Travelling waves in biology: Lecture 2

1 The spatially extended FitzHugh-Nagumo model

The following system of PDEs is the FitzHugh-Nagumo caricature of the Hodgkin-Huxley equations modelling the nerve impulse propagation along an axon:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + f(v) - u$$
$$\frac{\partial u}{\partial t} = \beta v$$

where $v(x,t)$ represents the membrane potential and $u(x,t)$ is a phenomenological *recovery* variable; $f(v) = v(\alpha - v)(v-1)$, $1 > \alpha > 0$, $\beta > 0$, $-\infty < x < \infty$, $t > 0$. Travelling waves are solutions of the form

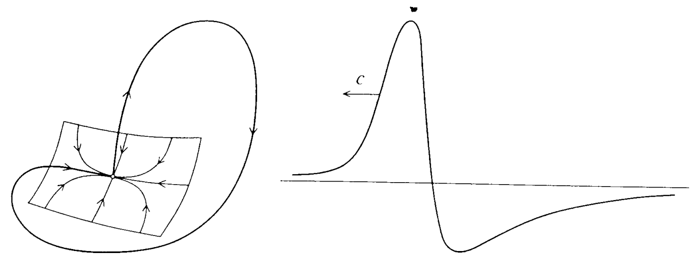
$$v(x,t) = V(\xi), \qquad u(x,t) = U(\xi), \qquad \xi = x + ct$$

for some unknown c. They satisfy the following first order system of ODEs

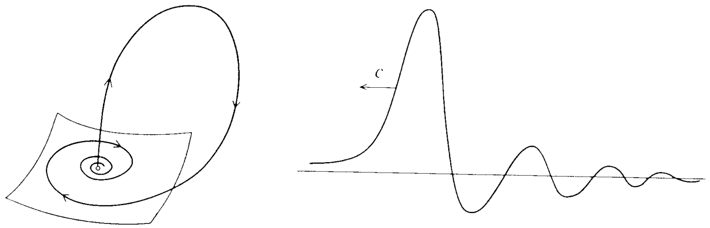
$$\dot{V} = W$$
$$\dot{W} = cW - f(V) + U$$
$$\dot{U} = \frac{\beta}{c} V$$

where the dot denotes differentiation with respect to ξ. Any bounded orbit corresponds to a travelling wave such that $c = c(\alpha, \beta)$.

For all $c > 0$ the wave system has a unique equilibrium at $(V, W, U) = (0, 0, 0)$ with one positive eigenvalue λ_1 and two eigenvalues $\lambda_{2,3}$ with negative real parts. (To show this first verify this assuming the eigenvalues are real. Then show that the characteristic equation cannot have roots on the imaginary axis, and finally, use the continuous dependence of the eigenvalues on the parameters). The equilibrium can either be a saddle or a saddle-focus with a 1D unstable and a 2D stable manifold. The transition between saddle and saddle-focus is caused by the presence of a double negative eigenvalue.

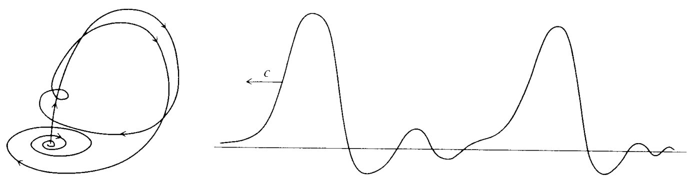


(a)



(b)

Impulses with (a) monotone and (b) oscillating tails.



A double pulse.

The piecewise linear FitzHugh-Nagumo model

Consider the FitzHugh-Nagumo system in the form

$$\epsilon \frac{\partial v}{\partial t} = \epsilon^2 \frac{\partial^2 v}{\partial x^2} + f(v,w)$$
$$\frac{\partial w}{\partial t} = g(v,w)$$

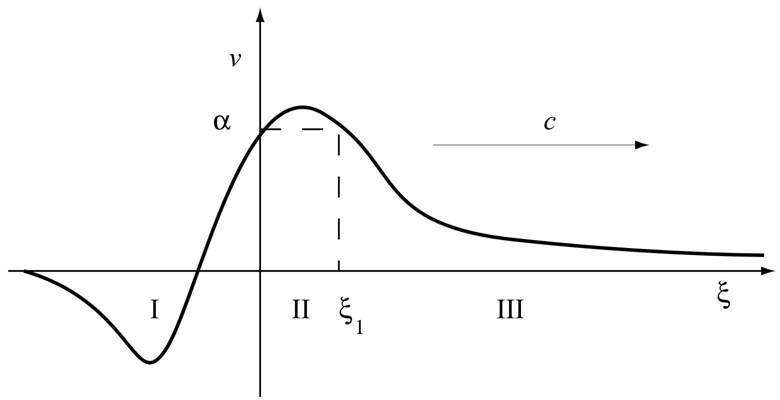
so that travelling waves are specified by the ODEs:

$$\epsilon^2 v_{\xi\xi} + c\epsilon v_\xi + f(v,w) = 0$$
$$c w_\xi + g(v,w) = 0$$

where $\xi = x - ct$. Analysis of the wave and its speed is possible for the piecewise linear dynamics

$$f(v,w) = \Theta(v-\alpha) - v - w, \qquad g(v,w) = v$$

where $\Theta(x)$ is a step function such that $\Theta(x) = 1$ if $x \geq 0$ and is zero otherwise. Consider solutions of the form shown in the figure with $\lim_{\xi\to\pm\infty} v(\xi) = 0$.



Regions I, II and III are defined respectively by $\xi < 0$, $0 < \xi < \xi_1$ and $\xi_1 < \xi$, and in the travelling coordinate frame $v(0) = v(\xi_1) = \alpha$. Note that ξ_1 and c are undetermined. In regions I and III look for solutions of the form $v = A\exp(\lambda\xi)$ and $w = B\exp(\lambda\xi)$. This gives the characteristic equation

$$p(\lambda) = \epsilon^2\lambda^3 + c\epsilon\lambda^2 - \lambda + 1/c \equiv \epsilon^2(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$$

Since $p(0) > 0$ then at least one root (say λ_1) is real and negative. If the remaining two roots $\lambda_{2,3}$ are real they are positive. (All this can be seen from a plot of $p(\lambda)$). In region II $f(v, w) = 1 - v - w$ so that

$$\epsilon^2 v_{\xi\xi} + c\epsilon v_\xi - v - w = -1$$
$$c w_\xi + v = 0$$

The inhomogeneous solution solution is $(v, w) = (0, 1)$. We now look for general solutions of the form *inhomogeneous solution* + *homogeneous solution* which decay at $\pm\infty$. A solution with this form is

$$w(\xi) = \begin{cases} Ae^{\lambda_1\xi} & \xi \geq \xi_1 \\ 1 + \sum_{i=1}^{3} B_i e^{\lambda_i\xi} & 0 \leq \xi \leq \xi_1 \\ \sum_{i=2}^{3} C_i e^{\lambda_i\xi} & \xi \leq 0 \end{cases}$$

where $v = -c w_\xi$. Note that although the solution and its first derivative are required to be continuous there must be a jump in $v_{\xi\xi}$ (due to the nature of the nonlinearity $f(v, w)$) at $\xi = 0$ and $\xi = \xi_1$ where $(v = \alpha)$. In fact

$$v_{\xi\xi}\big|_{0^-}^{0^+} = -1 \qquad \text{and} \qquad v_{\xi\xi}\big|_{\xi_1^-}^{\xi_1^+} = +1$$

Continuity of the solution and its first derivative gives

$$v(0^+) = v(0^-) = \alpha \qquad v(\xi_1^+) = v(\xi_1^-) = \alpha$$
$$v_\xi(0^+) = v_\xi(0^-) \qquad v_\xi(\xi_1^+) = v_\xi(\xi_1^-)$$
$$v_{\xi\xi}(0^+) = v_{\xi\xi}(0^-) - 1 \qquad v_{\xi\xi}(\xi_1^+) = v_{\xi\xi}(\xi_1^-) + 1$$

At $\xi = 0$, and using $v = -cw_\xi$ gives

$$\begin{aligned}
\lambda_1 B_1 + \lambda_2(B_2 - C_2) + \lambda_3(B_3 - C_3) &= 0 \\
\lambda_1^2 B_1 + \lambda_2^2(B_2 - C_2) + \lambda_3^2(B_3 - C_3) &= 0 \\
\lambda_1^3 B_1 + \lambda_2^3(B_2 - C_2) + \lambda_3^3(B_3 - C_3) &= 1/c
\end{aligned}$$

or $\Lambda(B_1, B_2 - C_2, B_3 - C_3) = (0, 0, 1/c)$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix}$$

which may be solved using Cramer's rule. Now

$$\begin{aligned}
\det \Lambda &= \lambda_1\lambda_2\lambda_3 \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix} = \lambda_1\lambda_2\lambda_3 \begin{vmatrix} 1 & 0 & 0 \\ \lambda_1 & \lambda_2 - \lambda_1 & \lambda_3 - \lambda_1 \\ \lambda_1^2 & \lambda_2^2 - \lambda_1^2 & \lambda_3^2 - \lambda_1^2 \end{vmatrix} \\
&= \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \begin{vmatrix} 1 & 0 & 0 \\ \lambda_1 & 1 & 1 \\ \lambda_1^2 & \lambda_2 + \lambda_1 & \lambda_3 + \lambda_1 \end{vmatrix} = \lambda_1\lambda_2\lambda_3(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)
\end{aligned}$$

Introduce

$$\Lambda_1 = \begin{vmatrix} 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_2^2 & \lambda_3^2 \\ 1/c & \lambda_2^3 & \lambda_3^3 \end{vmatrix} = \lambda_2\lambda_3(\lambda_3 - \lambda_2)/c, \qquad \Lambda_2 = \lambda_1\lambda_3(\lambda_1 - \lambda_3)/c, \qquad \Lambda_3 = \lambda_1\lambda_2(\lambda_2 - \lambda_1)/c$$

Remembering that $\epsilon^2 p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, then

$$p'(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) + (\lambda - \lambda_1)(\lambda - \lambda_3) + (\lambda - \lambda_2)(\lambda - \lambda_3)$$

so that

$$\epsilon^2 p'(\lambda_1) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3), \qquad \epsilon^2 p'(\lambda_2) = (\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3), \qquad \epsilon^2 p'(\lambda_3) = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)$$

Finally we may write the solution as

$$\boxed{\begin{bmatrix} B_1 \\ B_2 - C_2 \\ B_3 - C_3 \end{bmatrix} = \frac{1}{\det \Lambda} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{bmatrix} = \frac{1}{c\epsilon^2} \begin{bmatrix} \frac{1}{\lambda_1 p'(\lambda_1)} \\ \frac{1}{\lambda_2 p'(\lambda_2)} \\ \frac{1}{\lambda_3 p'(\lambda_3)} \end{bmatrix}}$$

$$\boxed{\xi = 0}$$

At $\xi = \xi_1$

$$\begin{aligned}
\lambda_1(A - B_1)e^{\lambda \xi_1} - \lambda_2 B_2 e^{\lambda_2 \xi_1} - \lambda_3 B_3 e^{\lambda_3 \xi_1} &= 0 \\
\lambda_1^2(A - B_1)e^{\lambda \xi_1} - \lambda_2^2 B_2 e^{\lambda_2 \xi_1} - \lambda_3^2 B_3 e^{\lambda_3 \xi_1} &= 0 \\
\lambda_1^3(A - B_1)e^{\lambda \xi_1} - \lambda_2^3 B_2 e^{\lambda_2 \xi_1} - \lambda_3^3 B_3 e^{\lambda_3 \xi_1} &= -1/c
\end{aligned}$$

A similar piece of algebra shows that

$$\boxed{\begin{bmatrix} (A - B_1)e^{\lambda_1 \xi} \\ B_2 e^{\lambda_2 \xi} \\ B_3 e^{\lambda_3 \xi} \end{bmatrix} = \frac{1}{c\epsilon^2} \begin{bmatrix} -\frac{1}{\lambda_1 p'(\lambda_1)} \\ \frac{1}{\lambda_2 p'(\lambda_2)} \\ \frac{1}{\lambda_3 p'(\lambda_3)} \end{bmatrix}}$$

$$\boxed{\xi = \xi_1}$$

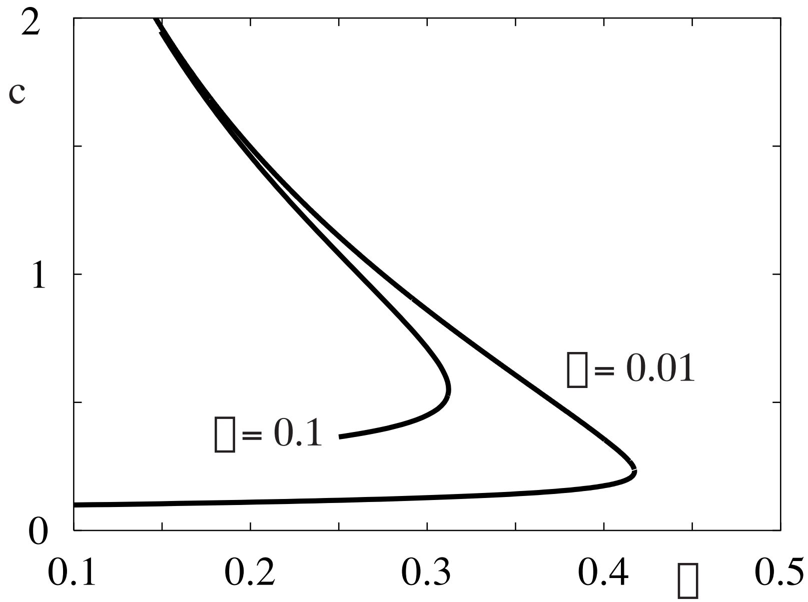
Together with the equations $\nu(0) = \alpha$ and $\nu(\xi_1) = \alpha$ we have eight equations in the six unknowns $A, B_1, B_2, B_3, C_1, C_2, \xi_1, c$. We shall now eliminate all but ξ_1 and c to obtain $(\nu, \xi_1) = (\nu(\alpha), \xi_1(\alpha))$. From $\nu(0) = \alpha$ and $\nu(\xi_1) = \alpha$ we have

$$\begin{aligned}
\alpha &= -c(\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3) \\
\alpha &= -cA\lambda_1 e^{\lambda_1 \xi_1}
\end{aligned}$$

Using $\xi = 0$ and $\xi = \xi_1$ we may then obtain the two equations

$$e^{\lambda_1 \xi_1} + \epsilon^2 p'(\lambda_1)\alpha - 1 = 0$$
$$\frac{e^{-\lambda_2 \xi_1}}{p'(\lambda_2)} + \frac{e^{-\lambda_3 \xi_1}}{p'(\lambda_3)} + \frac{1}{p'(\lambda_1)} + \epsilon^2 \alpha = 0$$

which can be solved numerically.



Speed of the travelling pulse solution in the piecewise linear FitzHugh-Nagumo model. A linear stability analysis shows that the fast wave is stable and the slow wave unstable.

2 An integrate-and-fire model

Here we replace the local excitable dynamics by a simpler threshold process. The integrate-and-fire (IF) model obeys a linear ODE that *resets* whenever it reaches a threshold h:

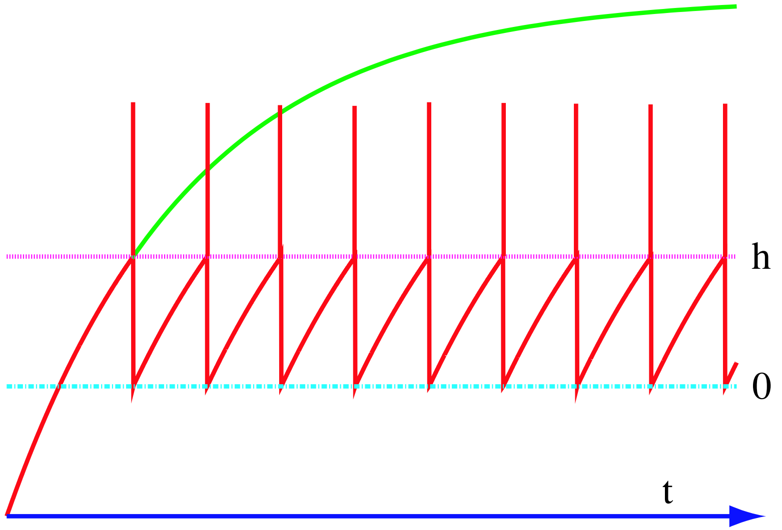
$$\tau \frac{du}{dt} = -u + v, \quad T_m < t < T_{m+1},$$

and

$$\lim_{\delta \to 0^+} u(T_m + \delta) = 0.$$

The *firing times* T_n are determined iteratively according to

$$T_n = \inf\{t \mid u(t) \geq h \; ; \; t \geq T_{n-1}\}.$$



A simple model of an excitable fibre is one that feels an effect whenever the local dynamics reaches threshold. The fibre itself may have a simple description as a cable equation:

$$\frac{\partial v}{\partial t} = -\frac{v}{\tau} + D\frac{\partial^2 v}{\partial x^2} + I(x,t),$$

whilst the "effect" could take the form $I(x,t) = \sum_m \eta(t - T_m(x))$, for some universal shape $\eta(t)$ (mimicking that of the pulse or action potential in the FitzHugh-Nagumo model).

A travelling solitary wave may be described with an ansatz of the form $T(x) = x/c$, where c denotes the speed of the wave. In the travelling frame co-ordinate system $\xi \equiv ct - x$ the wave (with constant profile $v(\xi)$) is described with the second order ordinary differential equation:

$$D v_{\xi\xi} - c u_{\xi} - \frac{v}{\tau} = -\eta(\xi/c) \tag{1}$$

where $v_\xi \equiv dv/d\xi$. Let us make the choice of a square pulse

$$\eta(t) = \frac{\sigma}{\tau_R} H(t) H(\tau_R - t).$$

For travelling pulse solutions which satisfy $\lim_{\xi \to \pm\infty} v(\xi) = 0$ the solution to (1) takes the form

$$v(\xi) = \begin{cases} \alpha_1 \exp(m_+ \xi) & -\infty < \xi < 0 \\ \alpha_2 \exp(m_+ \xi) + \alpha_3 \exp(m_- \xi) + \tau\sigma/\tau_R & 0 < \xi < c\tau_R \\ \alpha_4 \exp(m_- \xi) & \xi > c\tau_R \end{cases}$$

with

$$m_\pm = \frac{1}{2D}\left[c \pm \sqrt{c^2 + 4D/\tau}\right].$$

By ensuring the continuity of the solution and its first derivative at $\xi = 0$ and $\xi = c\tau_R$ one may solve for the unknowns $\alpha_1 \ldots \alpha_4$ as

$$\begin{aligned} \alpha_1 &= \alpha_3 \frac{m_-}{m_+}[1 - \exp(-m_+ c\tau_R)] \\ \alpha_2 &= -\alpha_3 \frac{m_-}{m_+} \exp(-m_+ c\tau_R) \\ \alpha_3 &= \frac{\tau\sigma}{\tau_R} \frac{m_+}{(m_- - m_+)} \\ \alpha_4 &= \alpha_3 [1 - \exp(-m_- c\tau_R)]. \end{aligned}$$

The self-consistent speed of the travelling wave may be determined by demanding that $u(0) = h$. In the travelling wave frame,

$$\tau c \frac{du}{d\xi} = -u + v.$$

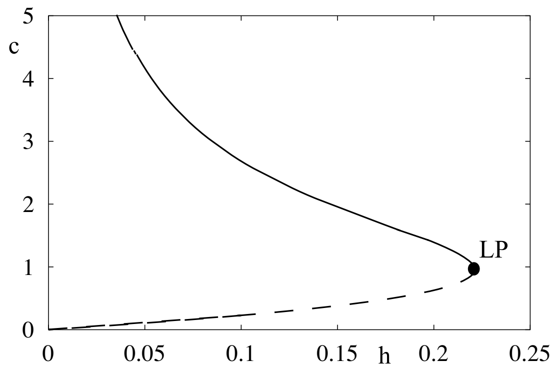
Solving:

$$\frac{d}{d\xi} u e^{\xi/c\tau} \Big|_{-\infty}^{0} = \frac{1}{c\tau} \int_{-\infty}^{0} v(\xi') e^{\xi'/c\tau} d\xi'.$$

Hence the speed of the travelling pulse satisfies

$$h = \frac{\alpha_1}{m_+ c\tau + 1}.$$

This is an implicit equation for $c = c(h)$, that we can plot numerically.



Speed of the travelling pulse solution in the IF fibre model. A linear stability analysis shows that the fast wave is stable and the slow wave unstable. Here $\tau = \tau_R = D = 1$.

For further discussion about stability (faster branch is stable) see [1, 2].

References

[1] S Coombes and P C Bressloff. Solitary waves in a model of dendritic cable with active spines. *SIAM Journal on Applied Mathematics*, 61:432–453, 2000.

[2] S Coombes. The effect of ion pumps on the speed of travelling waves in the fire-diffuse-fire model of Ca^{2+}. *Bulletin of Mathematical Biology*, 63:1–20, 2001.