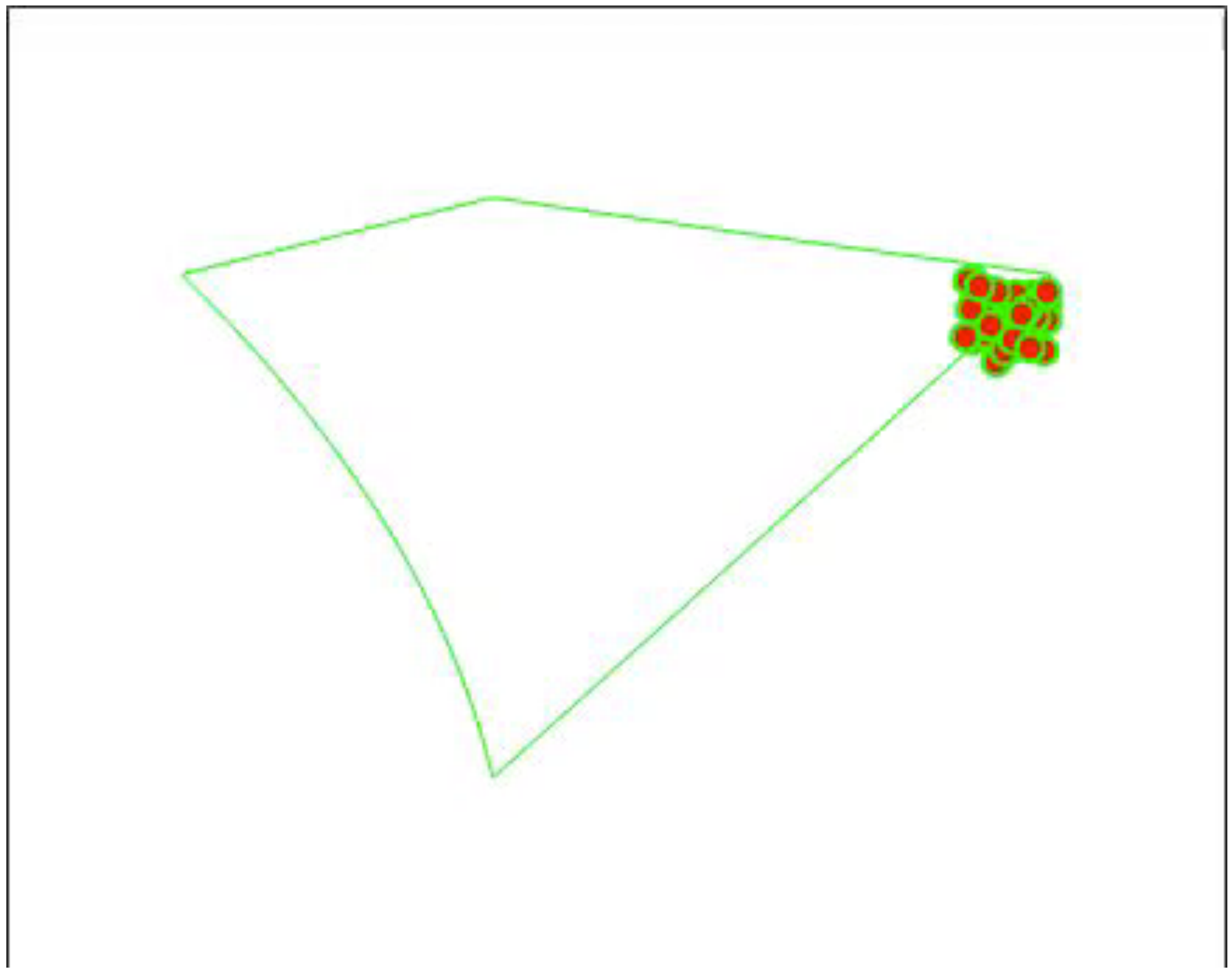


$$\sum_{j=1}^{N} \mathcal{W}_{ij}^{\alpha\beta} = w^{\alpha\beta}$$

$$\frac{du_i}{dt} = -u_i + H\left(I_u + \sum_{j=1}^{N} \mathcal{W}_{ij}^{uu} u_j - \sum_{j=1}^{N} \mathcal{W}_{ij}^{vu} v_j\right),$$

$$\tau \frac{dv_i}{dt} = -v_i + H\left(I_v + \sum_{j=1}^{N} \mathcal{W}_{ij}^{uv} u_j - \sum_{j=1}^{N} \mathcal{W}_{ij}^{vv} v_j\right)$$

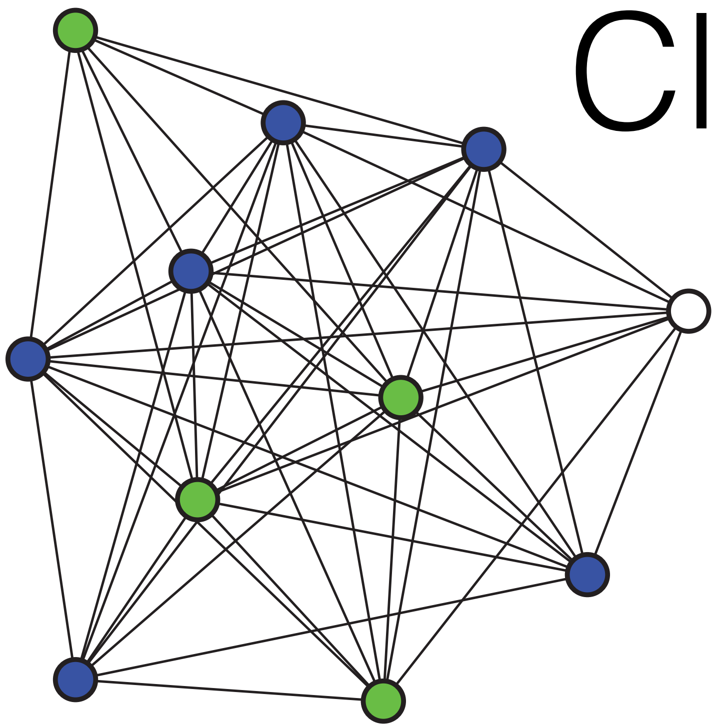
New variables (U,V); switching manifolds U=0 and V=0



MSF easily constructed

Mex-hat
ring network
(N=31)

Clusters (and Computational Group Theory)

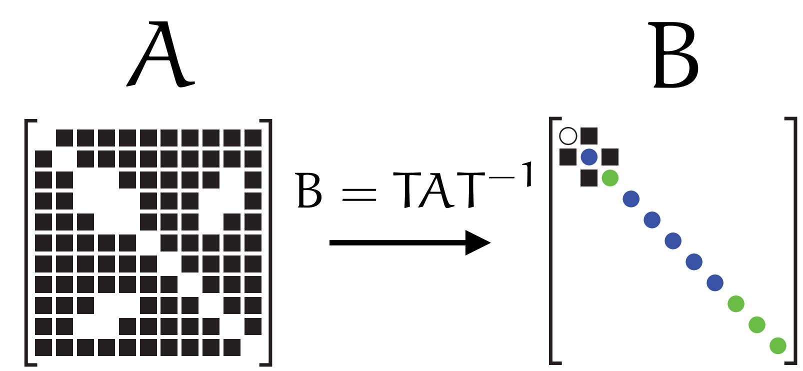


5,760 symmetries
3 clusters

GAP - Groups, Algorithms, Programming:
a System for Computational Discrete Algebra
http://www.gap-system.org

$$\dot{\mathbf{z}}_i = \mathbf{F}(\mathbf{z}_i) + \sigma \sum_j A_{ij}\mathbf{H}(\mathbf{z}_j)$$

Irreducible representations of the graph automorphism



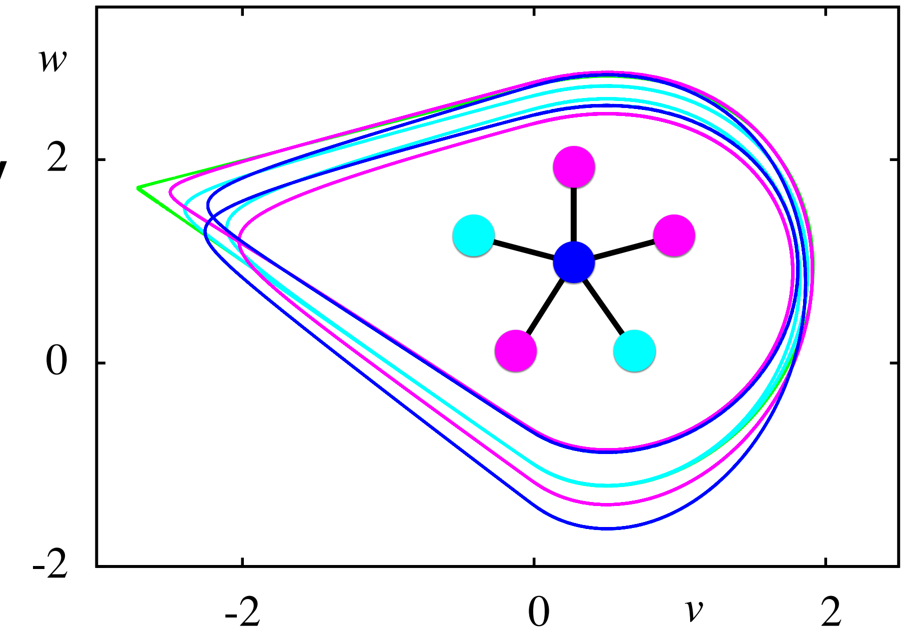
Nice variational formulation for M clusters

$$\dot{\mathbf{y}} = \left[\sum_{m=1}^{M} E^{(m)} \otimes \mathbf{DF}(\mathbf{s}_m) + \sigma B \otimes I_n \sum_{m=1}^{M} J^{(m)} \otimes \mathbf{DH}(\mathbf{s}_m)\right]\mathbf{y}$$

L M Pecora, *et al*. Cluster synchronization and isolated desynchronization in complex networks with symmetries. Nature Communications, 5(4079), 2014.

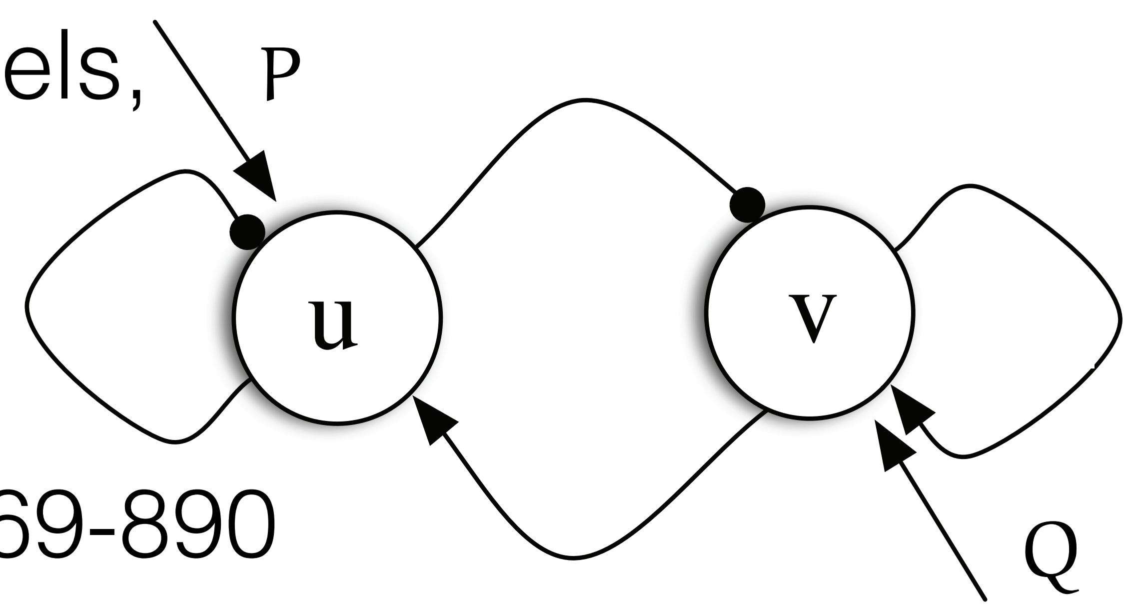
Papers

S Coombes and R Thul 2016 Synchrony in networks of coupled nonsmooth dynamical systems: Extending the master stability function



European Journal of Applied Mathematics, Vol 27(6), 904–922

S Coombes, Y-M Lai, M Sayli and R Thul 2018 Networks of piecewise linear neural mass models,



European Journal
of Applied Mathematics, Vol 29, 869-890