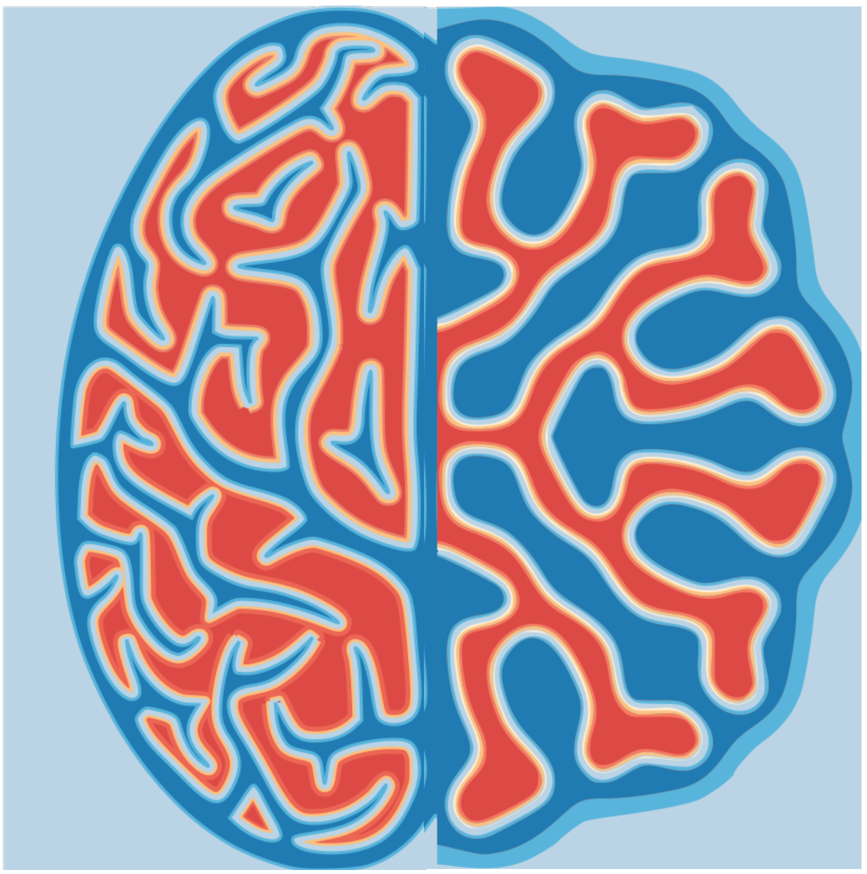
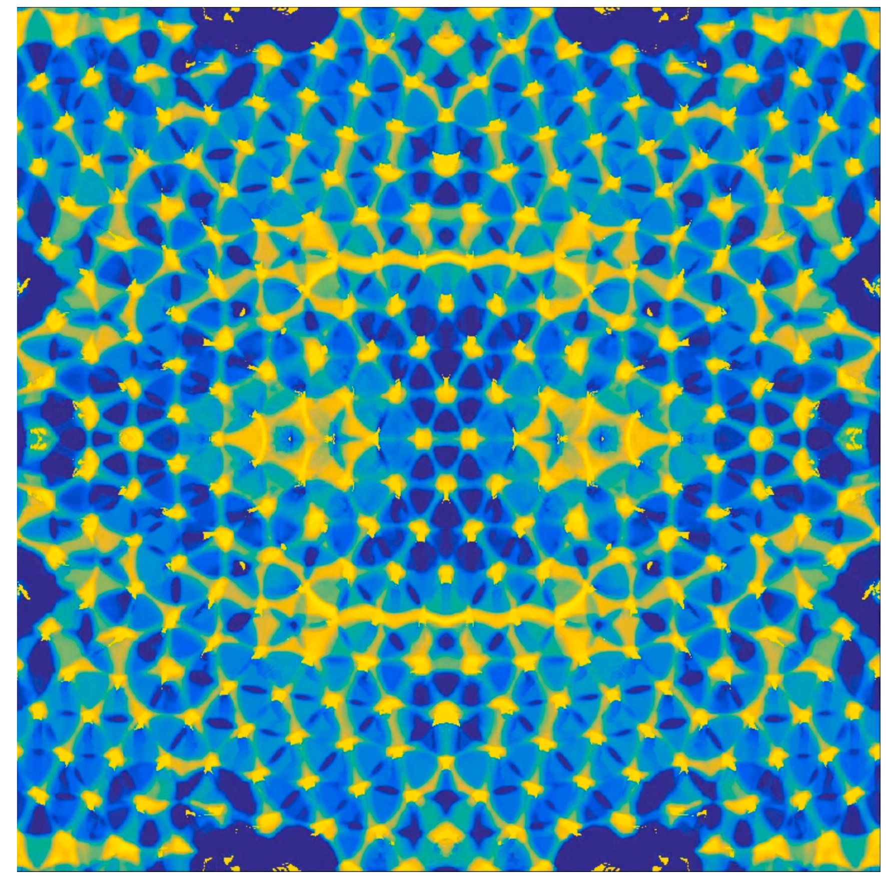
Neural fields with switches and spikes





Steve Coombes

CMMB Centre for Mathematical Medicine and Biology

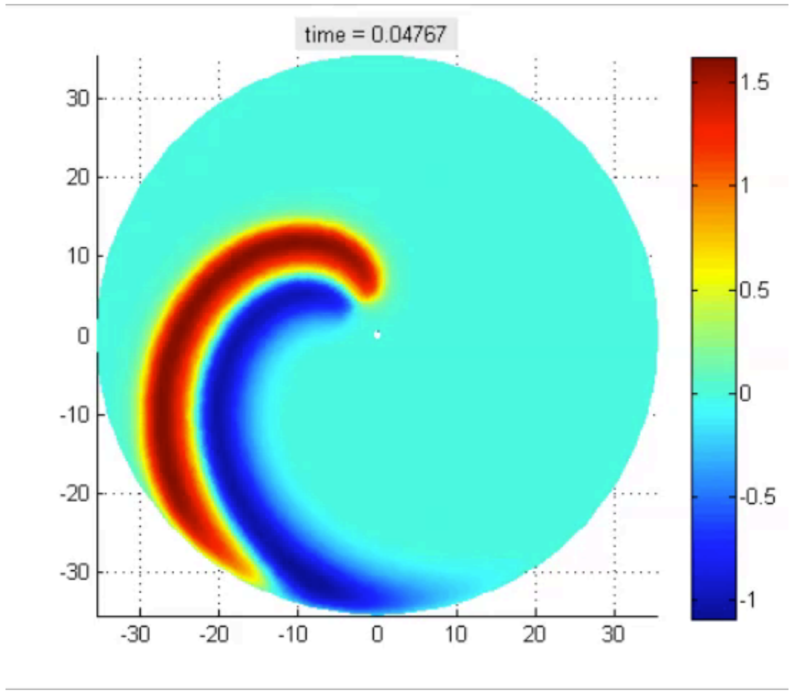
The University of Nottingham

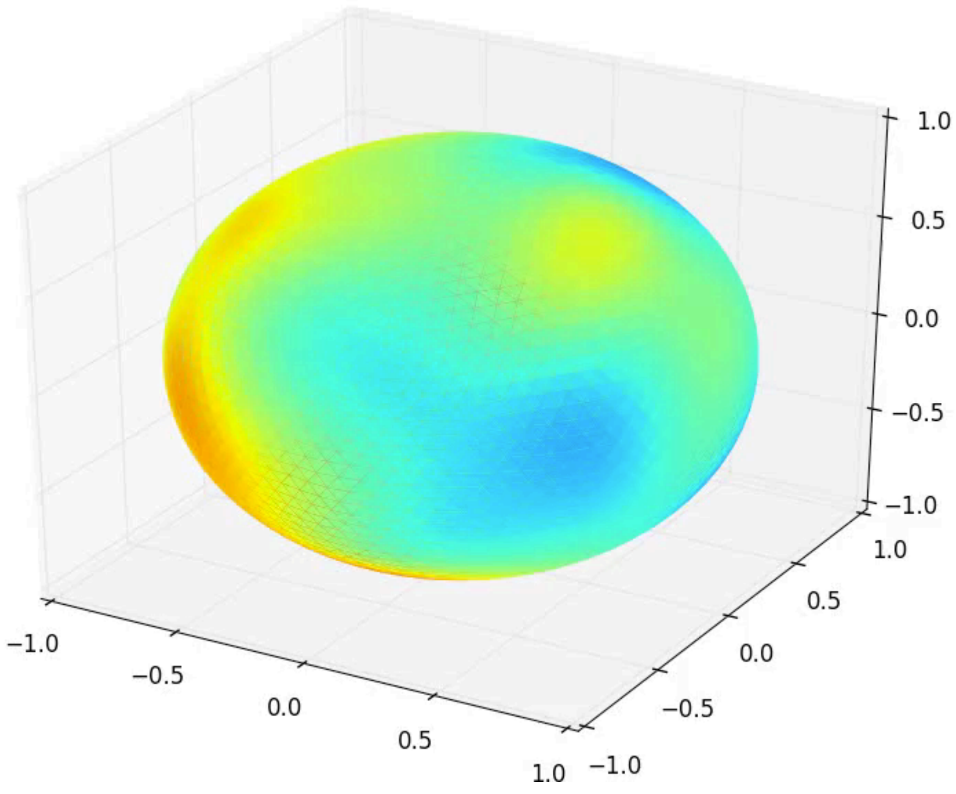
Neural fields

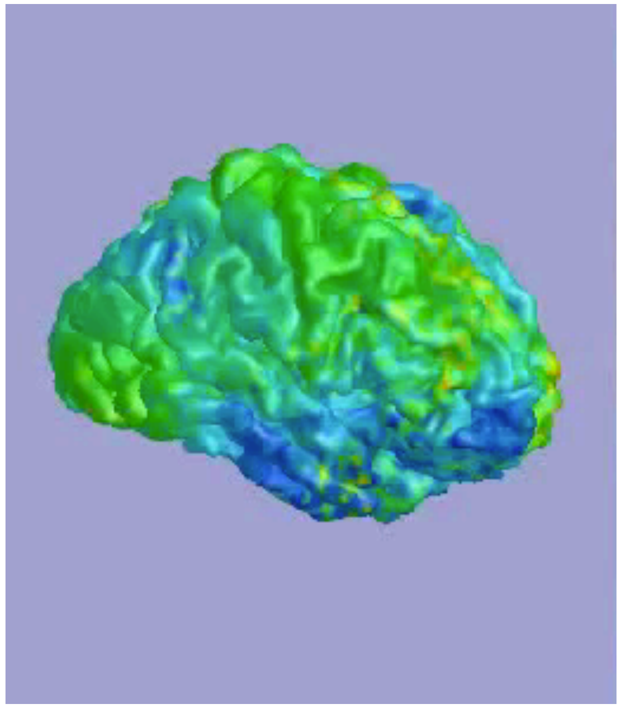
Ask Zack!

$$\mathrm{Q}\mathrm{g} = w \otimes \mathrm{f}(\mathrm{g})$$

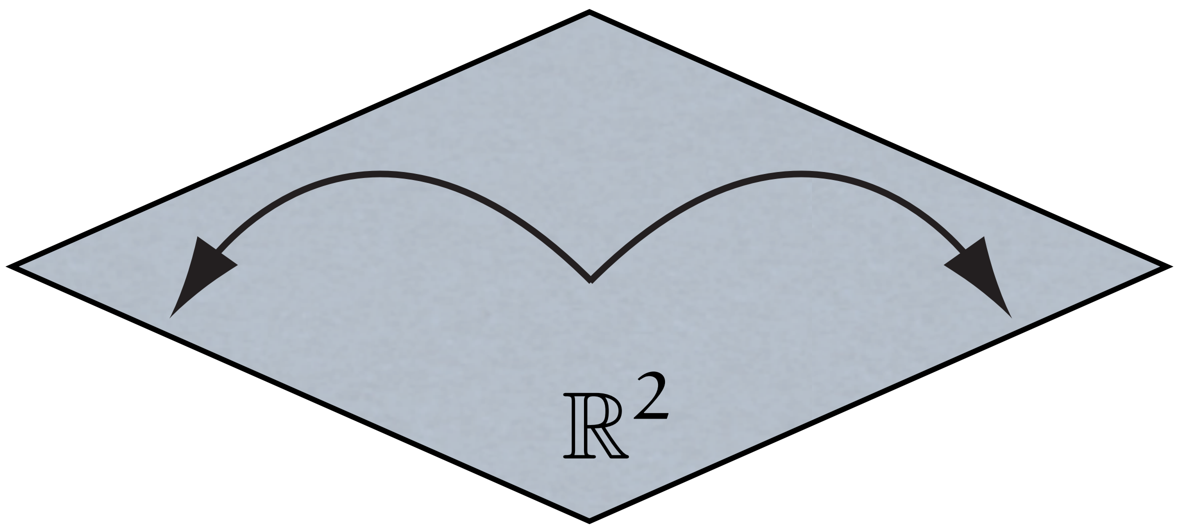
$$[w \otimes \mathrm{f}(\mathrm{g})](\mathbf{r}, \mathrm{t}) = \int_{\Omega} \mathrm{d}\mathbf{r}' w(|\mathbf{r} - \mathbf{r}'|) \mathrm{f} \circ \mathrm{g}(\mathbf{r}', \mathrm{t})$$

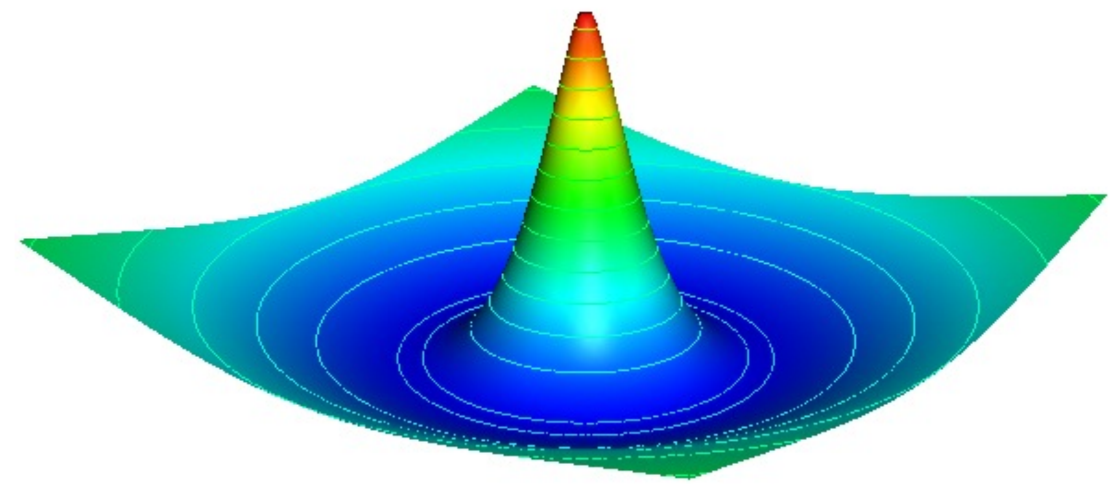






A simple 2D neural field model

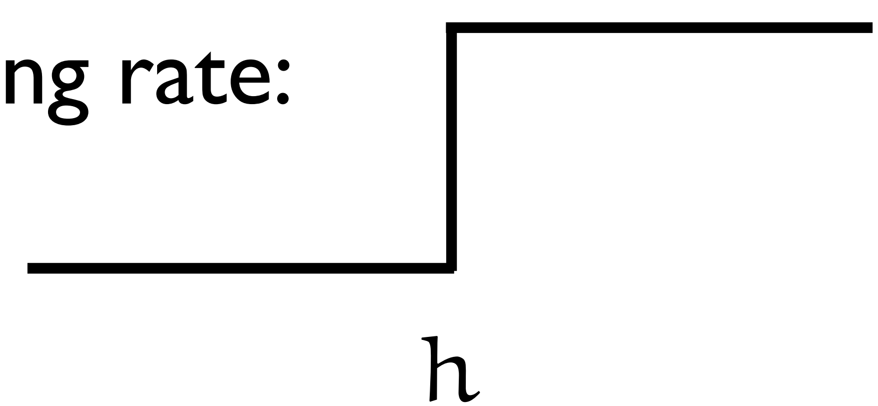




$$u_t(\mathbf{x}, t) = -u(\mathbf{x}, t) + \int_{\mathbb{R}^2} w(\mathbf{x} - \mathbf{x}') H[u(\mathbf{x}', t) - h] d\mathbf{x}'$$

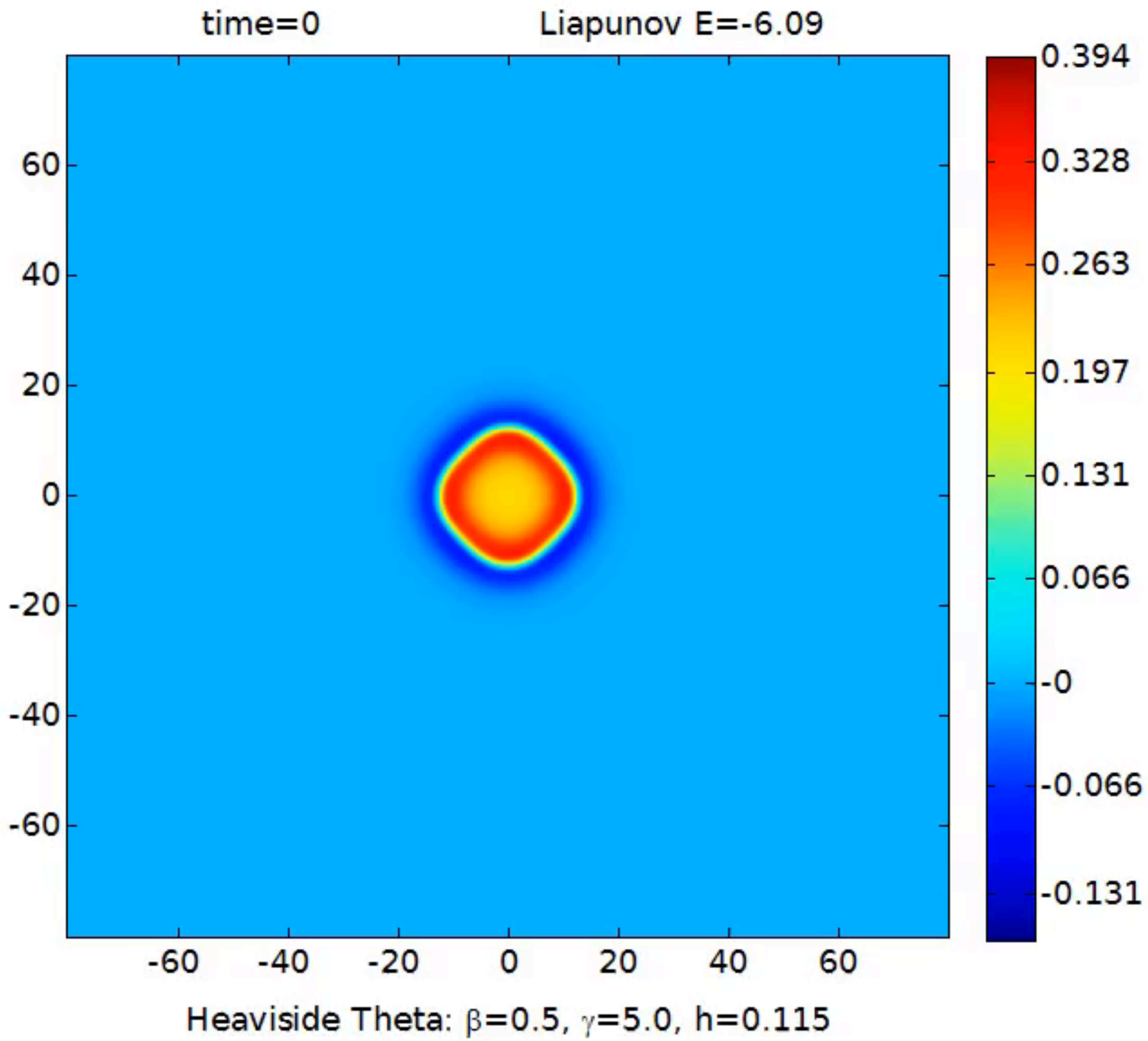
2D Amari model

Piece-wise constant firing rate:
Heaviside



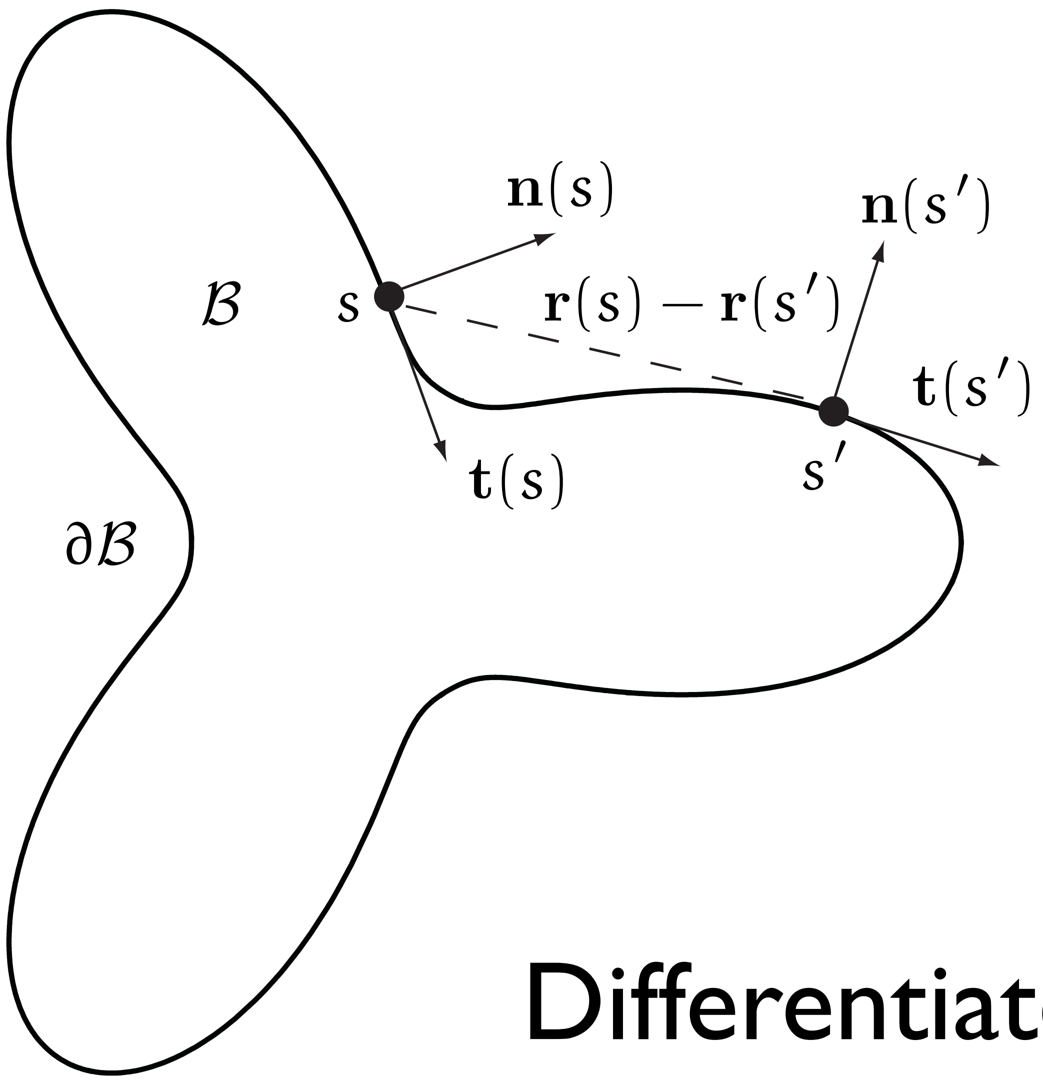
S Coombes, H Schmidt and I Bojak 2012 Interface dynamics in planar neural field models. **JMN**.

A simulation



An *interface* is easily identified

Interface dynamics in 2D



$$\mathbf{n} = -\nabla_{\mathbf{x}} u / |\nabla_{\mathbf{x}} u|$$

$$u_t(\mathbf{x}, t) = -u(\mathbf{x}, t) + \psi(\mathbf{x}, t)$$

$$\psi(\mathbf{x}, t) = \int_{\mathcal{B}(t)} w(|\mathbf{x} - \mathbf{x}'|) d\mathbf{x}'$$

Differentiate $u(\mathbf{x}, t) = h$ along $\partial\mathcal{B}(t)$

$$\nabla_{\mathbf{x}} u \cdot \frac{d\mathbf{r}}{dt} + \frac{\partial u}{\partial t} = 0$$

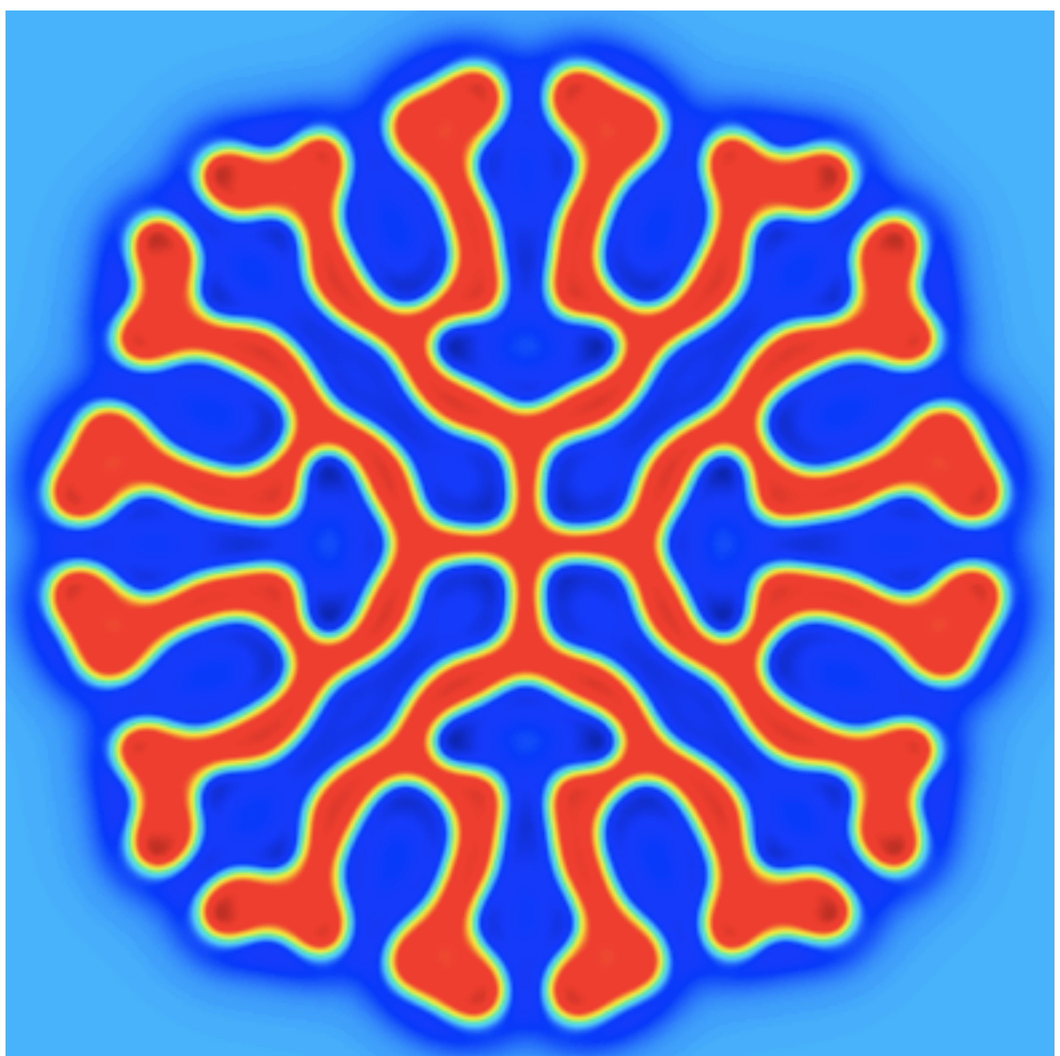
Normal velocity

$$\mathbf{n} \cdot \frac{d\mathbf{r}}{dt} = \frac{u_t}{|z|} \qquad z \equiv \nabla_{\mathbf{x}} u(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{r}}$$

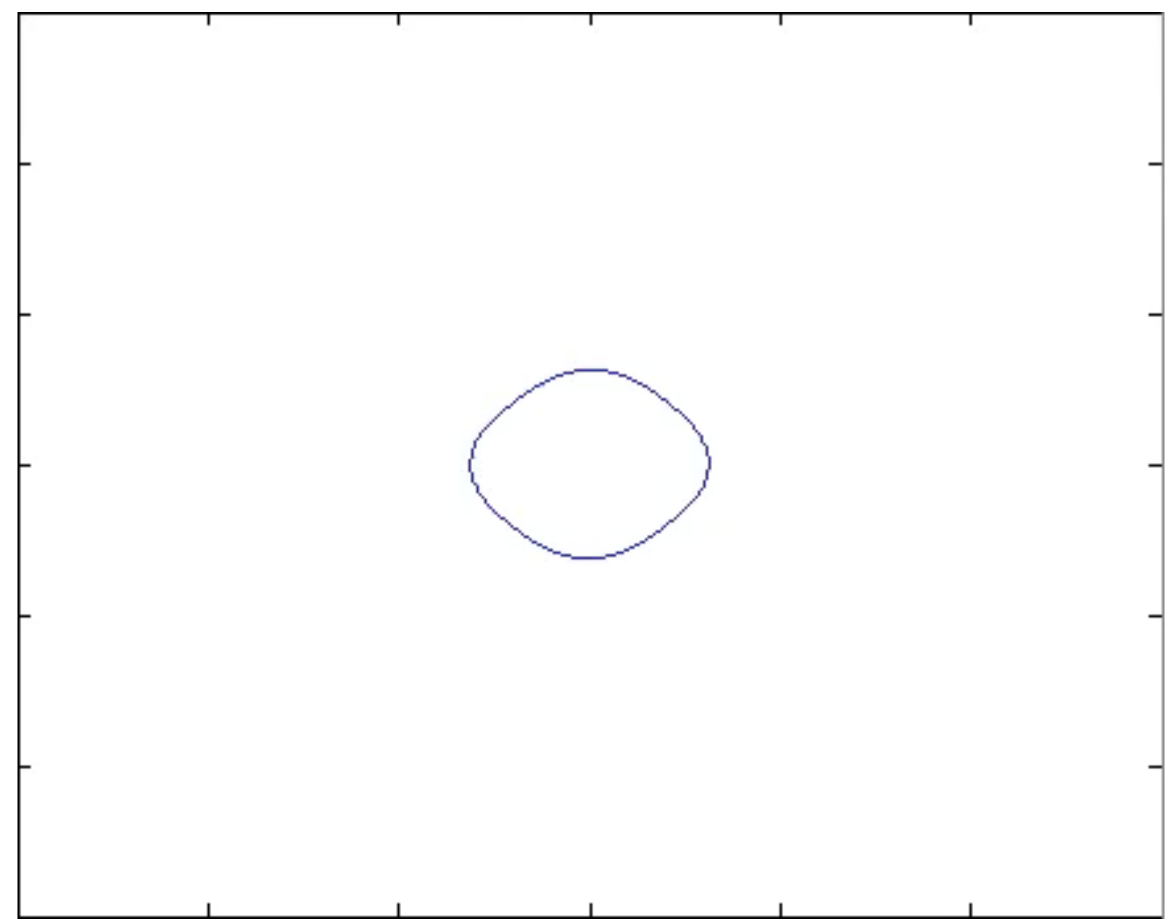
Dynamics from data on the boundary only

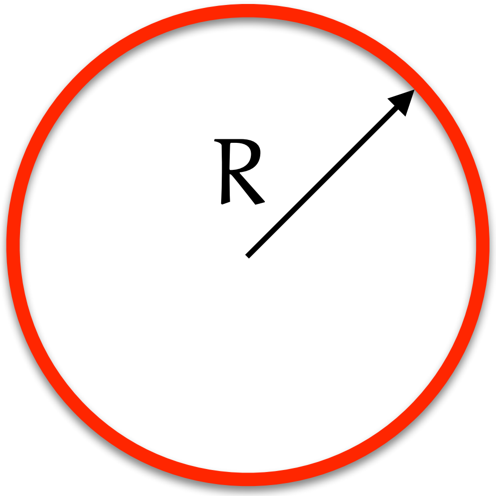
normal velocity $c_n = \frac{a}{|\mathbf{b}|}$ $(a, \mathbf{b}) = (\mathcal{F}(c_n), \mathcal{F}(\mathbf{n}))$

$$\mathcal{F}(x(s,t)) = \int_0^\infty dt' \, e^{-t'} \oint_{\partial\mathcal{B}(t-t')} ds' \, w(\mathbf{y}(s,t), \mathbf{y}(s',t')) x(s', t-t')$$

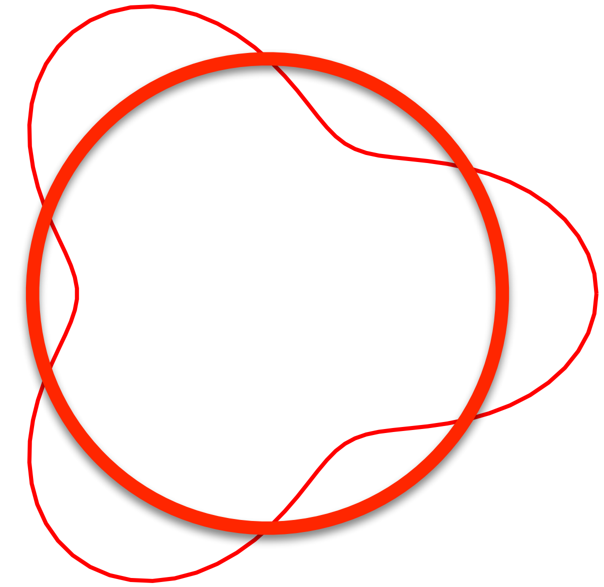


Reynold's transport theorem





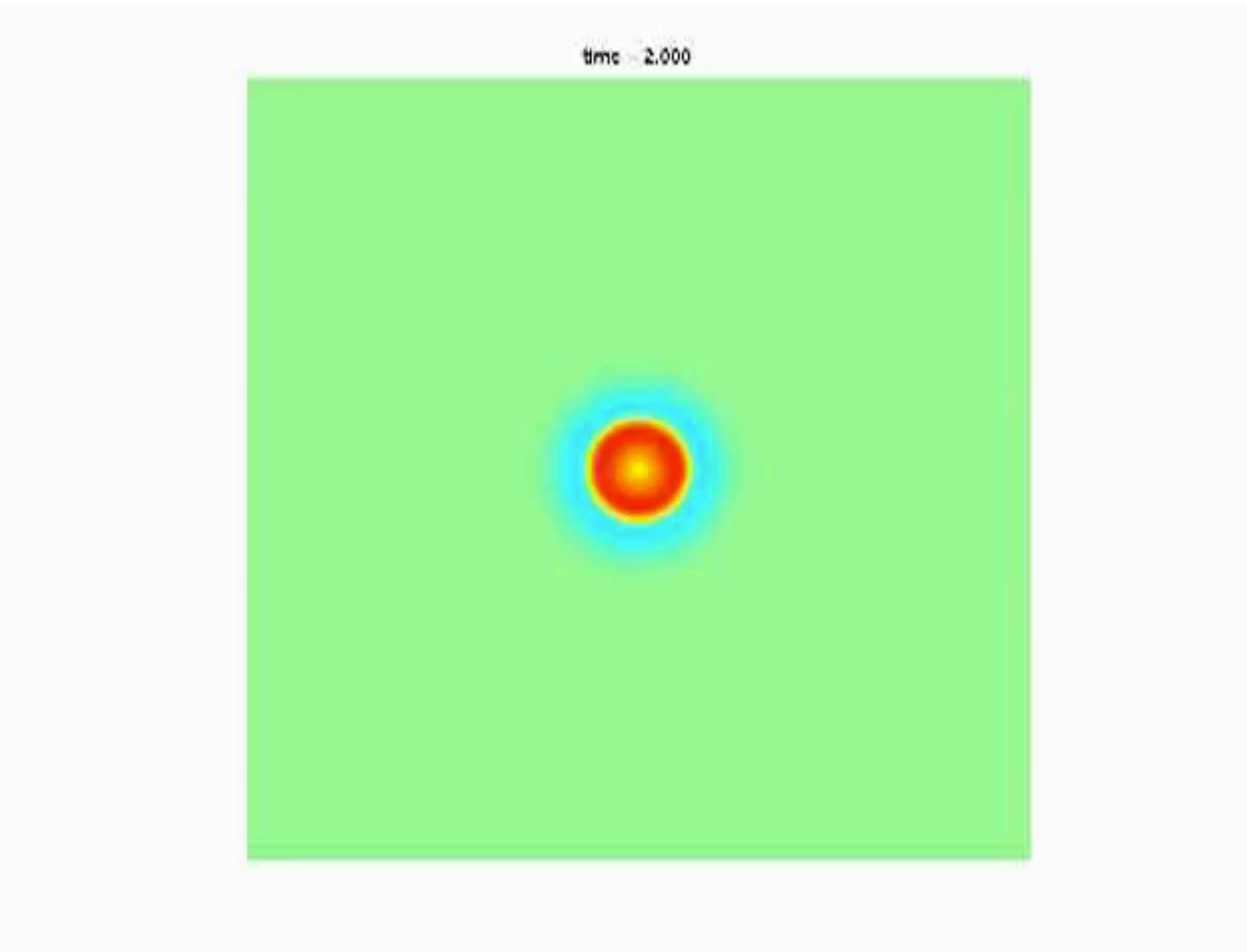
Spots and Stability

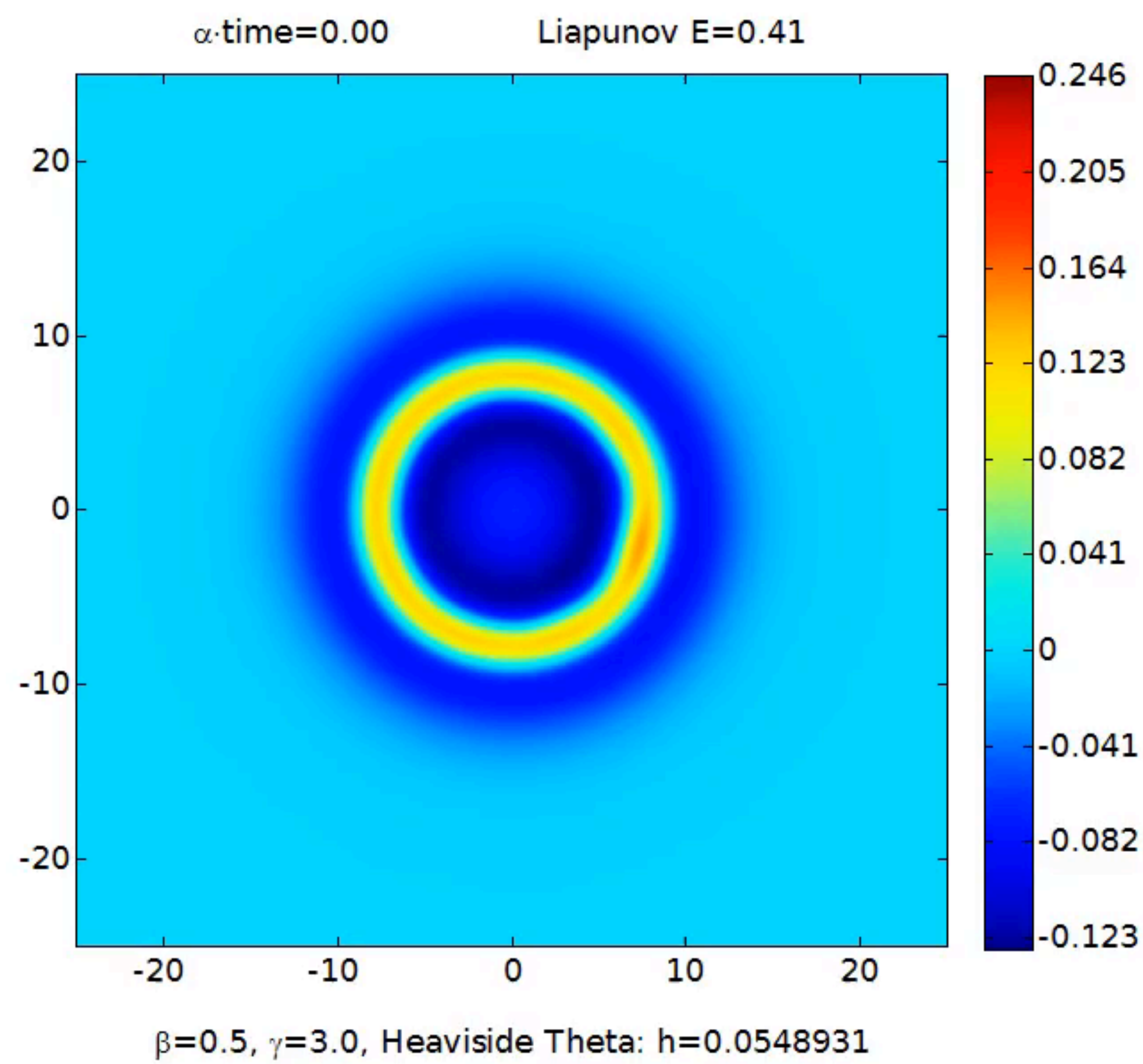


Zero normal velocity

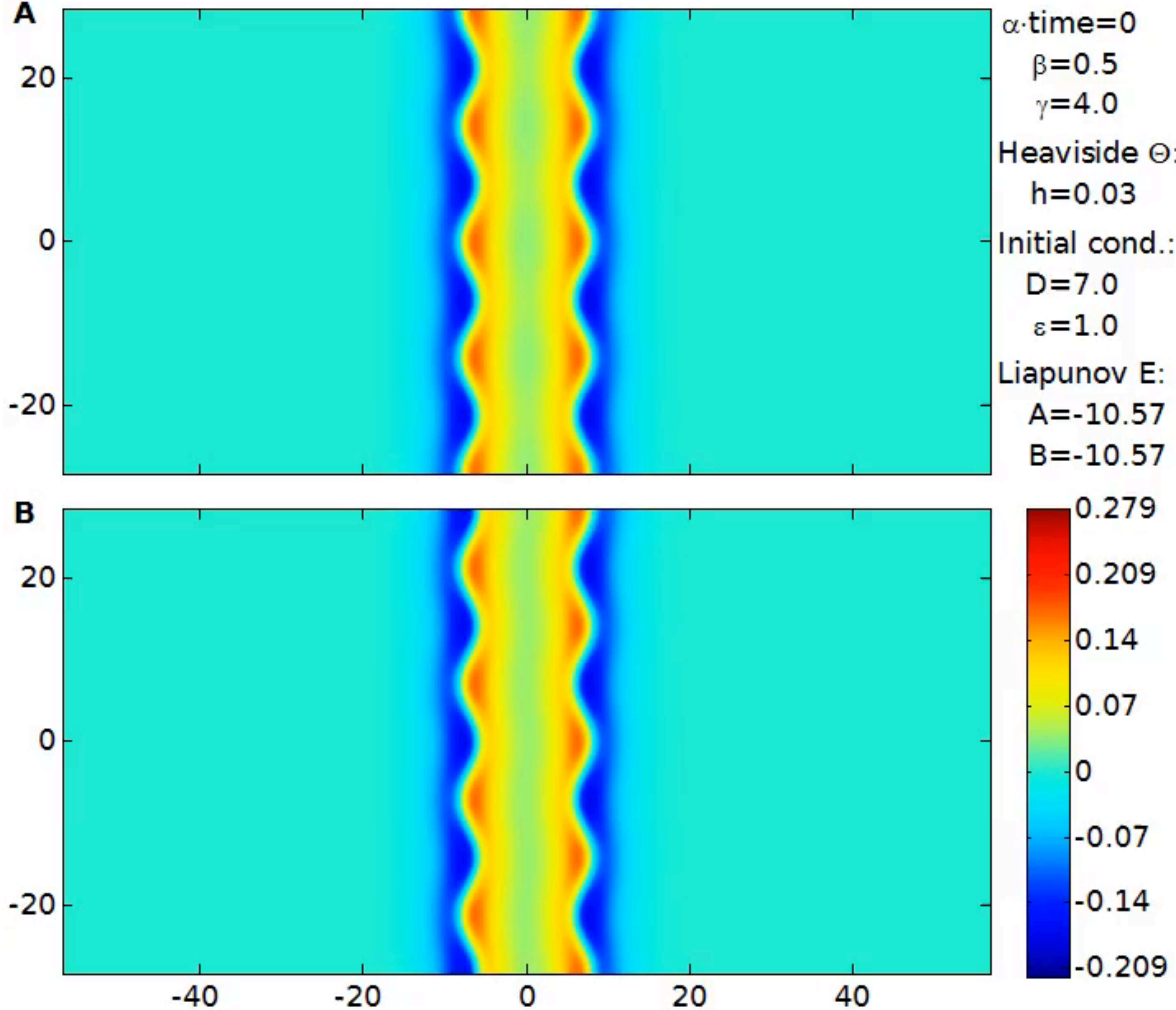
$$u(R) = h$$

R





h


A
B
α·time=0
β=0.5
γ=4.0
Heaviside Θ:
h=0.03
Initial cond.:
D=7.0
ε=1.0
Liapunov E:
A=-10.57
B=-10.57
20
0
-20
-40
-20
0
20
40
0.279
0.209
0.14
0.07
0
-0.07
-0.14
-0.209

With axonal delays

$$Qu = \psi \qquad Q = (1 + \partial_t)$$

$$\psi(x,t) = \int_{\mathbb{R}} dy\, w(|y|) H\left[u(x-y, t-|y|/v) - h\right]$$

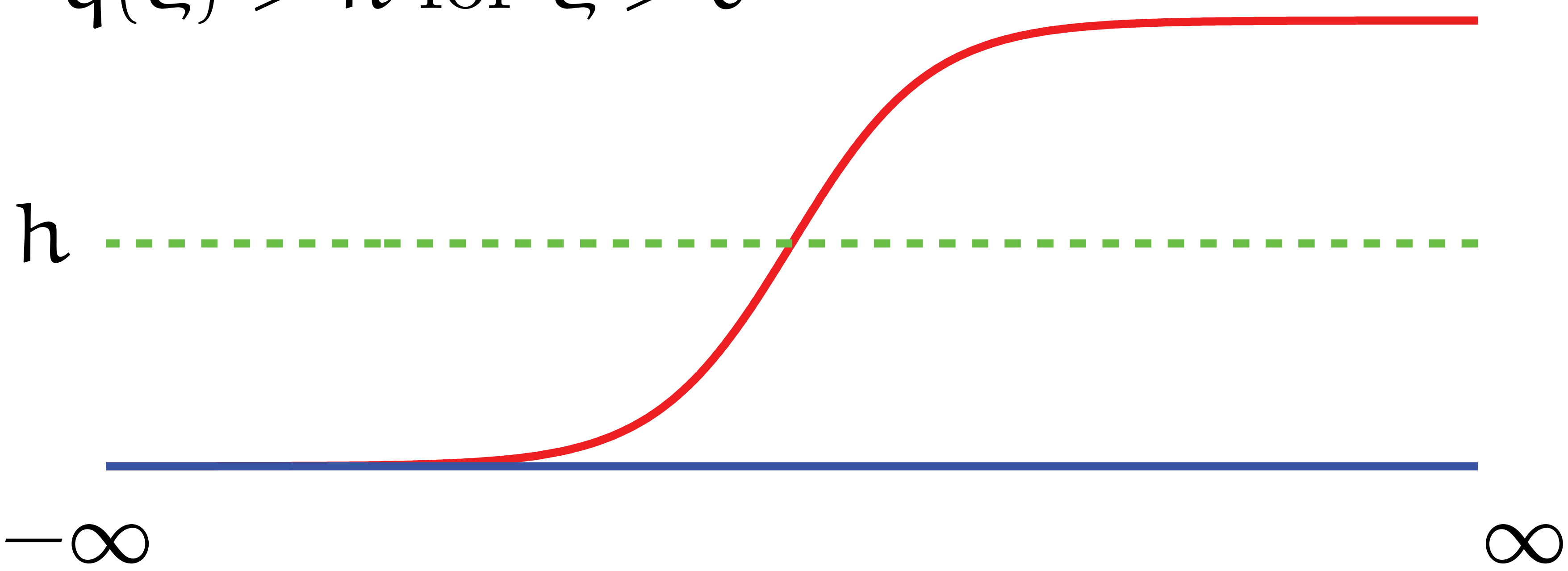
Travelling wave frame $\xi = x - ct$

$$u(x,t) = q(\xi) = \int_0^\infty ds\, \eta(s) \psi(\xi + cs)$$

$$\psi(\xi) = \int_{\mathbb{R}} dy\, w(|y|) H\left[q(\xi - y + c|y|/v) - h\right]$$

Travelling fronts

$q(\xi) > h$ for $\xi > 0$

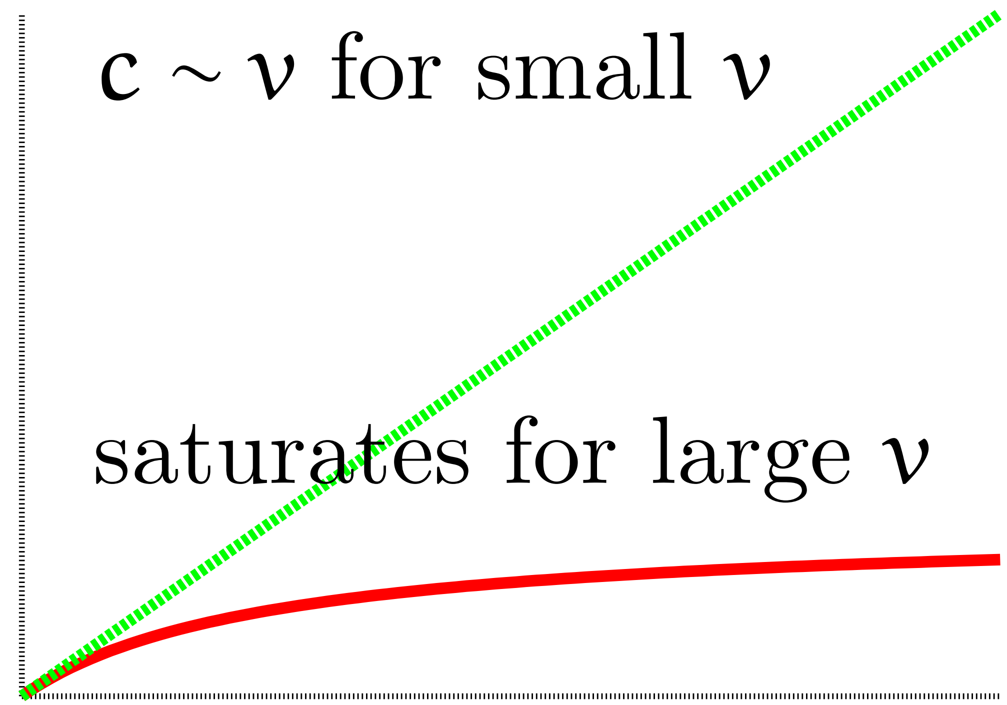


Self-consistent speed

$q(0) = h$

$$\psi(\xi) = \begin{cases} \int_{-\infty}^{\xi/(1-c/v)} w(y)dy & \xi > 0 \\ \int_{-\infty}^{\xi/(1+c/v)} w(y)dy & \xi < 0 \end{cases}$$

$$c = v\frac{2h-1}{2h-1-2hv/\alpha}$$



Linear stability

Linearise about the steady state: $U(\xi, t) = q(\xi) + u(\xi)e^{\lambda t}$

TW is linearly stable if $\mathrm{Re}(\lambda) < 0 \ (\lambda \neq 0)$

Eigenvalues as zeros of the **Evans** function

- Order of the roots = multiplicity of eigenvalues
- $\mathcal{E}(\lambda)$ is analytic

Essential spectrum in left half plane, so not a problem.

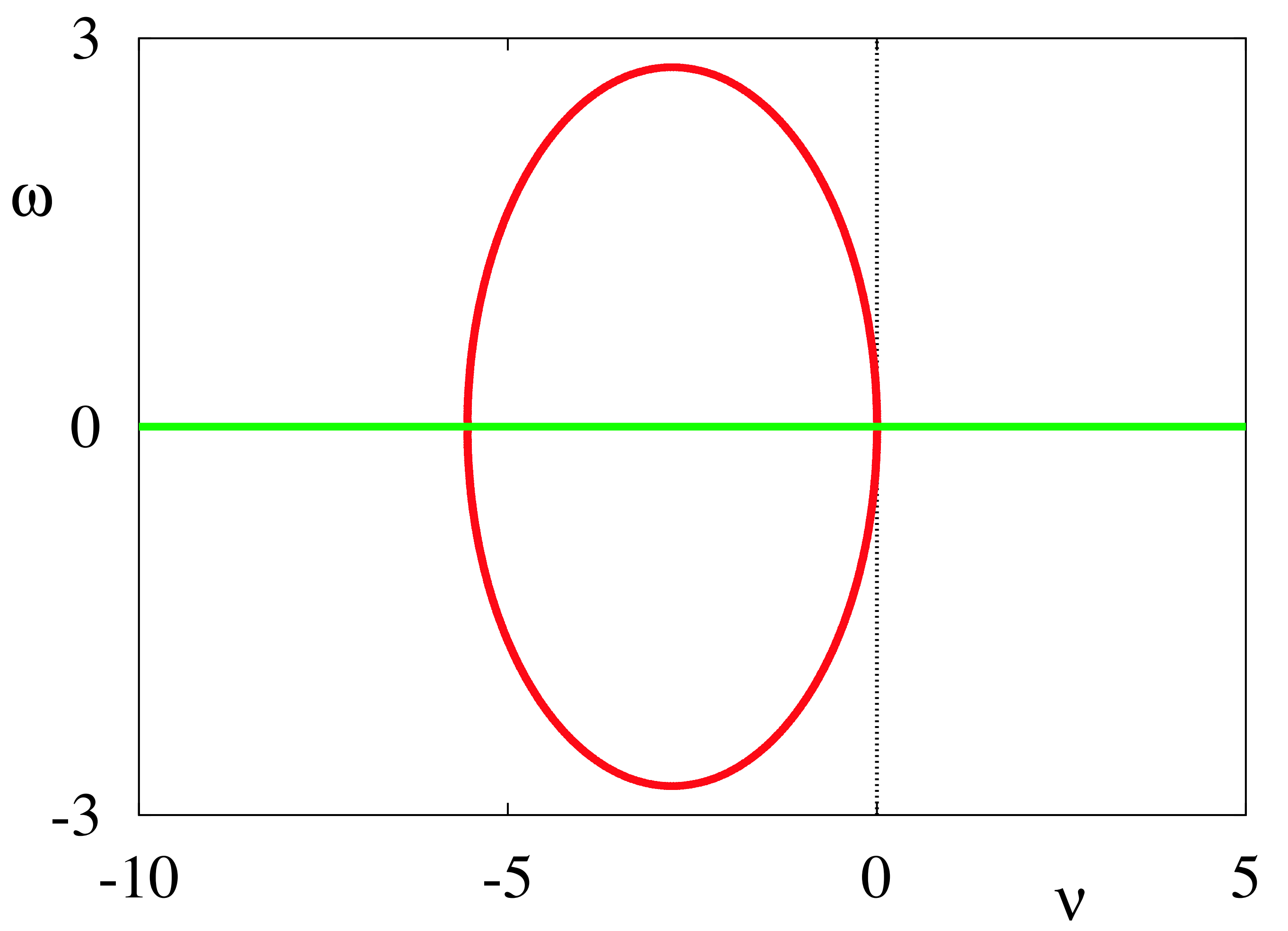
e.g. for a front

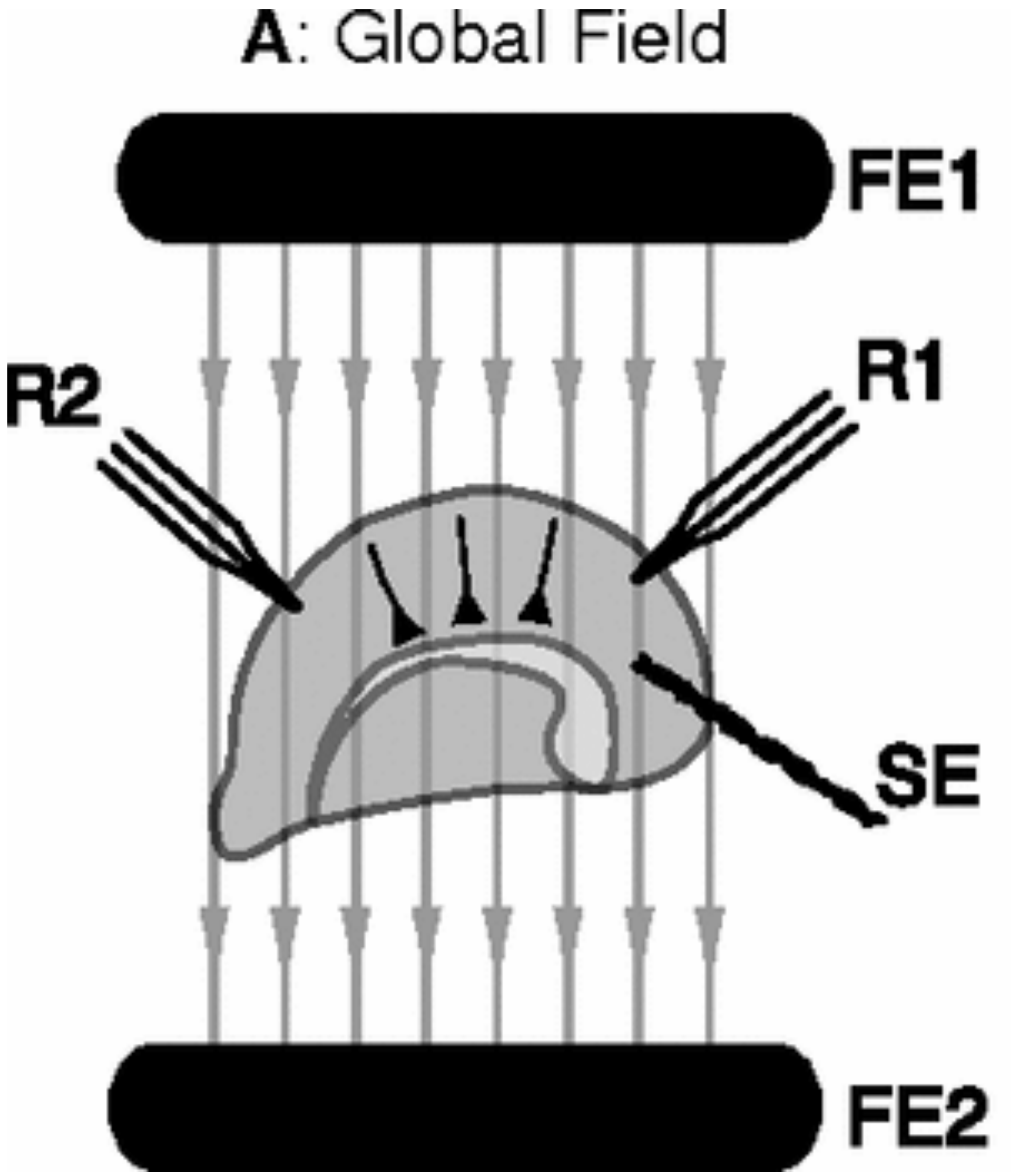
$$\mathcal{E}(\lambda) = 1 - \frac{1}{c|q'(0)|} \int_{-\infty}^{\infty} dy w(y) \eta(y/c - |y|/\nu) e^{-\lambda y/c}$$

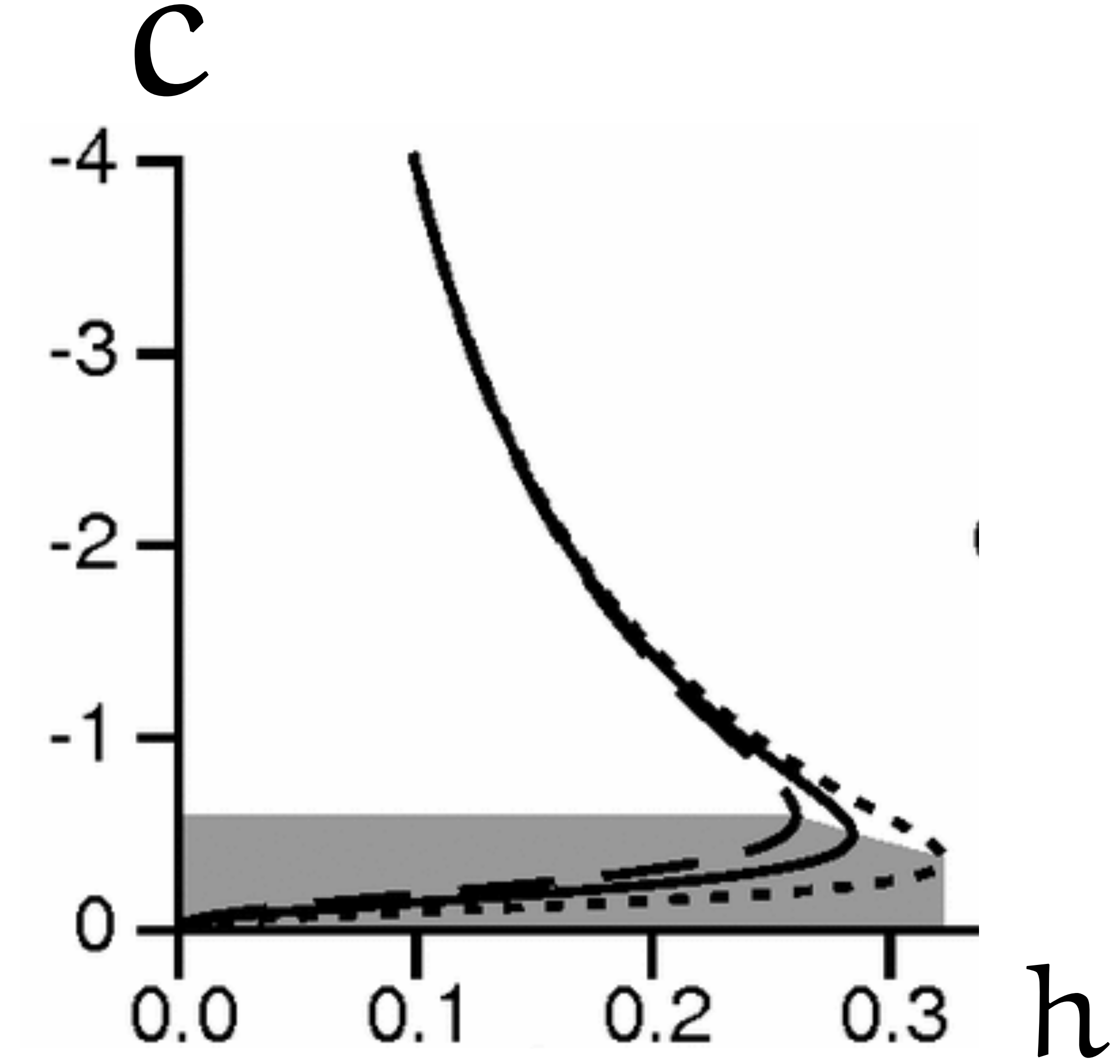
For this example the front is stable.

$$\mathcal{E}(\lambda) = \frac{\lambda}{\frac{c}{\sigma} + \alpha\left(1 - \frac{c}{\nu}\right) + \lambda}$$

Let $\lambda = \nu + i\omega$ and plot $\mathrm{Re}\ \mathcal{E}(\lambda) = 0 = \mathrm{Im}\ \mathcal{E}(\lambda)$







Control of Traveling Waves in the Mammalian Cortex. Kristen A. Richardson, Steven J. Schiff, and Bruce J. Gluckman. Phys. Rev. Lett. 94, 028103 (2005)

Modulating excitability in the cortical network: impact on emergent activity and traveling waves. Sanchez-Vives MV and Mattia M.
Non Linear Theory and its Applications (2014)

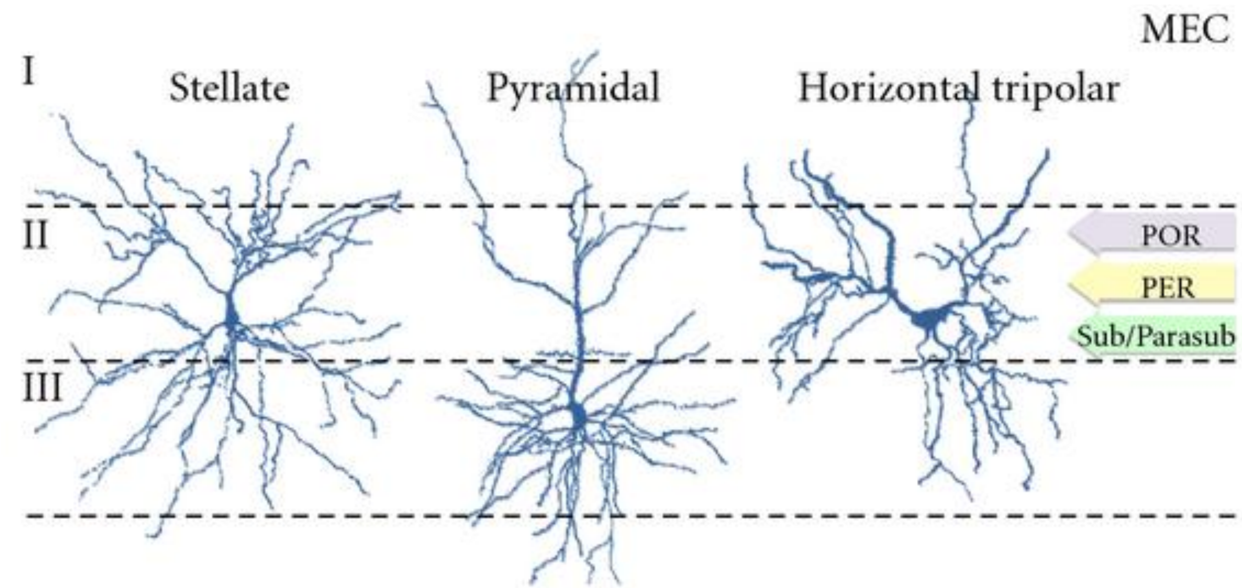
Rebound currents

Rebound responses are well known in deep cerebellar nuclear neurons … and elsewhere!

Two classes of ion channels play crucial roles in cellular excitability

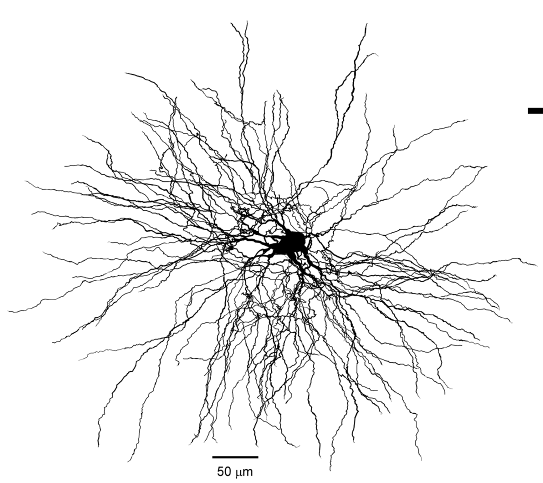
Hyperpolarization-activated cyclic-nucleotide (HCN) channels

I_h Layer II stellate cells in MEC

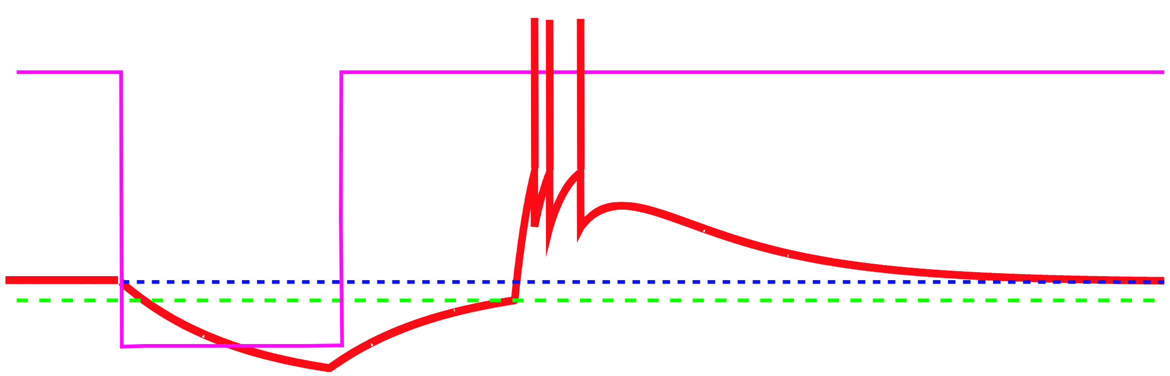


T-type calcium channels

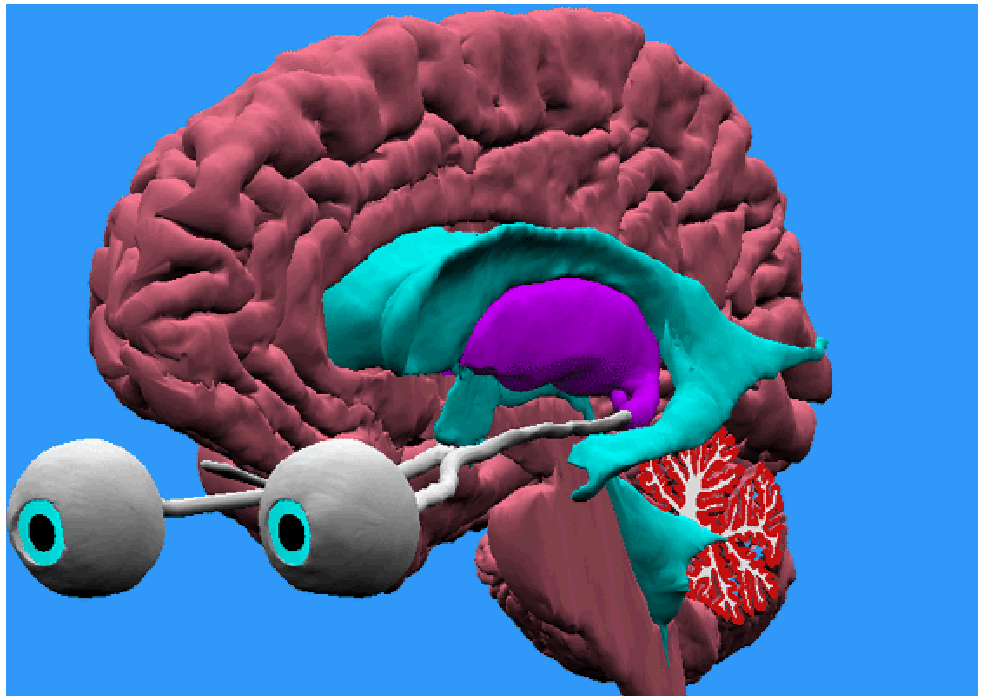
I_T

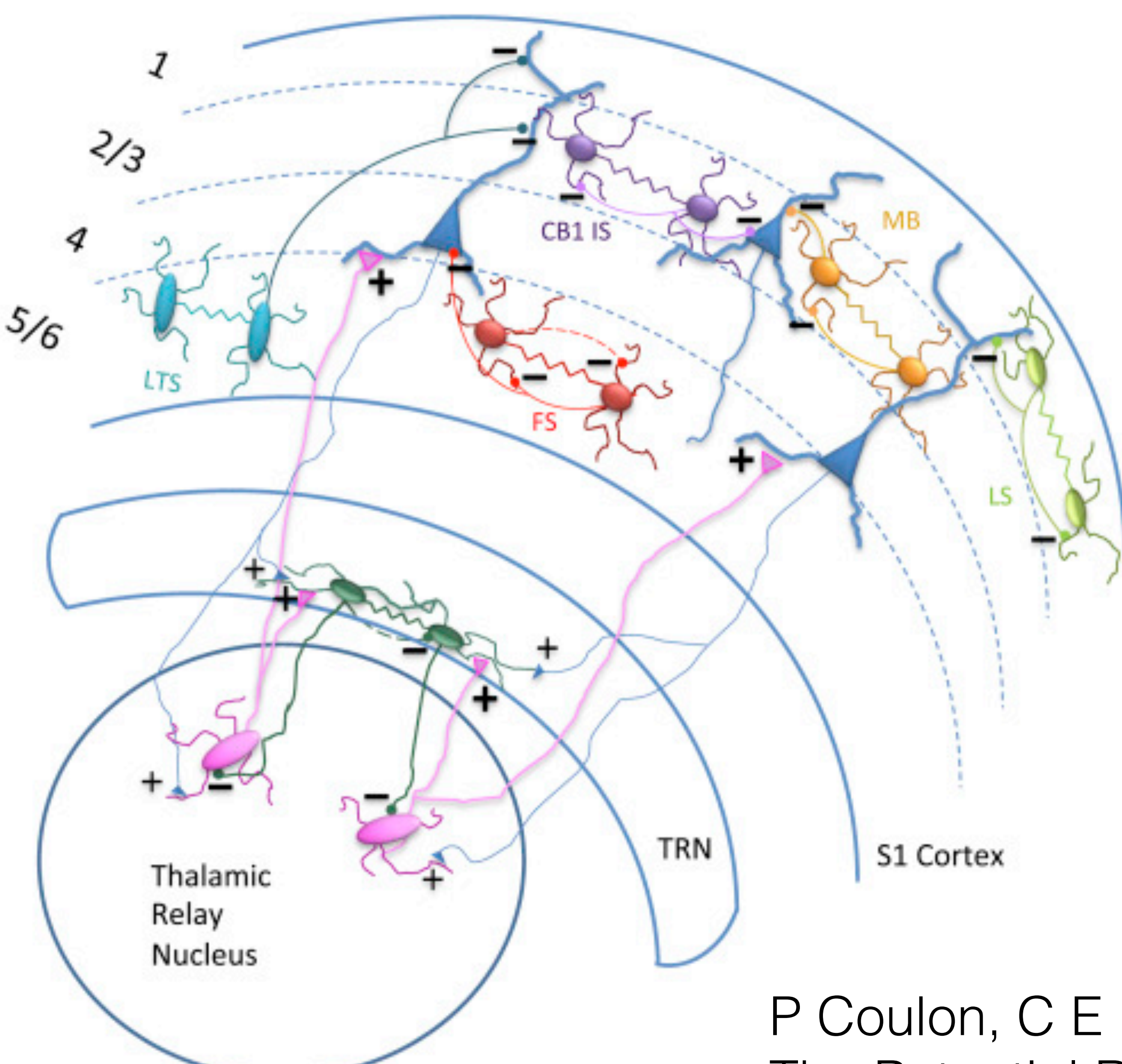


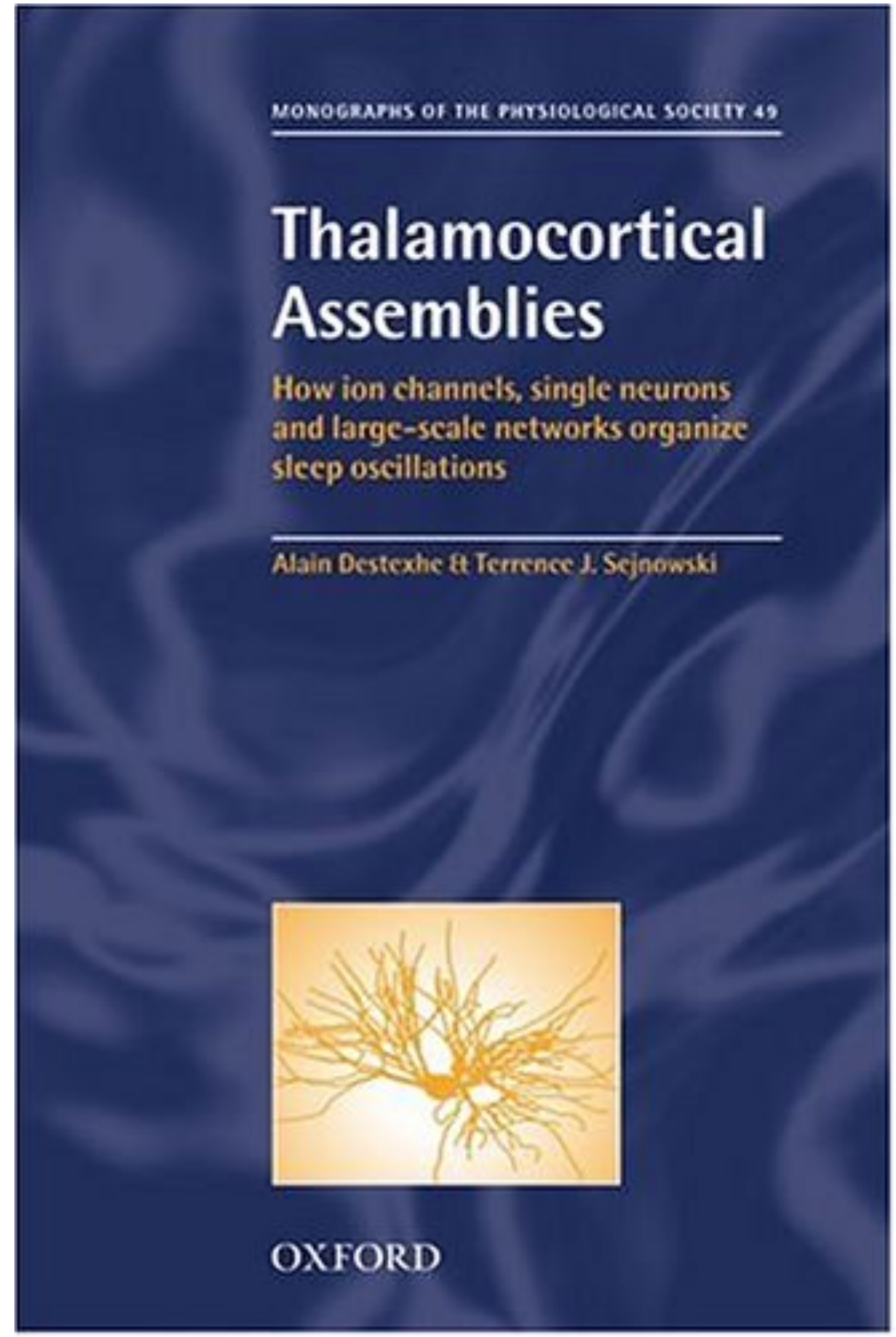
Thalamo-cortical relay cell



Role at the **network** level?

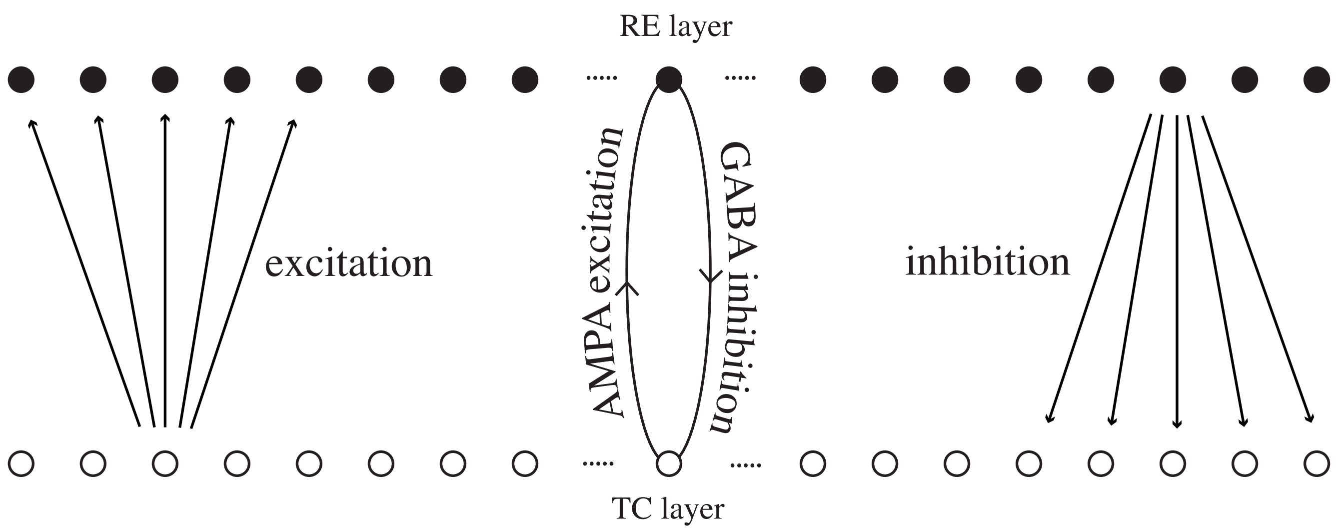


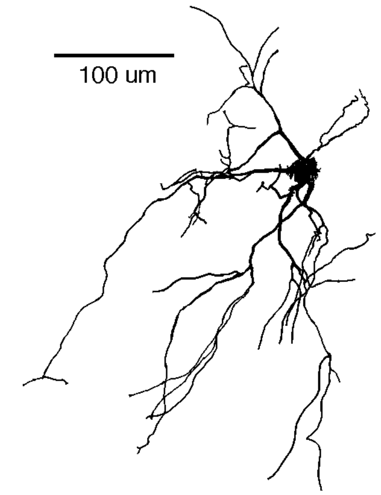


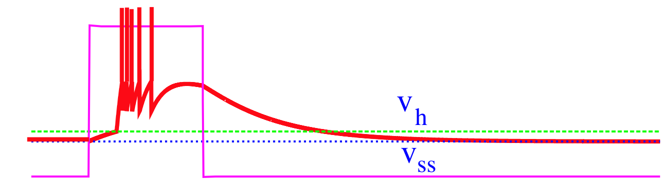


P Coulon, C E Landisman (2017) Neuron
The Potential Role of Gap Junctional Plasticity in the Regulation of State

Thalamic modelling

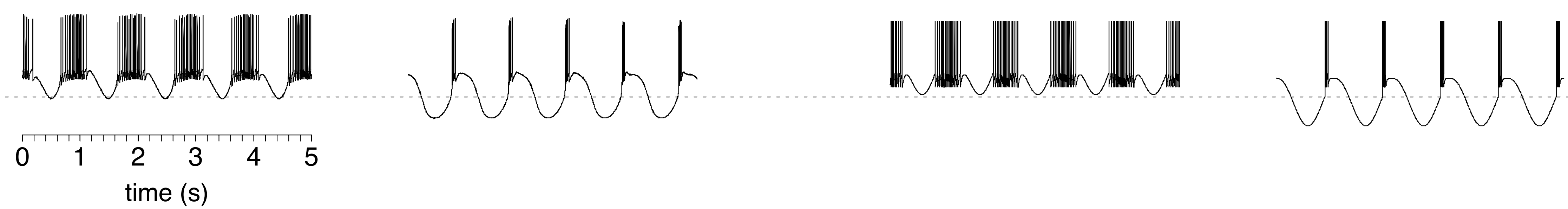




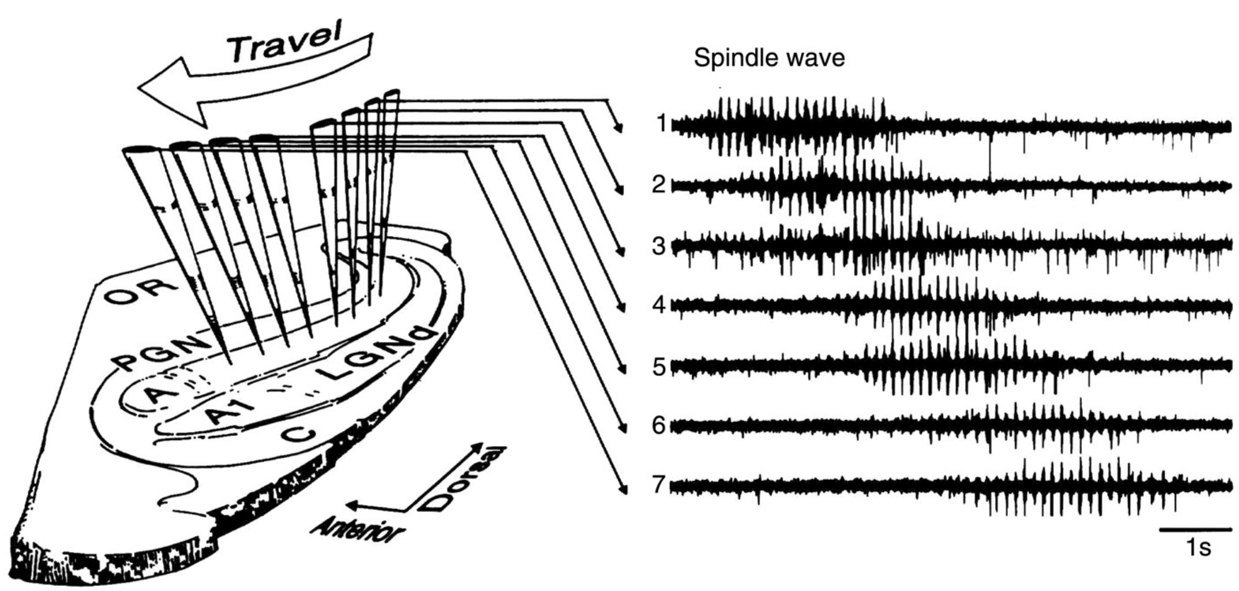


Experiment

IFB model



Fourier Analysis of Sinusoidally Driven Thalamocortical Relay Neurons and a Minimal **Integrate-and-Fire-or-Burst** Model, J Neurophys 2000, Gregory D. Smith, Charles L. Cox, S. Murray Sherman, and John Rinzel

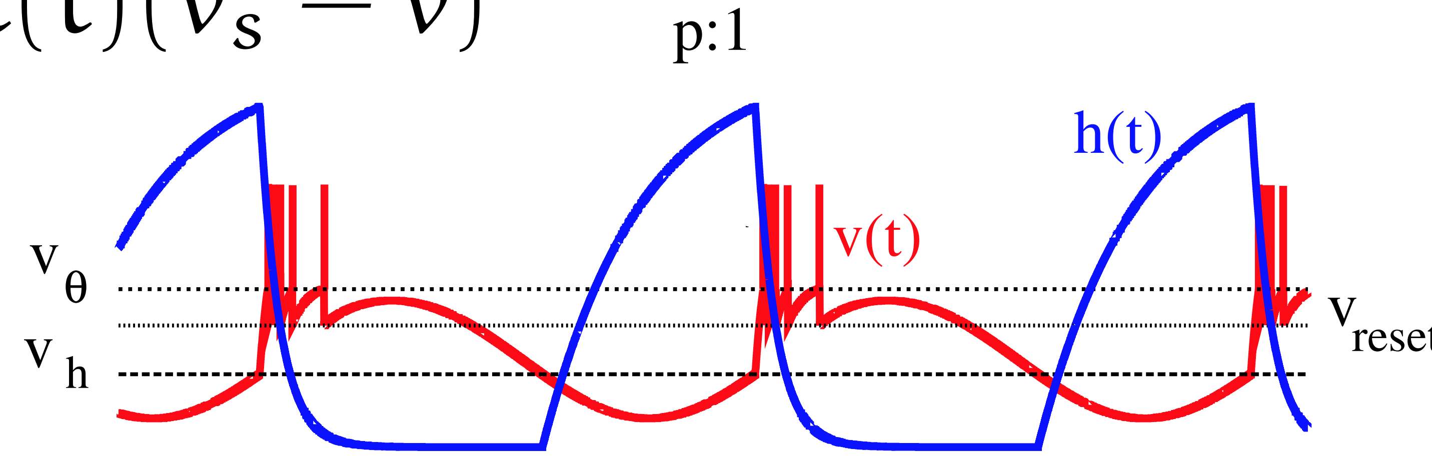


Spindle waves
in thalamic slices
mm/s

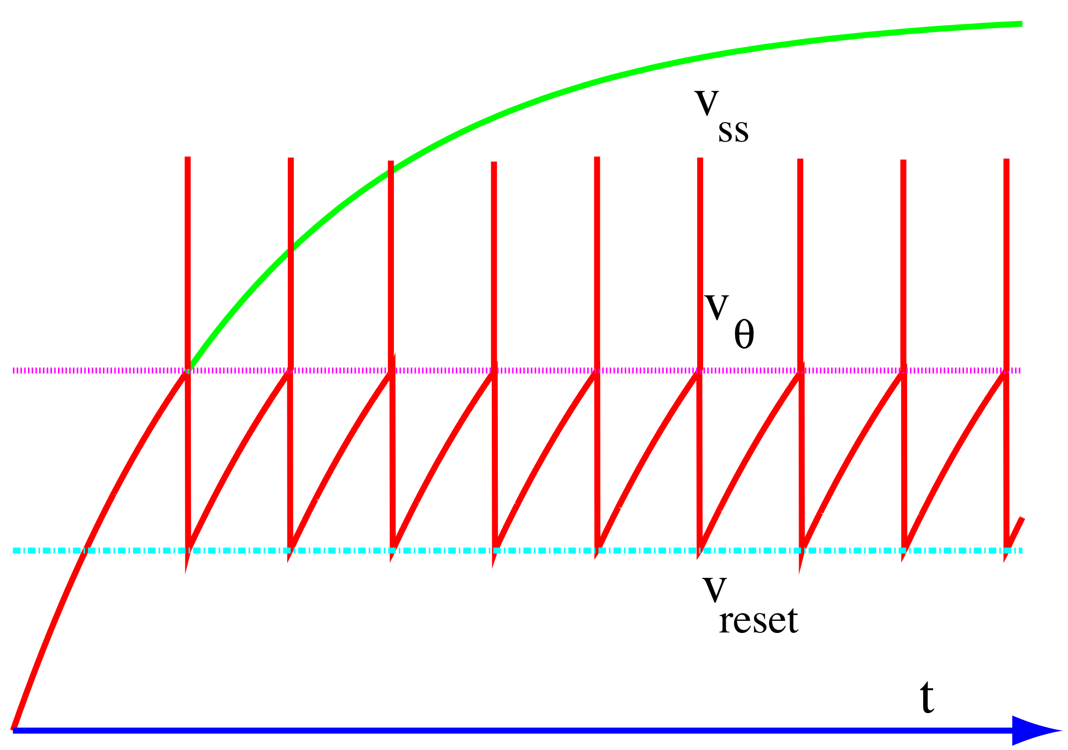
From spike to rate (phenomenology)

$$\dot{v} = -\frac{v}{\tau} + gh\Theta(v - v_h) + u(t)(v_s - v)$$

$$\dot{h} = \begin{cases} -h/\tau_h^- & v \geq v_h \\ (1-h)/\tau_h^+ & v < v_h \end{cases}$$



Slow drive: $$v_{ss}(h,u) = \frac{v_s u + gh\chi}{\tau^{-1} + u}$$ $$\chi \in \{0, 1\}$$

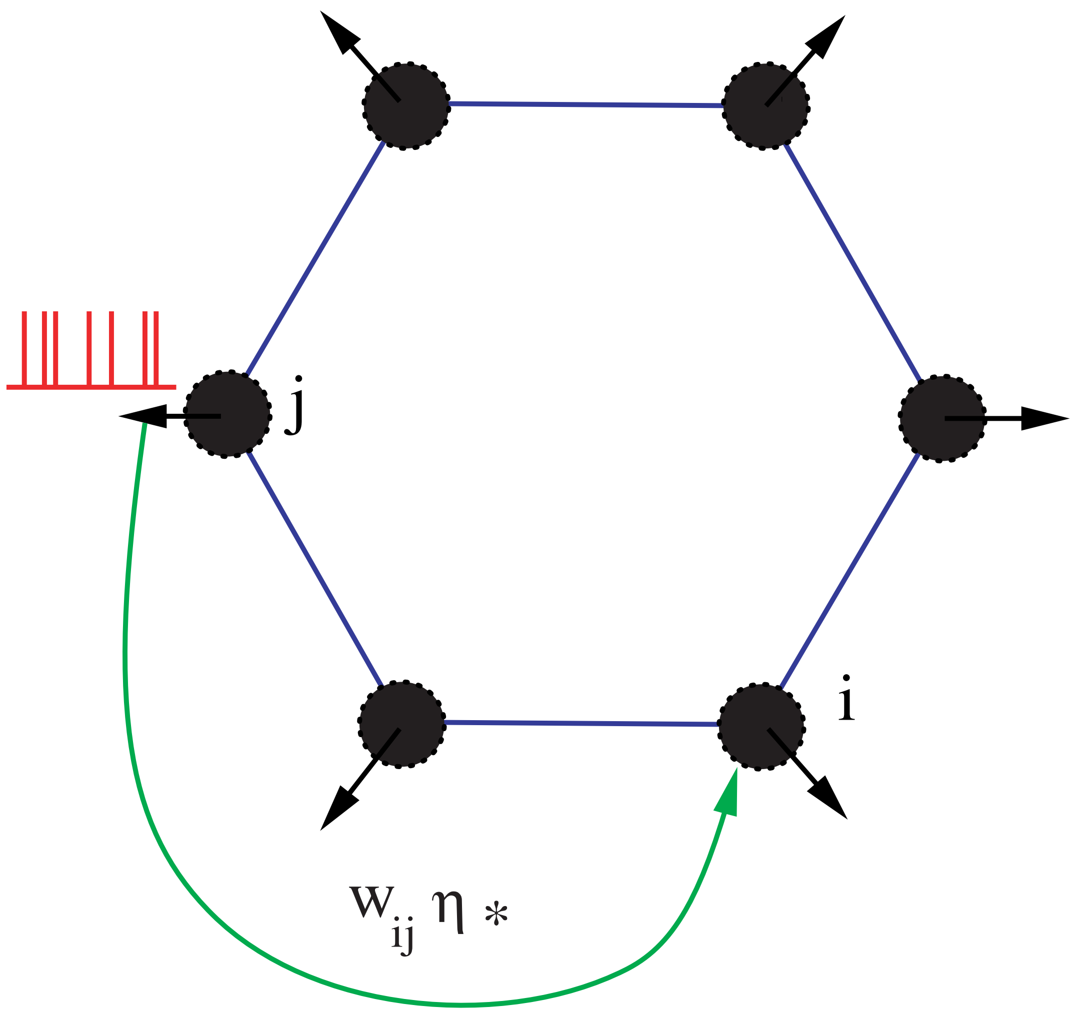


Firing rate : $f(v_{ss}(h,u))$

$$f(v) = \left\{\tau_R + \tau \log\left[\frac{v - v_{reset}}{v - v_\theta}\right]\right\}^{-1} \Theta(v - v_\theta)$$

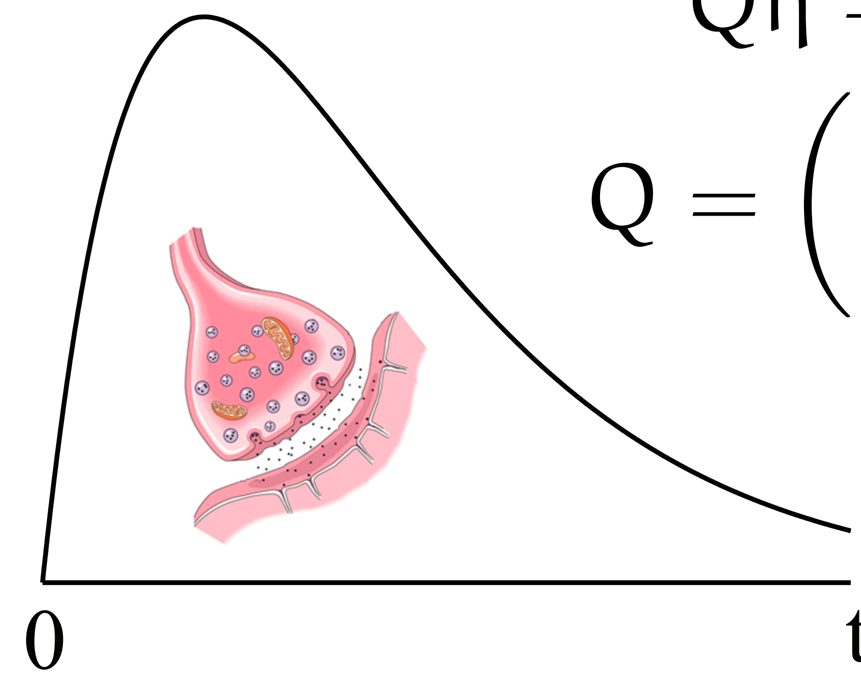
$$\rightarrow \frac{1}{\tau_R}\Theta(v - v_\theta)$$

$$u_i(t) = \epsilon \sum_j w_{ij} \sum_m \eta(t - T_j^m) = \epsilon \sum_j w_{ij} \int_0^\infty \eta(s) \sum_m \delta(s - t + T_j^m) ds$$



$$Q\eta = \delta$$

$$Q = \left(1 + \frac{1}{\alpha}\frac{d}{dt}\right)^2$$

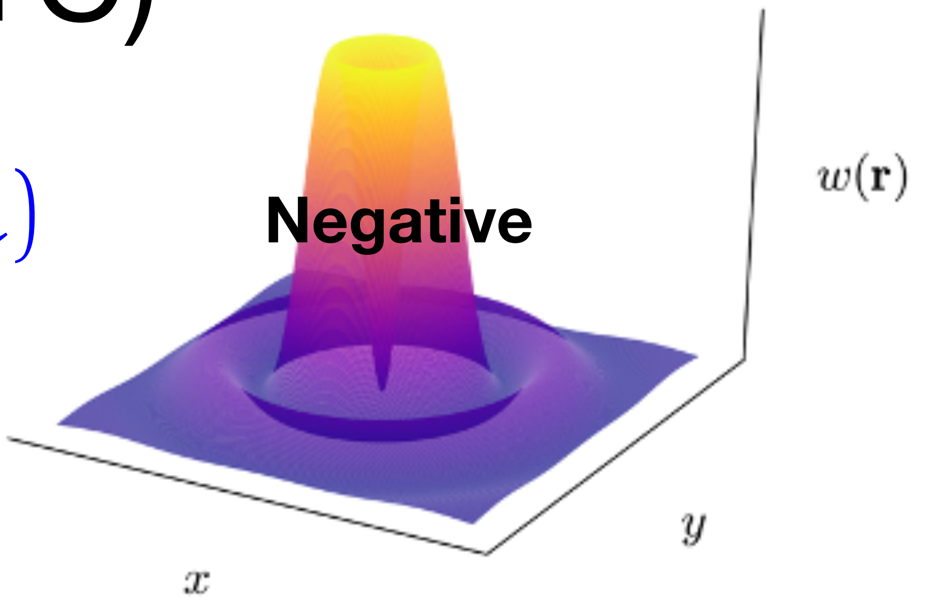


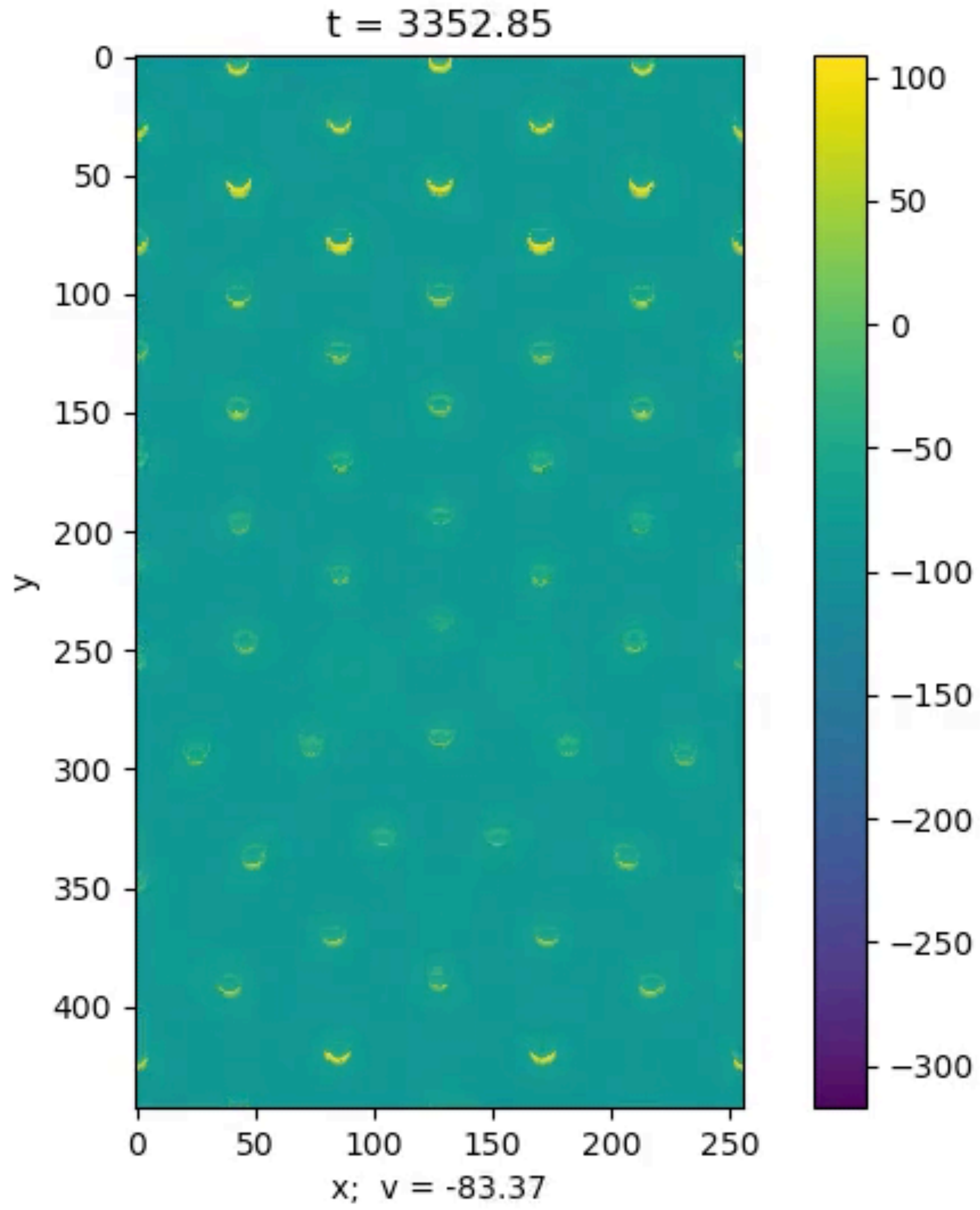
$$\eta(t) = \alpha^2 t e^{-\alpha t}$$

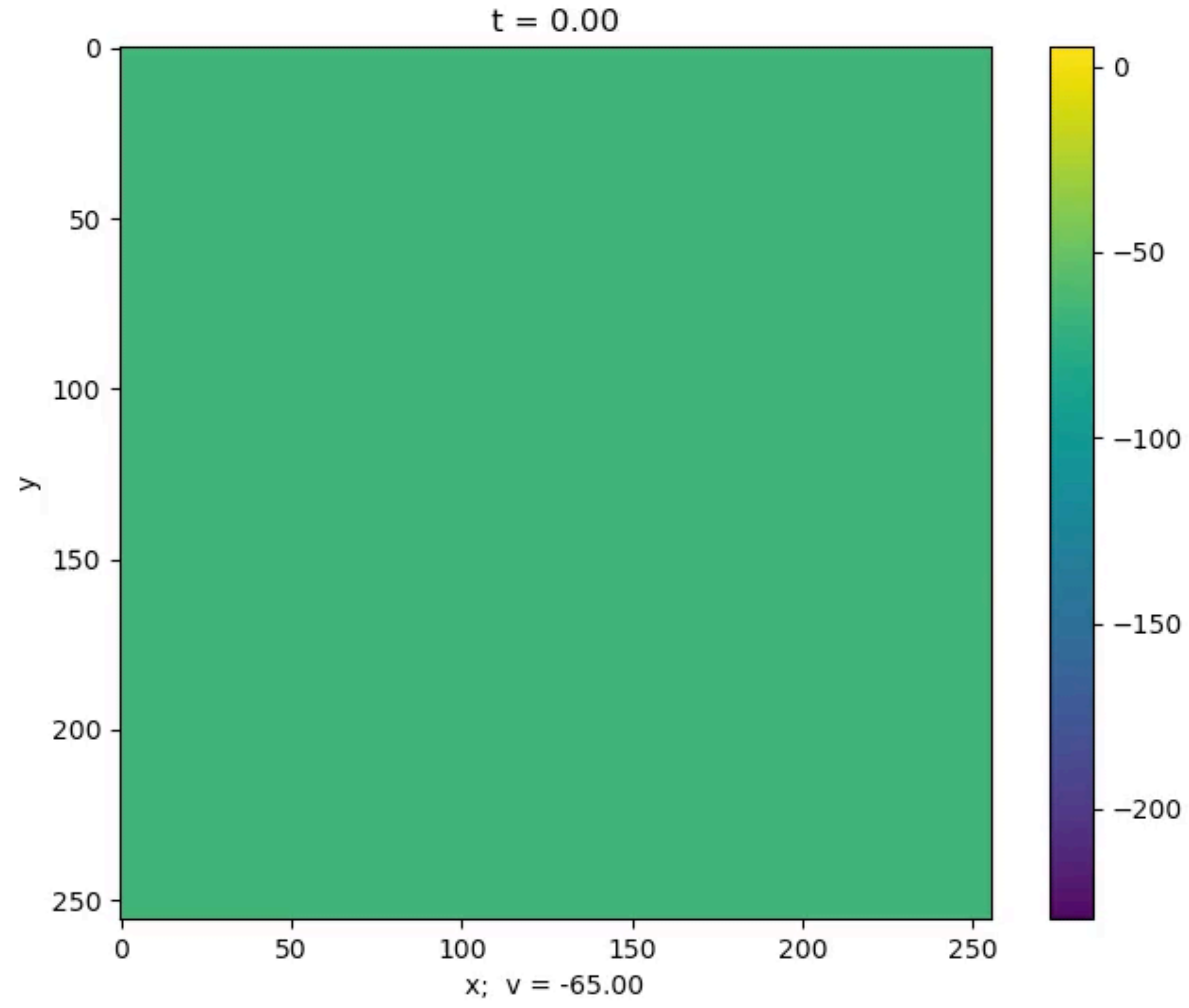
$$Qu_i(t) = \epsilon \sum_j w_{ij} f(v_j(t))$$

Continuum simulations (TC-TC)

$$\mathrm{Qu} = w \otimes \mathrm{f}(v); \quad (\mathrm{small})v_t = \mathrm{F}(v, \mathrm{h}, \mathrm{u})$$







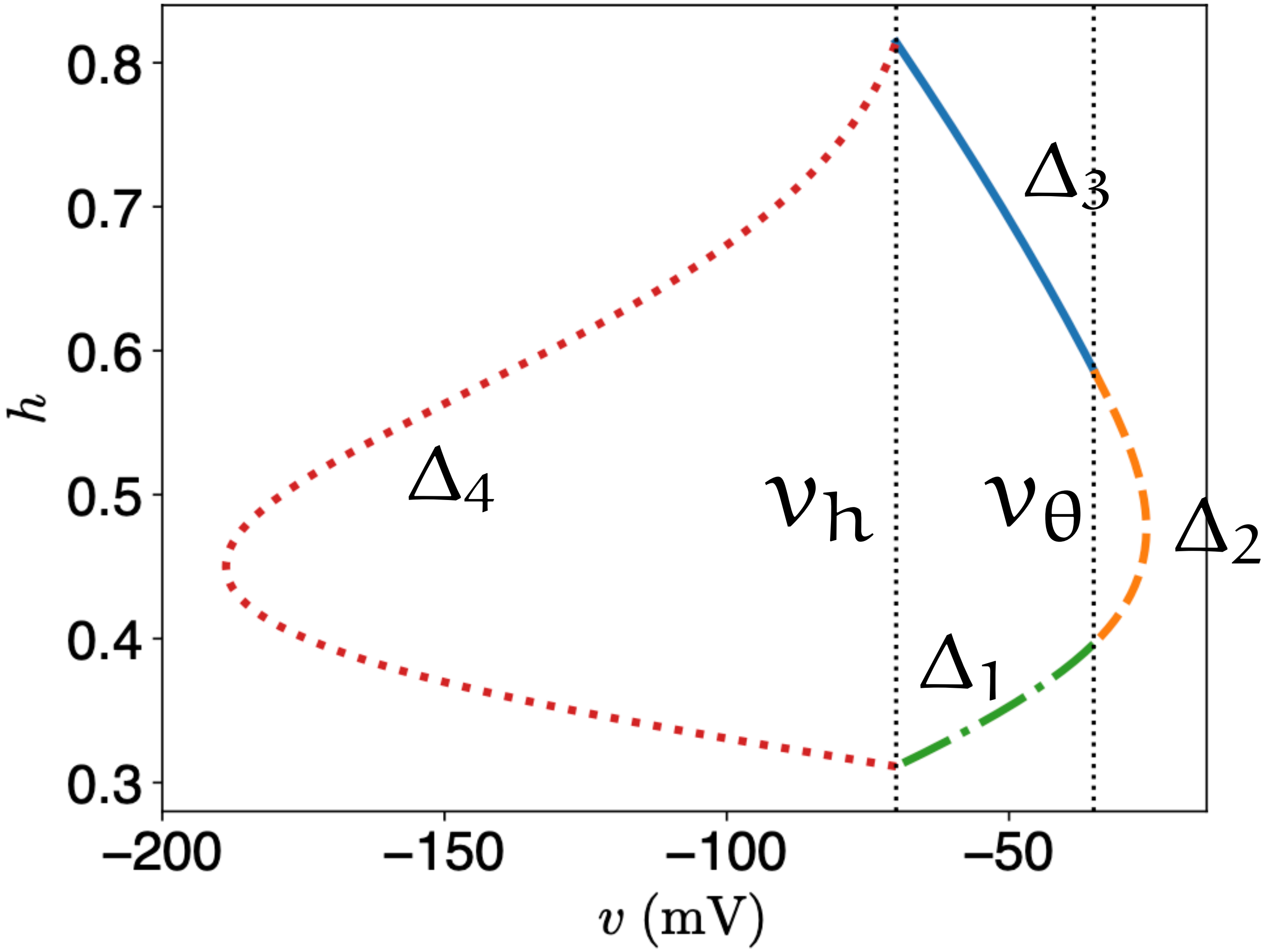
Understanding patterning (1D)

State vector: $z = (v, h, u, u') \in \mathbb{R}^4$

Synchrony = homogeneous oscillation: $z(x,t) = z(t) = z(t + \Delta)$

… in case you didn't spot it yet - this system is PWL

$$z_t = Jz + I$$



Patching / switching

$$v(T_1) = v_\theta$$
$$v(T_2) = v_\theta$$
$$v(T_3) = v_h$$
$$v(T_4) = v_h$$

Times of flight Δ_i

Time of events T_i

A reminder of the model

$$C v_t = -g_L(v - v_L) - g_T h \Theta(v - v_h) - g_{syn} u,$$
$$u_t = \alpha(-u + r),$$
$$r_t = \alpha\left(-r + \int_{-\infty}^{\infty} w(x,y) f(v(y,t))dy\right),$$
$$h_t = \frac{h_\infty(v) - h}{\tau_h(v)},$$

$$z = (v, u, r, h)$$

$$K(T) = \mathbb{I}_4 + \frac{1}{\dot{v}(T^-)}\begin{pmatrix} \dot{v}^+ - \dot{v}^- & 0 & 0 & 0 \\ \dot{u}^+ - \dot{u}^- & 0 & 0 & 0 \\ \frac{\alpha}{\tau_R}\widehat{w}(k) & 0 & 0 & 0 \\ \dot{h}^+ - \dot{h}^- & 0 & 0 & 0 \end{pmatrix}$$

$$\widehat{w}(k) = \text{FT}[w]$$

$$\delta z(x,t) = \delta z(t) e^{ikx}$$

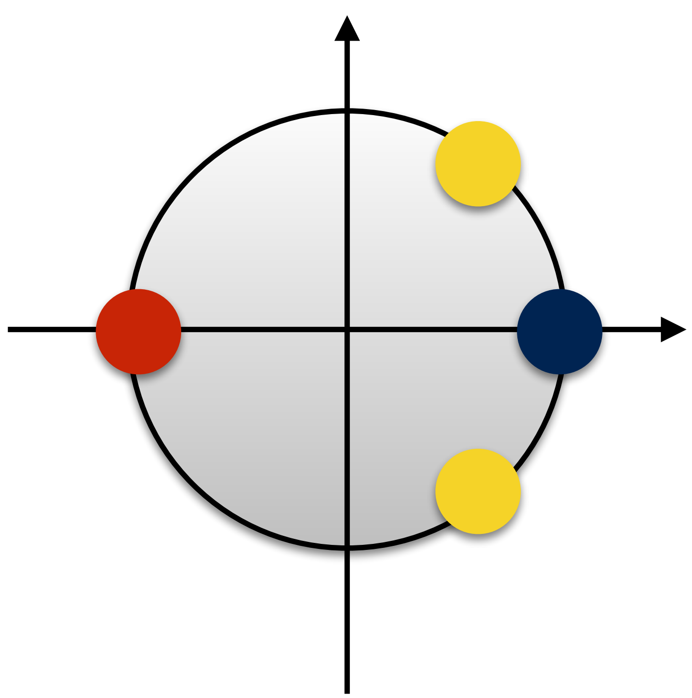
Putting it all together - monodromy

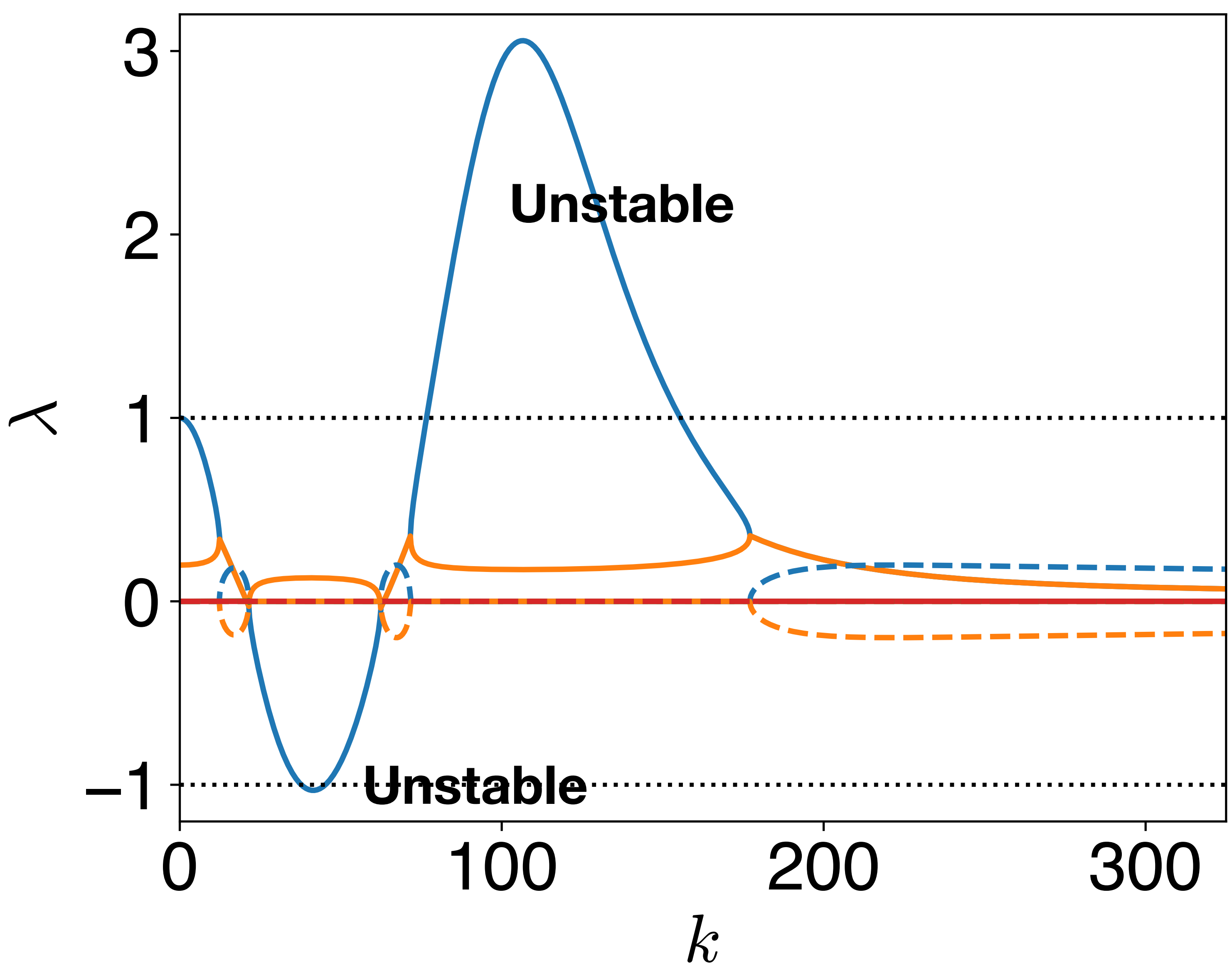
$$\delta z(\Delta) = \Psi \delta z(0)$$

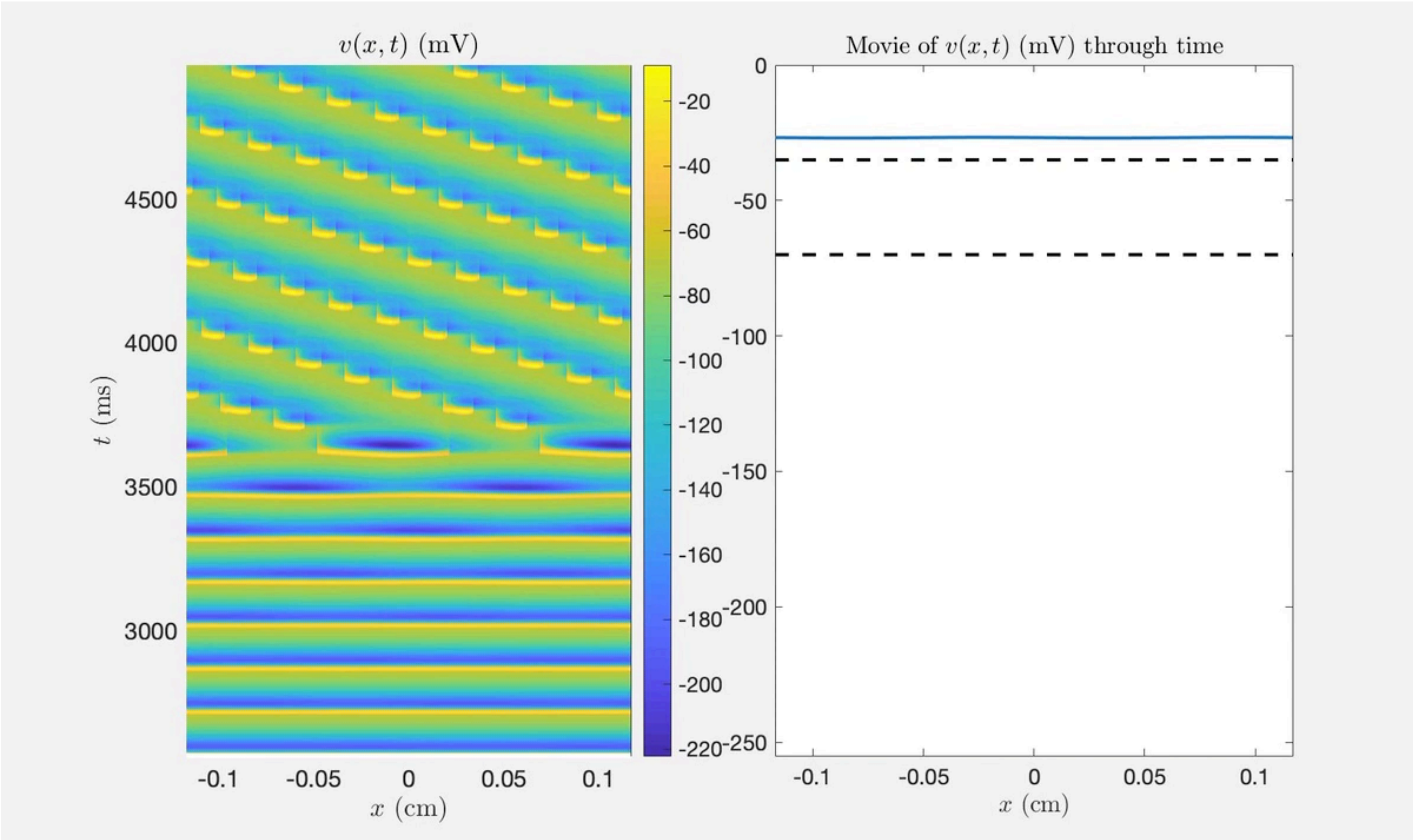
$$\Psi(k) = K(T_4)\exp(J_4\Delta_4)K(T_3)\exp(J_3\Delta_3)K(T_2)\exp(J_2\Delta_2)K(T_1)\exp(J_1\Delta_1)$$

saltate.propagate … saltate.propagate

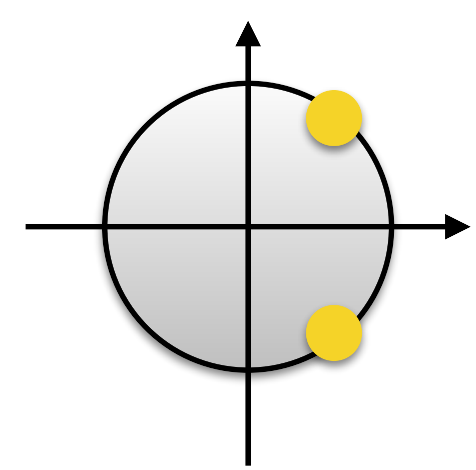
Eigenvalues of Ψ





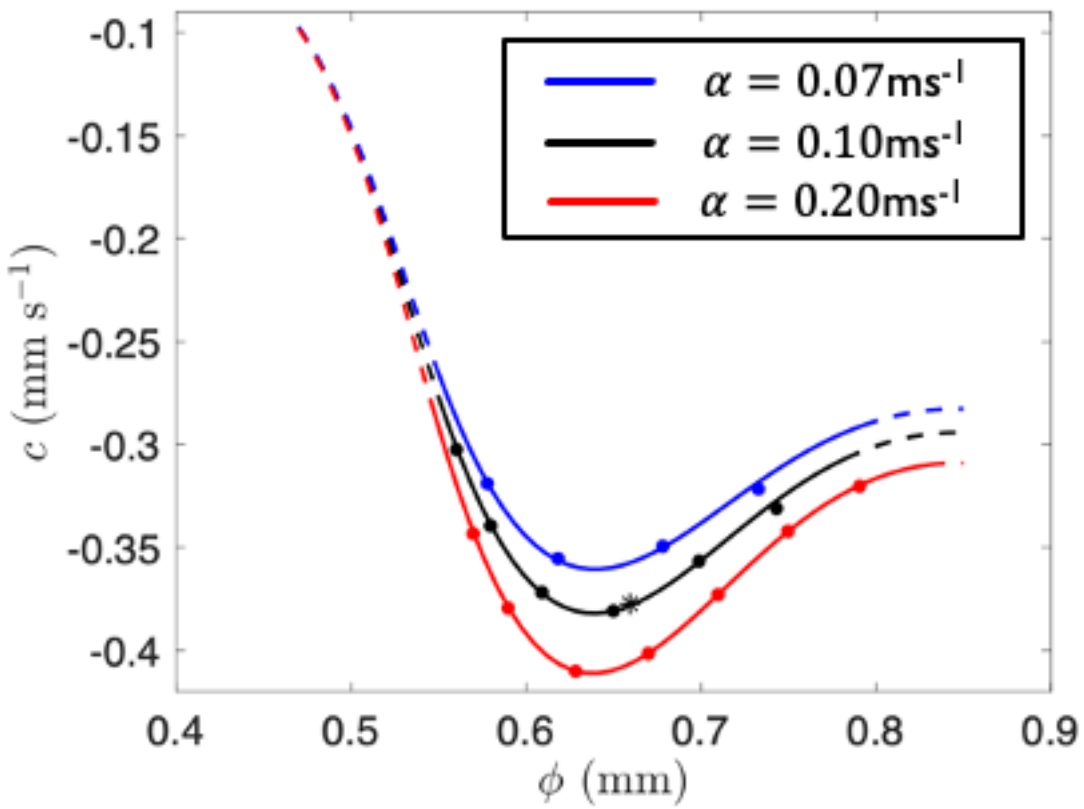


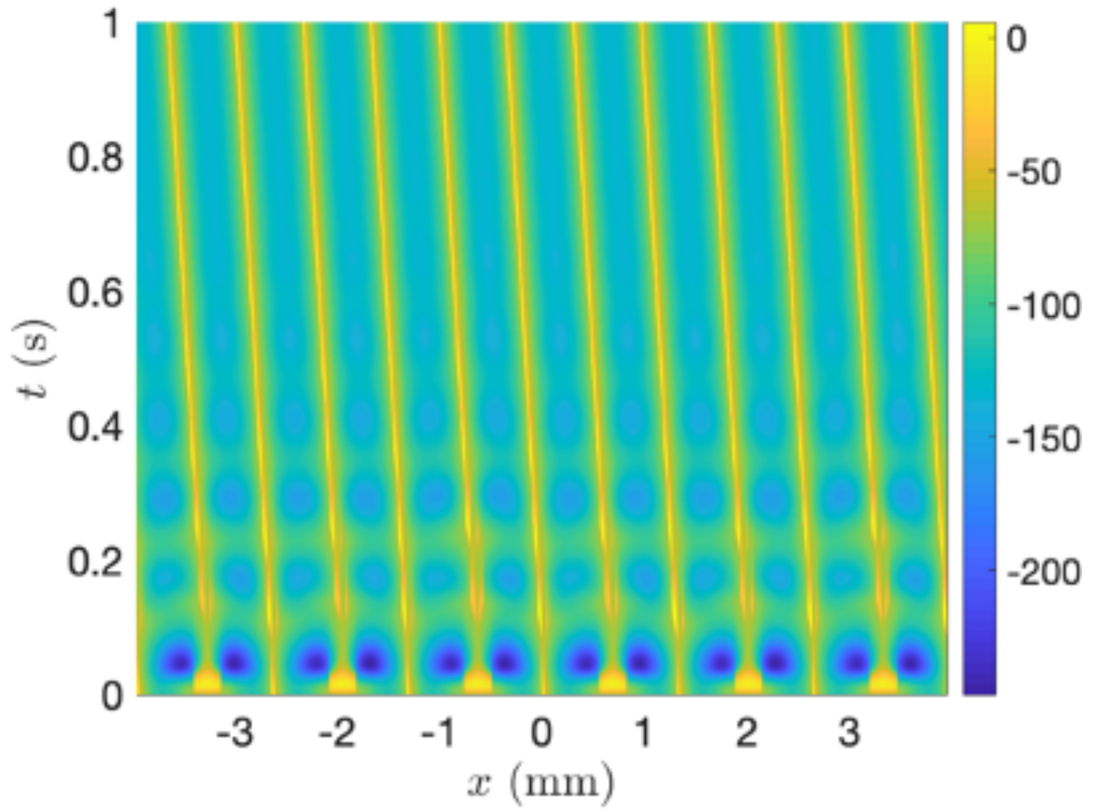
Lurching via Neimark-Sacker



Rinzel, Terman, Wang, Ermentrout 1998 Propagating activity patterns in large-scale inhibitory neuronal networks, Science 279: 1351-1355.

Periodic travelling waves - similar





Dispersion curve for speed (c) and period (ϕ)

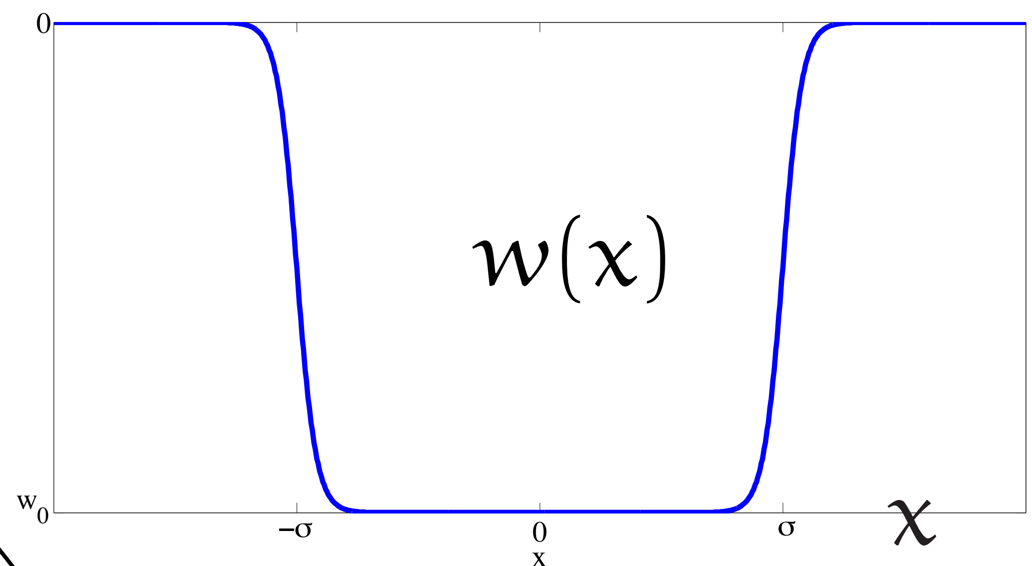
Stability via Evans function

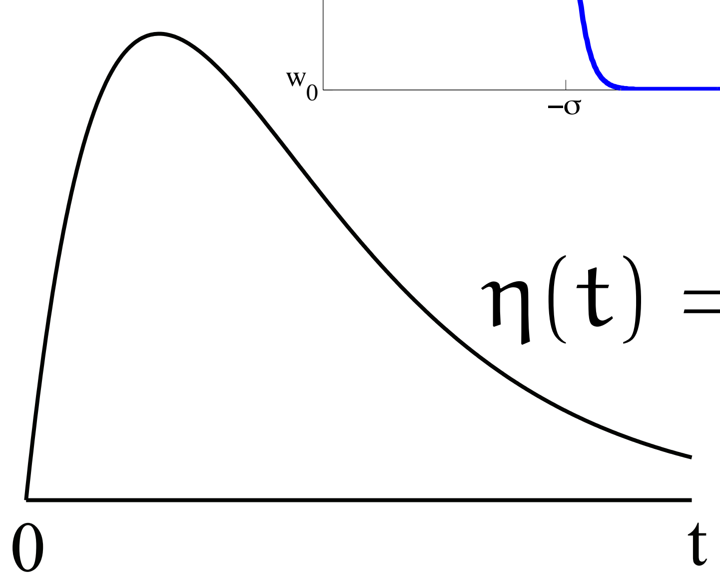
S Modhara, Y-M Lai, R Thul and S Coombes 2021 Neural fields With rebound currents: Novel routes to patterning, SIAM Journal on Applied Dynamical Systems, Vol 20, 1596–1620.

IF spiking model with an h current

$$C\frac{\partial}{\partial t}V(\mathbf{r},t) = -g_l V(\mathbf{r},t) + g_h n(\mathbf{r},t) + I_{syn}(\mathbf{r},t) + I_{hd}(\mathbf{r},t)$$

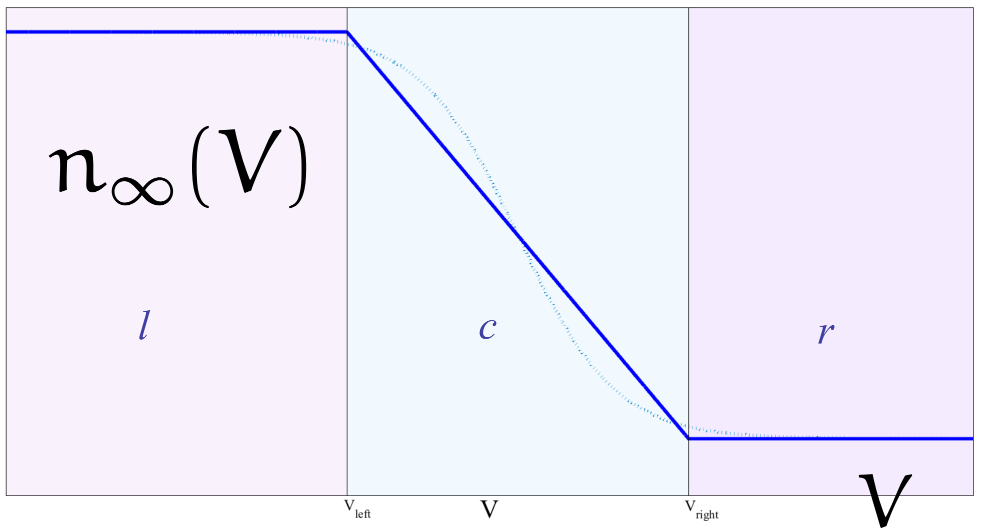
$$I_{syn}(\mathbf{r},t) = \int_{\mathbb{R}^2} w(|\mathbf{r}-\mathbf{r}'|) \sum_{m\in\mathbb{Z}} \eta(t - T^m(\mathbf{r}'))d\mathbf{r}'$$



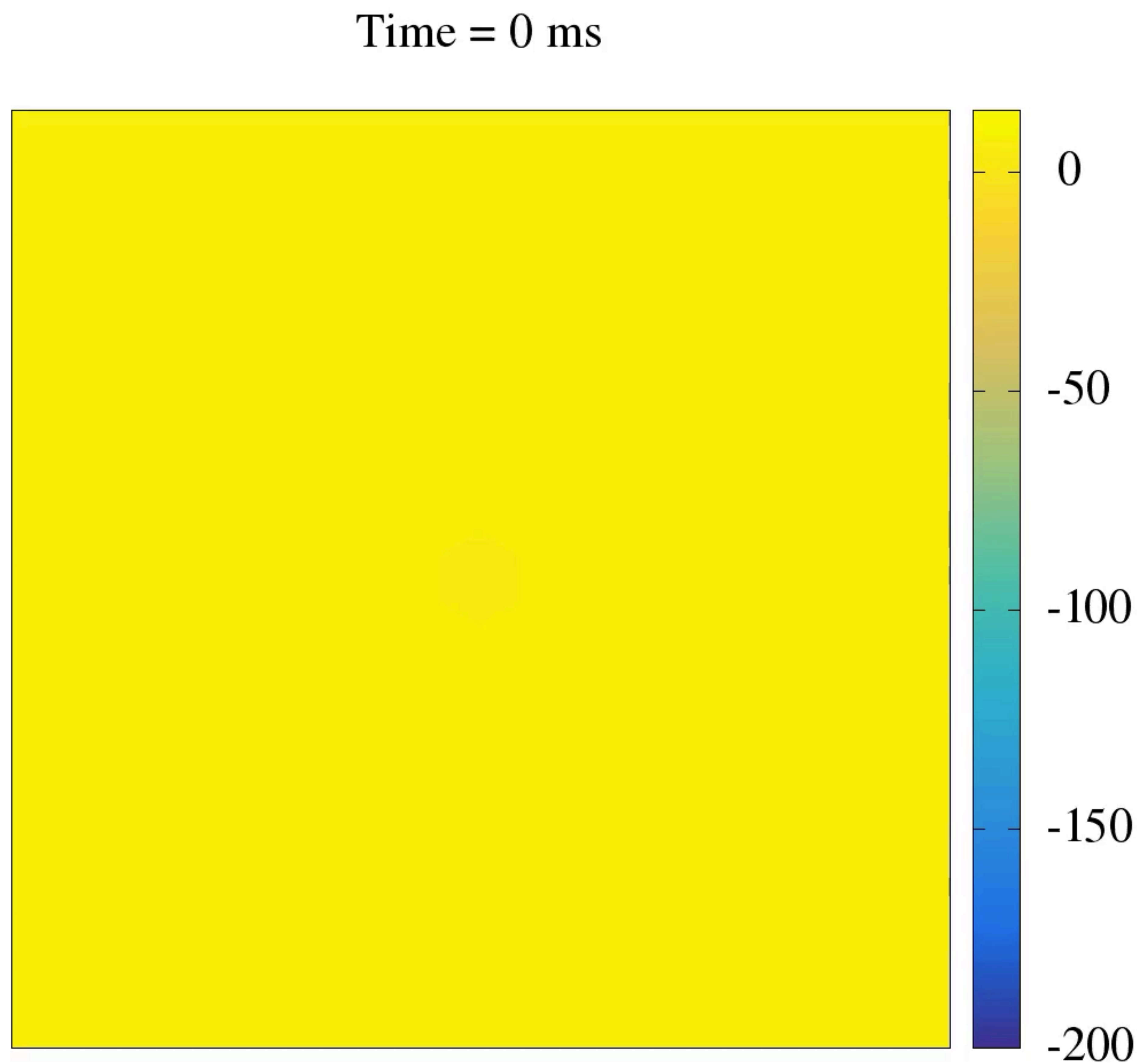


$$\eta(t) = \alpha^2 t e^{-\alpha t}$$

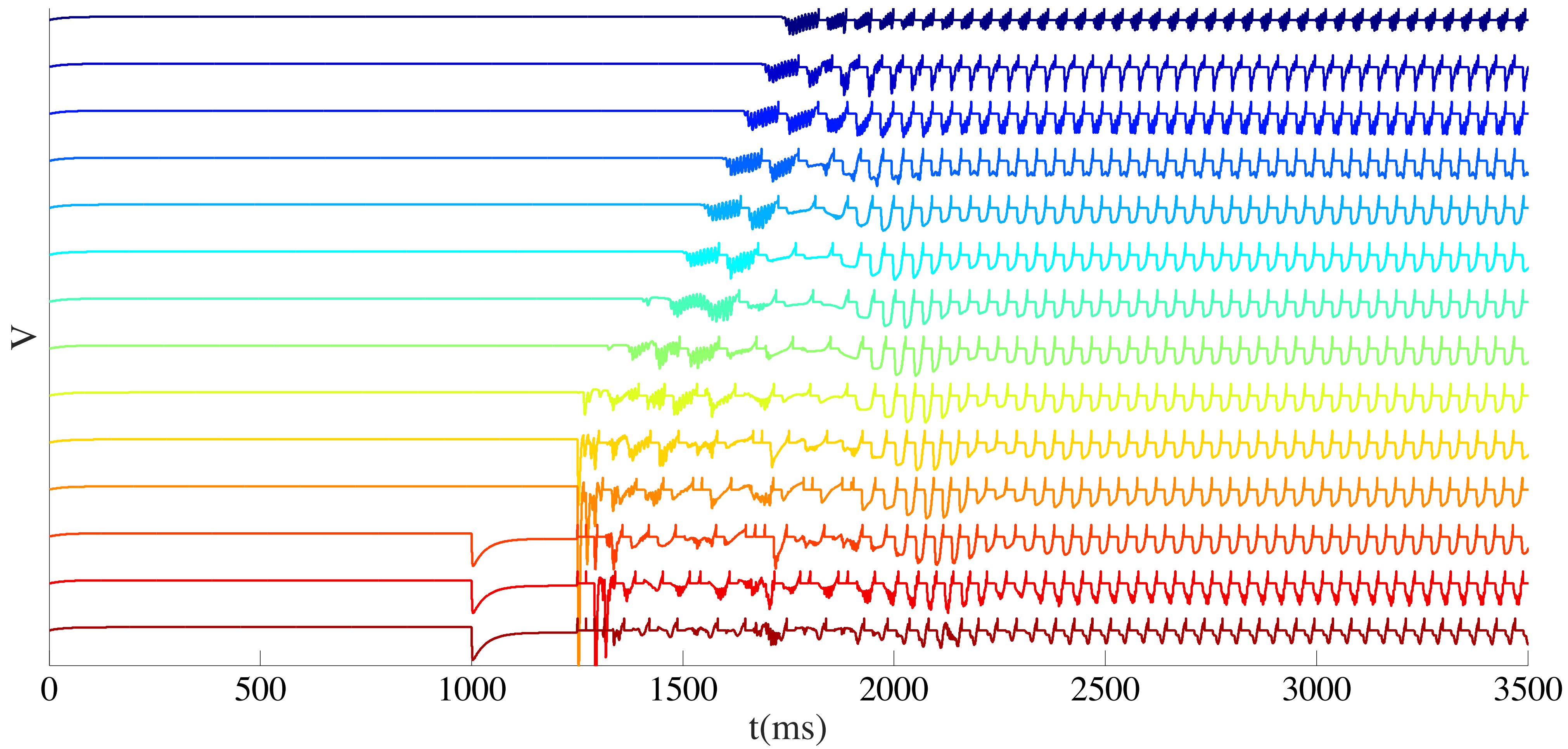
$$\tau_h \frac{dn}{dt} = n_\infty(V) - n$$

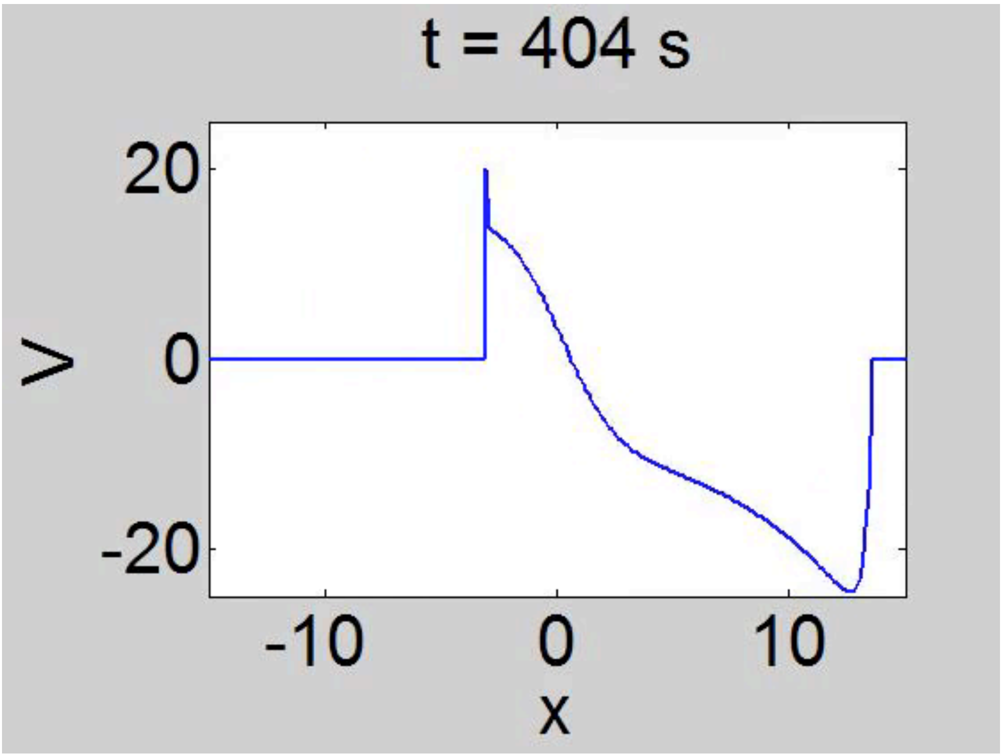


Cool 2D dynamics!



Pattern analysis 1D



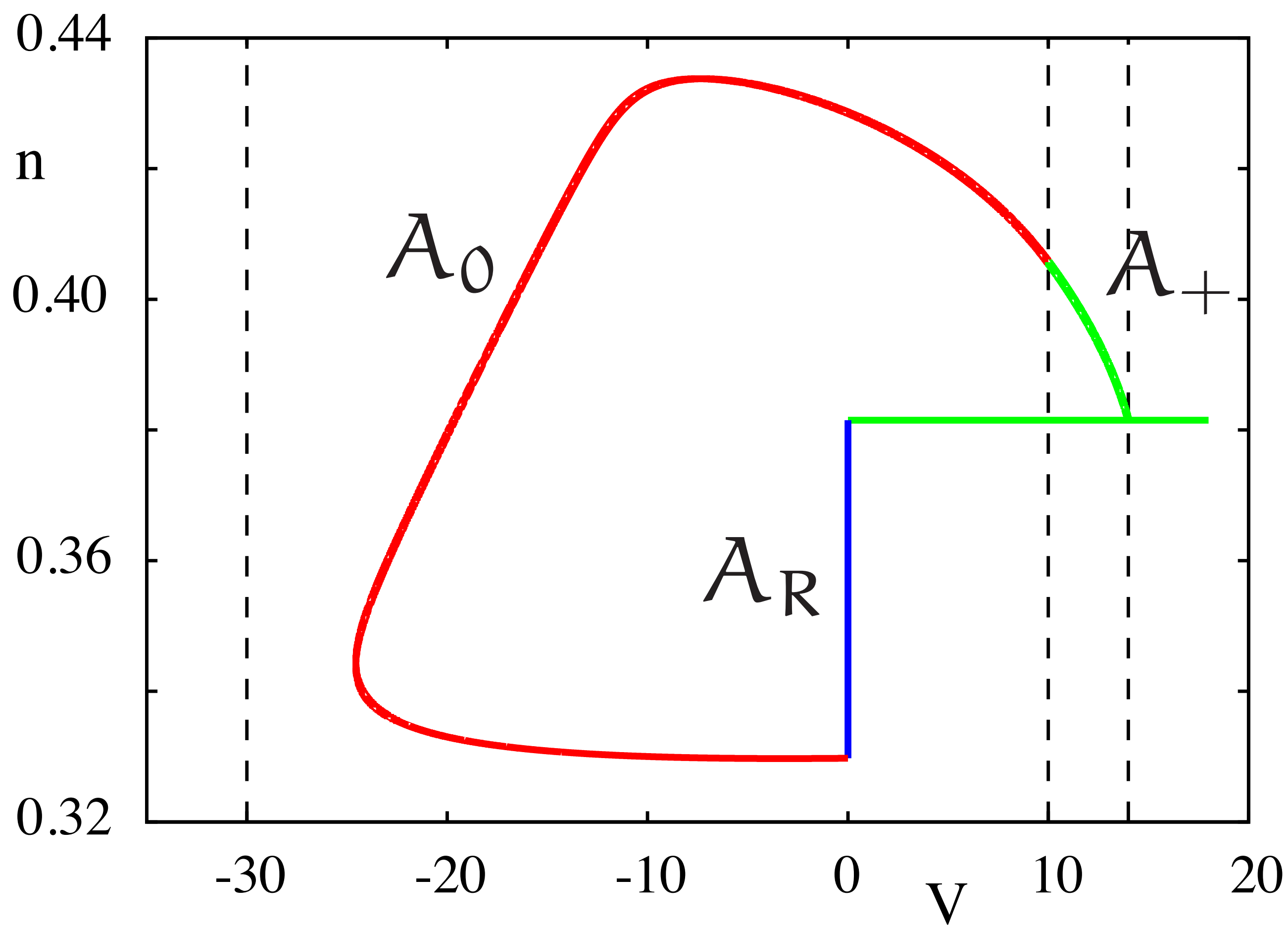


Theory (existence and stability)

Exploit **pwl** nature of model

$$X = (V, n_h) \in \mathbb{R}^2$$

$$\frac{\partial}{\partial t} X(x,t) = AX(x,t) + \Psi(x,t),$$



$$T^{m-1}(x) \le t < T^m(x)$$

TW frame $\xi = t - x/c$

Stationary solution $X \to Q(\xi)$

$$\frac{dQ}{d\xi} = AQ(\xi) + \widehat{\Psi}(\xi)$$

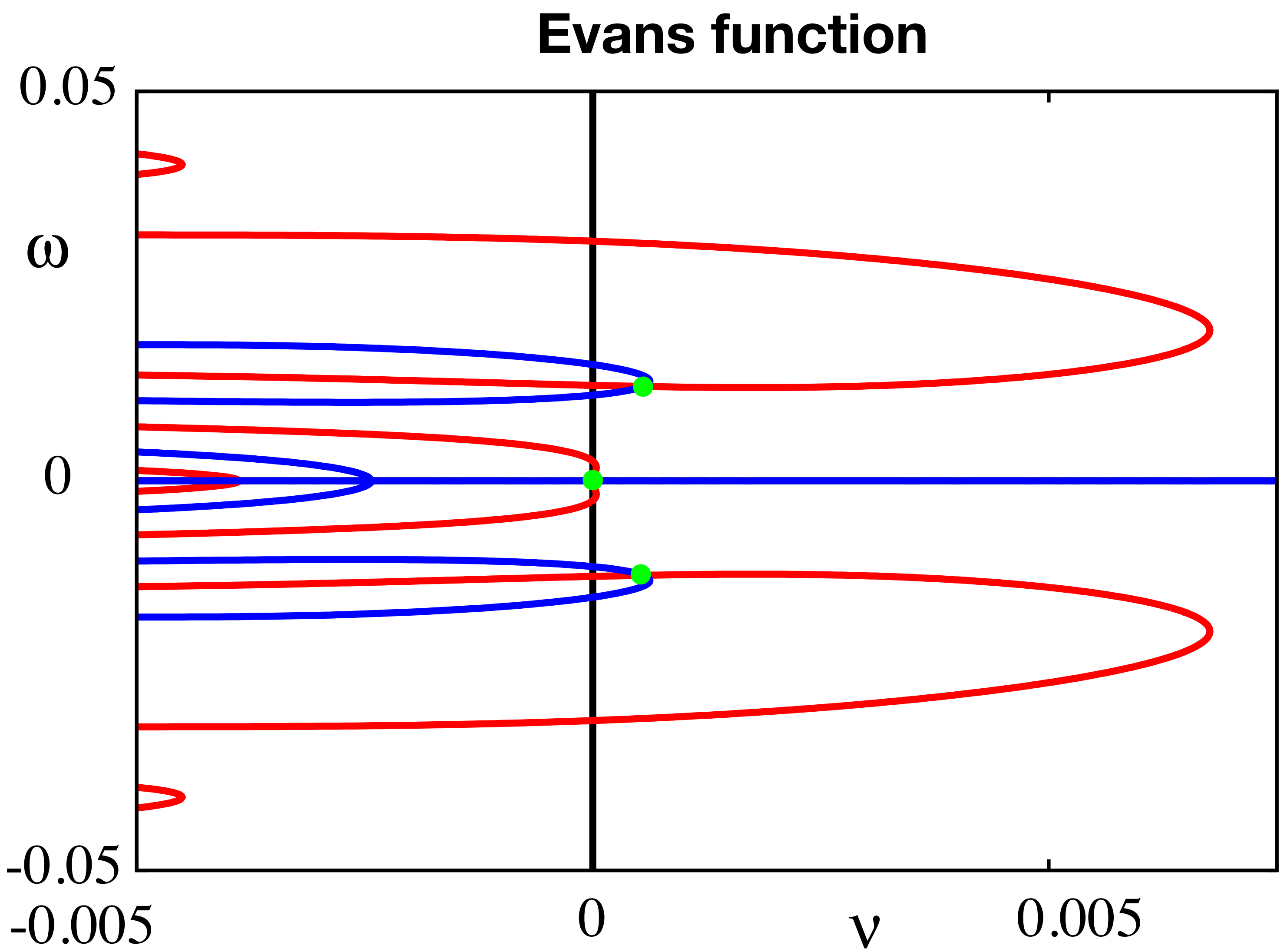
$$\widehat{\Psi}(\xi) = c \sum_{m \in \mathbb{Z}} \int_0^\infty ds\, \eta(s) w(|c(s-\xi) + cm\Delta|)$$

be mindful of nonsmooth dynamics - *switch, fire, refract.*

Saltation rule at event time T :

$$\exp(A(T - t_0)) \to K(T)\exp(A(T - t_0))$$

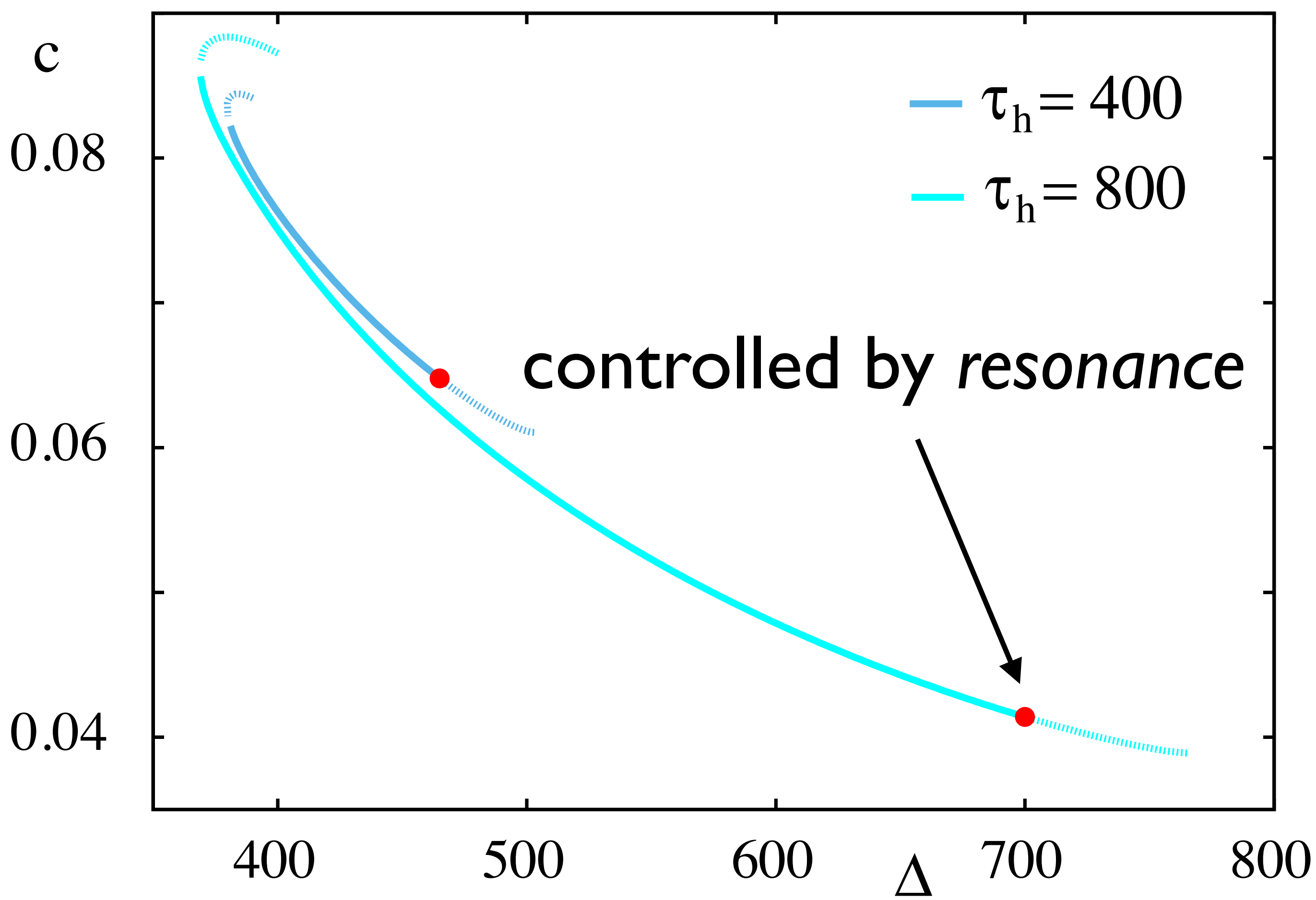
$$K(T) = Dg(\mathbf{X}(T^-)) + \frac{[\dot{\mathbf{X}}(T^+) - Dg(\mathbf{X}(T^-))\dot{\mathbf{X}}(T^-)][\nabla_{\mathbf{X}} h(\mathbf{X}(T^-))]^\top}{\nabla_{\mathbf{X}} h(\mathbf{X}(T^-)) \cdot \dot{\mathbf{X}}(T^-)}$$



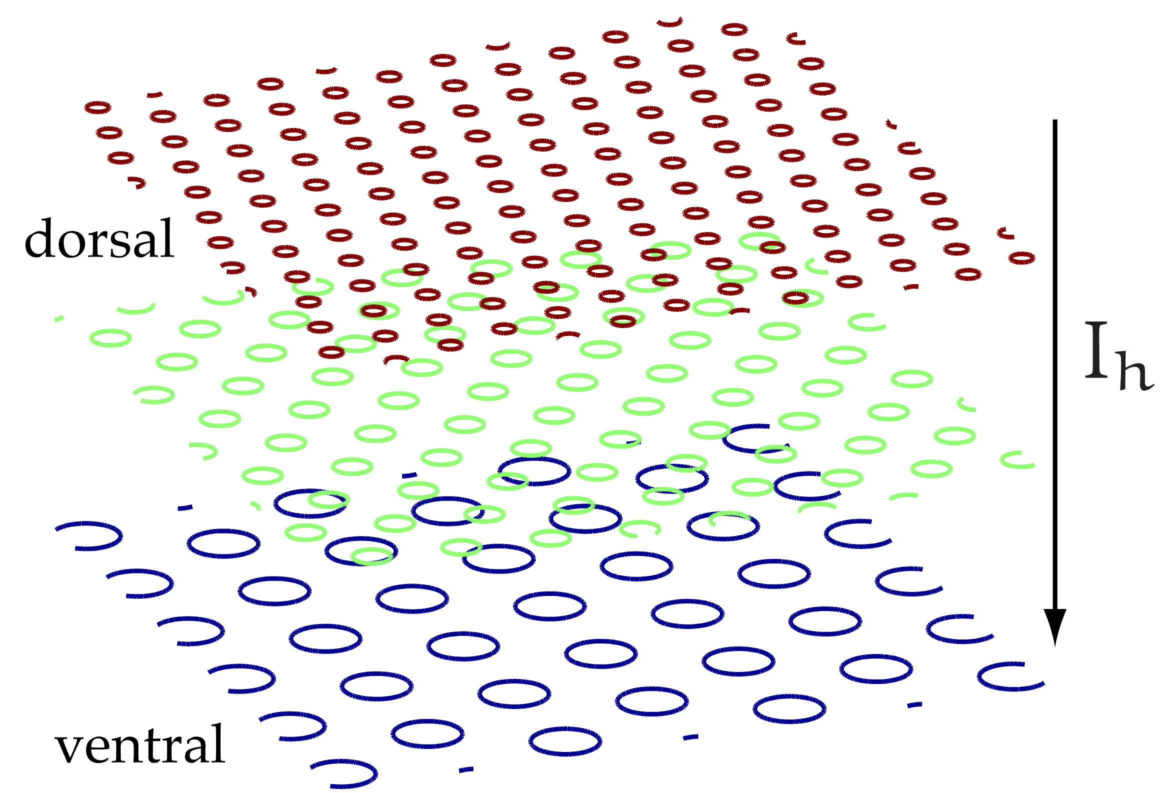
h: indicator function for event

g: rule for action at event

Local control by I_h



Spatial scale *not so* hardwired



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Journal of Mathematical Neuroscience, 7:9

Next: Modelling large scale brain dynamics