MCMC2: Lab Session 3 – SIR Models (2) (Supplementary Material)

Phil O'Neill & Theo Kypraios

We are interested in fitting the non-Markovian model to some observed data.

When writing an MCMC algorithm the first thing to do is to write what is the target distribution's density, ie the density of the distribution that we want to sample from. In our case it will the posterior distribution of the infection rate (β), the scale parameter of the Gamma distribution (γ) and the set of the infection times $\boldsymbol{I}$.

$$
\begin{aligned}
\pi(\beta, \gamma, \boldsymbol{I}_{-\alpha}, I_\alpha | \boldsymbol{R}) \quad \propto \quad & \prod_{j \neq a}^{n} (\beta/N) I_{i_j} \exp\left\{ - \int_{I_a}^{\max(\boldsymbol{R})} (\beta/N) S_t I_t \, \mathrm{d}t \right\} \\
\times \quad & \prod_{j=1}^{n} f(R_j - I_j | \gamma) \\
\times \quad & \pi(\beta)\, \pi(\gamma)
\end{aligned}
$$

where $f(\cdot)$ denotes the pdf of a Gamma distribution. Note that we have implicitly assumed an improper prior for the initial infection time I_α and a uniform prior on the integers $\{1, \ldots, n\}$ for the label of the initial infection time α. That means

$$
\begin{aligned}
\pi(\beta, \gamma, \boldsymbol{I} | \boldsymbol{R}) \quad \propto \quad & \prod_{j \neq a}^{n} (\beta/N) I_{i_j} \exp\left\{ - \int_{i_a}^{r_b} (\beta/N) S_t I_t \, \mathrm{d}t \right\} \\
\times \quad & \prod_{j=1}^{n} \left(\frac{1}{\Gamma(\alpha)} \gamma^{\alpha} (R_j - I_j)^{\alpha - 1} \exp\{-(R_j - I_j)\gamma\} \right) \\
\times \quad & \frac{1}{\Gamma(\lambda_\beta)} \nu_\beta^{\lambda_\beta} \beta^{\lambda_\beta - 1} \exp\{-\beta \nu_\beta\} \, \frac{1}{\Gamma(\lambda_\gamma)} \nu_\gamma^{\lambda_\gamma} \gamma^{\lambda_\gamma - 1} \exp\{-\gamma \nu_\gamma\}
\end{aligned}
$$

Full conditional distributions

1. $$\pi(\beta|\gamma,\boldsymbol{I},\boldsymbol{R}) \propto \beta^{n-1}\exp\left\{-\beta\left(\int_{i_a}^{r_b}\frac{1}{N}S_tI_t\right)\right\}\beta^{\lambda_\beta-1}\exp\{-\beta\nu_\beta\}$$

2. $$\pi(\gamma|\beta,\boldsymbol{I},\boldsymbol{R}) \propto \gamma^{\alpha n}\exp\left\{-\gamma\left(\sum_{j=1}^{n}(R_j-I_j)\right)\right\}\gamma^{\lambda_\gamma-1}\exp\{-\gamma\nu_\gamma\}$$

3. $$\pi(\boldsymbol{I}|\boldsymbol{R},\beta,\gamma) \propto \prod_{j\neq a}^{n} I_{i_j}\prod_{j=1}^{n}(R_j-I_j)^{\alpha-1}\exp\left\{-\int_{i_a}^{r_b}\frac{\beta}{N}S_tI_t\ \mathrm{d}t\right\}\exp\left\{-\gamma\left(\sum_{j=1}^{n}(R_j-I_j)\right)\right\}$$

It is then easy to see that

$$\beta|\gamma,\boldsymbol{I},\boldsymbol{R} \sim \text{Ga}\left(n-1+\lambda_\beta,\nu_\beta+\int_{i_a}^{r_b}\frac{1}{N}S_tI_t\right)$$

and therefore can do a Gibbs step for β.

Likewise for the scale parameter of the Gamma infectious period distribution

$$\gamma|\beta,\boldsymbol{I},\boldsymbol{R} \sim \text{Ga}\left(\alpha n+\lambda_\gamma,\nu_\gamma+\sum_{j=1}^{n}(R_j-I_j)\right)$$

and hence, can do a Gibbs step.

Unfortunately, the full conditional distribution of the infection times given the removal times and β and γ does not look like that it comes from a well known distribution and therefore we can do a Metropolis-Hastings step as we did for the Markovian epidemic model.