

# NONDEMOLITION PRINCIPLE OF QUANTUM MEASUREMENT THEORY

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ABSTRACT. We give an explicit axiomatic formulation of the quantum measurement theory which is free of the projection postulate. It is based on the generalized nondemolition principle applicable also to the unsharp, continuous-spectrum and continuous-in-time observations. The “collapsed state-vector” after the “objectification” is simply treated as a random vector of the *a posteriori* state given by the quantum filtering, i.e., the conditioning of the *a priori* induced state on the corresponding reduced algebra. The nonlinear phenomenological equation of “continuous spontaneous localization” has been derived from the Schrödinger equation as a case of the quantum filtering equation for the diffusive nondemolition measurement. The quantum theory of measurement and filtering suggests also another type of the stochastic equation for the dynamical theory of continuous reduction, corresponding to the counting nondemolition measurement, which is more relevant for the quantum experiments.

## 1. THE STATUS OF QUANTUM MEASUREMENT THEORY

Quantum measurement theory, based on the ordinary von Neumann or a generalized reduction postulate, was never an essential part of quantum physics but rather of metaphysics. First, this was because the orthodox quantum theory had always dealt with a closed quantum system while the object of measurement is an open system due to the interaction with the measurement apparatus. Second, the superposition principle of quantum mechanics, having dealt with simple algebras of observables, is in contradiction with the von Neumann projection postulate while it may be not so in the algebraic quantum theory with the corresponding superselection rules. Third, due to the dynamical tradition in quantum theory going on from the deterministic mechanics, the process of the measurement was always considered by theoretical physicists as simply just an ordinary interaction between two objects while any experimentalist or statistician knows that this is a stochastic process, giving rise to the essential difference between a *priori* and a *posteriori* description of the states.

The last and most essential reason for such an unsatisfactory status of the quantum measurement theory was the limitations of the projection postulate applicable only to the instantaneous measurement of the observables with the discrete spectra, while the real experiments always have a finite duration and the most important observation is the measurement of the position having the continuous spectrum.

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There are many approaches to the theory of quantum measurement ranging from purely philosophical to qualitative and even quantitative theories in which the projection postulate apparently is not needed or is generalized to meet the indirect, or unsharp, measurements [1–10].

The most general, the philosophical level, of the discussion of these problems is of course the simplest and the appropriate one for the largest audience. But it provides room for unprofessional applications of the more sophisticated theoretical arguments, giving rise to different kinds of the speculations and paradoxes. I believe that the professional standard of quantum measurement theory ought to be an axiomatic and rigorous one and the quantum measurement problems must be formulated within it and solved properly instead of making speculations.

In order to examine the quantum paradoxes of Zeno type related to the continuous measurements, the study must be based on advanced mathematical methods of the quantum theory of compound systems with not regular but rather singular interaction, and this has recently received a stochastic treatment in the quantum theory of open systems and noise. It must use the tools of the quantum algebraic theory for the calculus of input fields of the apparatus, i.e., the quantum noises which usually have an infinite number of degrees of freedom, and for the superselection of output fields, i.e., commutative (classical) pointer processes which are usually the stochastic processes in continuous time.

Perhaps some philosophers and physicists would not like such a treatment of quantum measurement theory; the more mathematical a theory is the less philosophical it is, and the more rigorous it is, the less alive it is. But this is just an objective process of the development of any scientific theory and has already happened with the classical information and measurement theory.

The corresponding classical dynamical measurement theory, called the stochastic filtering theory, was developed in the beginning of the 60's by Stratonovich [11] and for the particular linear case by Kalman [12]. This theory, based on the notion of the partial (unsharp) observation and the stochastic calculus method, is optional for the classical deterministic physics, having dealt with the complete (sharp) observations of the phase trajectories and ordinary differential calculus, and is usually regarded as a part of the stochastic systems theory or, more precisely, the classical information and control theory. The main task of the filtering theory is to derive and solve a stochastic reduction equation for the present posterior state of the object of incomplete measurement, giving a means to calculate the conditional probabilities of the future observations with respect to the results of the past measurements. The corresponding filtering equation describes, for example, the continuous spontaneous localization of the classical Brownian particle under an unsharp observation as the result of the dynamical reduction of the statistical posterior state given by the classical conditional expectations under the continuous increase of the interval of the observation. The stochasticity of this nonlinear equation is generated either by the Wiener process or by the Poisson process, or by mixture of them, corresponding to the diffusive, counting, or mixed type of continuous measurement on the fixed output. It can be also written in the linear form in terms of the classical renormalized state vector (probability density), and is sometimes called “the Schrödinger equation of the classical systems theory” to emphasize its importance and the probabilistic interpretation.

Recently the corresponding quantum filtering theory was developed for the different types of continuous observations, [13,14], although the particular linear case of quantum Kalman filter was proposed by the author much earlier [6,7]. This gives rise to an axiomatic quantum measurement theory based on the new quantum calculus method to handle rigorously the singular interactions of the quantum object and input fields, and based on the generalized nondemolition principle to select properly the output observable processes. The mathematical quantum measurement theory plays the same central role in the general quantum theory of compound systems containing the information and control channels, as in the classical systems theory. But in distinction to the classical case it is not optional for the quantum physics due to the irreducible probabilistic nature of quantum mechanics which results in the absence of the phase trajectories. There is no need in this theory to use the projection or any other reduction postulate. But it does not contradict the quantum theory, as claimed in Ref. [15], and its application can be derived in the relevant cases simply as the result of state vector filtering by means of which the conditional probabilities of the future observation with respect to the results of the past measurements are calculated.

There is no need to postulate different nonlinear stochastic modifications of the Schrödinger equation in the phenomenological theories of spontaneous localization or of the nonstandard quantum theories of dynamical reduction and continuous collapse, [16–20] and to argue which type is more universal. They all are given as particular cases [21–24] of the general diffusive type quantum filtering equation, [25], rigorously derived by conditioning the corresponding Schrödinger equation for the uniquely determined minimal compound quantum system in Fock–Hilbert space.

The quantum filtering theory gives also a new type of phenomenological stochastic equations which are relevant to the quantum mechanics with spontaneous localization, [19,20], corresponding to the random quantum jumps, [26–28]. This pure discontinuous type is also rigorously derived from the Schrödinger equation [29] by conditioning the continuous-in-time counting measurement which contains the diffusive type as the central limit case [30].

Thus, the stochastic nature of measurement processes is reconciled with unitarity and deterministic interaction on the level of the compound system. But to account for the unavoidable noise in the continuous observation the unitary model necessarily involves a quantum system with infinitely many degrees of freedom and a singular interaction.

The purpose of this paper is to describe explicitly a new universal nondemolition principle for quantum measurement theory which makes possible the derivations of the reduction postulates from the quantum interactions. We show on simple examples what it means to derive rigorously the quantum filtering equation (thus the Hilbert stochastic process) by conditioning a Schrödinger equation for a compound system. Here, we demonstrate these derivations from the corresponding unitary interactions with the apparatus for the particular cases of the measurement of a single observable with the trivial Hamiltonian  $H = 0$  of the object using the operator quantum calculus method instead of the quantum stochastic one [21–23]. But if one wants to obtain such results in nontrivial cases related to the dynamical observables that are continuous in time and continuous in spectra and that do not commute with  $H \neq 0$ , one needs to use the appropriate mathematical tools, such as

quantum differential calculus and quantum conditional expectations, recently developed within the algebraic approach in quantum probability theory. Otherwise, one would be in the same situation as trying to study the Newton mechanics in nontrivial cases without using the ordinary differential calculus.

Note that the quantum filtering equation was first obtained in a global form [9] and then in the differential form [30] within the operational approach, [1,2], giving the reduced description of the open quantum systems and quantum continuous measurements. This was done by the stochastic representation of the continuous instrument, described by the semigroup of the operational valued measures which are absolutely continuous with respect to the standard Wiener or Poisson process. The most general approach [31] to these problems is based on the quantum stochastic calculus of nondemolition measurements and quantum conditional expectations. It clearly shows that the operational semigroup approach is restricted to only the Markovian case of the quantum stochastic object as an open system and to the conditionally independent nondemolition observations describing the output of the compound system.

## 2. CAUSALITY AND NONDEMOLITION PRINCIPLE

Let us begin with the discussion of the quantum nondemolition principle which forms the basis of the axiomatic formulation of the quantum measurement theory without the projection postulate, and which has been implicitly explored also in other approaches [1–10]. The term “nondemolition measurement” was first introduced into the theory of ultrasensitive gravitational experiments by Braginski and others [32–34] to describe the sequential observations in a quantum Weber antenna as a simultaneous measurement of some quantum observables. But the property of nondemolition has never been formalized or even carefully described other than by requiring the commutativity of the sequential observables in the Heisenberg picture, which simply means that the measurement process can be represented as a classical stochastic one by the Gelfand transformation. Therefore no essentially quantum, noncommutative results have been obtained, and no theorems showing the existence of such measurements in nontrivial time continuous models have been proved.

An operator  $X$  in a Hilbert space  $\mathcal{H}$  is said to be demolished by an observable  $Y = Y^\dagger$  in  $\mathcal{H}$  if the expectation  $\langle X \rangle$  is changed for  $\langle \tilde{X} \rangle \neq \langle X \rangle$  in an initial state when  $Y$  has been measured, although without reading. According to the projection postulate the demolished observable  $\tilde{X} = \delta[X]$  is described by the reduction operation  $\delta[X] = \sum P_i X P_i$  for a discrete observable  $Y = \sum y_i P_i$  given by the orthoprojectors  $P_i^2 = P_i = P_i^\dagger$ ,  $\sum P_i = I$  and eigenvalues  $\{y_i\}$ . The observable  $Y$  is nondemolition with respect to  $X$  if  $\delta[X]$  is compatible,  $\langle \delta[X] \rangle = \langle X \rangle$ , with respect to each initial state, i.e., iff  $\delta[X] = X$ . It follows immediately in this discrete case that the nondemolition condition is  $XY = YX$ , as the main filtering theorem says [30] even in the general case. Moreover, for each demolition observable  $Y$  there exists a nondemolition representation  $\tilde{Y} = \varrho[Y]$  in an extended Hilbert space  $\mathcal{H} \otimes \mathcal{F}$ , which is statistically equivalent to  $Y$  in the sense that  $\langle \tilde{X} \tilde{Y} \rangle = \langle X \hat{Y} \rangle$  for each input state in  $\mathcal{H}$  and corresponding output state in  $\mathcal{H} \otimes \mathcal{F}$ . This follows from the reconstruction theorem [35] for quantum measurements giving the existence of the nondemolition representation for any kind of observations, which might be even continuously distributed in the relativistic space-times  $\mathbb{R}^{1+d}$ . In the case of a

single discrete observable  $Y$  it proves the unitary reconstruction of the projection postulate, which is given in section 3.

Now we give several equivalent formulations of the dynamical nondemolition considered not just as a *possible* property for the quantum measurements but rather as the *universal* condition to handle such problems as the modeling of the unsharp measurements, the generalized reduction and instantaneous collapse for the continuous spectrum observables, the quantum sequential measurements, and the dynamical reduction and spontaneous localization under the continuous-in-time observation. This condition, based on the reconstruction theorem, was discovered in Ref. [7] and consists of a new principle of quantum axiomatic measurement theory for the proper representation of the observable process in a Hilbert space, such as the interaction representation of the object with the measurement apparatus.

On the philosophical level, one can say that the nondemolition principle is equivalent to the quantum causality principle of the statistical predictability for the present and all possible future observations and for all possible initial states from the a posteriori probability distributions which are conditioned by the results of the past measurements. This should be regarded rather as the physical content and purpose of this principle and not as a definition.

On the mathematical level the nondemolition principle must be formulated as a necessary and sufficient condition for the existence of the conditional expectations on the algebras generated by the present and future Heisenberg operators of the object of the measurement and all the output observables with respect to the subalgebras of the past measurements and arbitrary input states.

In the most general algebraic approach this formulation was first obtained in Ref. [7], (see also Refs. [13] and [14]) as the condition

$$(2.1) \quad [X(t), Y(s)] := X(t)Y(s) - Y(s)X(t) = 0, \quad \forall s \geq t$$

of compatibility of all system operators  $X(t)$  considered as the possible observables at a time instant  $t$  with all past observables  $Y(s)$ ,  $s \leq t$ , which have been measured up to  $t$ . It says that the Heisenberg operators  $X(t)$  of the quantum object of the measurement given, say, in the interaction representation with the apparatus must commute with all past output observables  $Y(s)$ ,  $s \leq t$ , of the pointer for any instant  $t$ . And according to the causality principle there is no restriction on the choice of the future observables  $Y(r)$ ,  $r \geq t$ , with respect to the present operators  $X(t)$  except the self-nondemolition  $[Y(r), Y(s)] = 0$  for the compatibility of the family  $\{Y(t)\}$ . Generalized then in [21–28] for arbitrary  $X$  and  $Y$ , these conditions define a stochastic process  $Y(t)$  which is nondemolition with respect to a given quantum process  $X(t)$ . Note that the condition (2.1) for clearly distinguished object and pointer observables does not reduce completely the algebra of the compound system to the commutative one as it does in the case of the direct observations  $Y = X$  when it reads as the self-nondemolition condition  $[X(t), X(s)] = 0$ ,  $\forall t, s$ . The nondemolition measurements considered in Refs. [32–34] were defined only by the self-nondemolition condition, corresponding to this trivial (Abelian) case  $X(t) = Y(t)$ .

In the operational approach [1,2], applicable for the reduced description of the quantum Markov open system, one might prefer to have a condition that is equivalent to the nondemolition principle in that case. It can be given in terms of the induced states on the reduced algebra, i.e., of the states given by the expectations  $\phi(Z) = \langle \psi, Z(t)\psi \rangle$  on the algebra of observables  $Z$  generated in the Heisenberg

picture  $Z(t) = U^\dagger(t)ZU(t)$  by all  $X(t)$  and  $Y(t)$  for a given initial state vector  $\psi$ . The nondemolition principle simply means that the induced current quantum state of the object coincides with the *a priori* one, as a statistical mixture of a posteriori states with respect to the past, but not the future, observations [30]. The a posteriori state as a quantum state of the object after the measurement, when a result has been read mathematically, will be defined in the next section. Here we only point out that the coincidence means that the induced state is not demolished by the measurement if the results have not been read. This justifies the use of the word nondemolition in the generalized sense.

One can call this coincidence the generalized reduction principle because it does not restrict the consideration to the projection valued operations only, corresponding to the von Neumann reduction of the quantum states, which is not applicable even for the relatively simple case of instantaneous measurements of the quantum observables with the continuous spectrum.

The equivalence of these two formulations in the quantum Markovian case and their relation to the projection postulate (see the next section) can be illustrated even in the case of the single operation corresponding to an instantaneous measurement, or a measurement with fixed duration.

Let  $\mathcal{H}$  and  $\mathcal{F}$  be the Hilbert spaces of state vectors  $\eta \in \mathcal{H}$ , and  $\varphi \in \mathcal{F}$  for the quantum object and the measurement apparatus, respectively, and let  $R$  be a self-adjoint operator in  $\mathcal{H}$ , representing a dynamical variable with the spectral values  $x \in \mathbb{R}$  to be measured by means of the measurement apparatus with a given observable  $\hat{y}$ , representing the pointer of the apparatus as a self-adjoint operator in  $\mathcal{F}$  with either discrete or continuous spectrum  $\Lambda \subseteq \mathbb{R}$ . The measurement apparatus has the fixed initial state  $\varphi_0 \in \mathcal{F}$ ,  $\|\varphi_0\| = 1$  and is coupled to the object by an interaction operator  $S^\dagger = V_0U^\dagger V_1$ , where  $U$  is a unitary evolution operator of the system in the product space  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ ,  $U^\dagger = U^{-1}$ , and  $V_0 = V \otimes \hat{1}$ ,  $V_1 = I \otimes \hat{v}$  are the unitarities given by the free evolution operators  $V : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\hat{v} : \mathcal{F} \rightarrow \mathcal{F}$  of the object and the apparatus, respectively, during the fixed measurement interval  $[0, t]$ . It is natural to suppose that the interaction does not disturb the variable  $R$  in the sense  $R_0 := R \otimes \hat{1} = S^\dagger R_0 S$ , or equivalently,  $\langle x|S = \hat{s}_x \langle x|$ , i.e.,

$$(2.2) \quad S : |x\rangle \otimes \varphi_0 \mapsto |x\rangle \otimes \varphi_x, \quad \forall x \in \mathbb{R}$$

in terms of (generalized) eigenvectors  $|x\rangle$  of  $R$ , where  $\varphi_x = \hat{s}_x \varphi_0$ . But it must disturb the input observable  $\hat{q} = \hat{v}^\dagger \hat{y} \hat{v}$  in order to get the distinguishable probability densities  $f_x(y) = |\varphi_x(y)|^2$  of the output observable  $Y = S^\dagger(\kappa I \otimes \hat{q})S$ , corresponding to the different eigen values  $x \in \mathbb{R}$  of the input states  $|x\rangle$  to be tested by the usual methods of mathematical statistics. Here  $\kappa > 0$  is a scaling parameter and we have assumed, for simplicity that the observable  $\hat{y}$  and hence  $\hat{q}$  has the nondegenerate spectral values  $y \in \Lambda$ , so that  $\varphi \in \mathcal{F}$  in the input representation is described by the (generalized) eigenvectors  $|y\rangle$  of  $\hat{q} : |y\rangle \mapsto y|y\rangle$  as a square integrable function  $\varphi(y) = \langle y|\varphi$ ,  $\|\varphi\|^2 = \int |\varphi(y)|^2 d\nu < \infty$  with respect to a given measure  $\nu$  on  $\Lambda$ .

The positive measure  $\nu$  is either discrete or continuous or can even be of mixed type normalizing the probability densities  $g(y) = \langle \psi(y), \psi(y) \rangle$  for the state vectors  $\psi \in \mathcal{G}$ :

$$(2.3) \quad \|\psi\|^2 = \int_\Lambda \langle \psi(y), \psi(y) \rangle d\nu = \int_\Lambda g(y) d\nu = 1$$

where  $\psi(y) = \langle y|\psi$  are the  $\mathcal{H}$ -valued wavefunctions of the system “quantum object plus measurement apparatus.” One can consider, for example, the standard Lebesgue measures  $d\nu = d\lambda$  on  $\Lambda = \mathbf{Z}$ ,  $d\lambda = 1$  and on  $\Lambda = \mathbb{R}$ ,  $d\lambda = dy$ :

$$\|\psi\|^2 = \sum \langle \psi(k), \psi(k) \rangle (d\lambda = 1) ; \quad \|\psi\|^2 = \int \langle \psi(y), \psi(y) \rangle dy (d\lambda = dy)$$

respectively for the discrete spectrum  $y \in \mathbf{Z}$  and for the continuous one  $y \in \mathbb{R}$ , given by the distributions  $f(y) = \sum \delta(y - k)$  and  $f(y) = 1$  as  $d\lambda = f(y)dy$ .

The output state vectors  $\chi = S(\xi \otimes \varphi_0) \in \mathcal{G}$ , corresponding to the arbitrary input ones  $\xi = V\eta$ ,  $\langle \xi, \xi \rangle = 1$ , are given by the vector-functions  $\chi : y \mapsto \chi(y) \in \mathcal{H}$  of  $y \in \Lambda$  with values

$$\chi(y) = \langle y|S(\xi \otimes \varphi_0) = \langle y|\chi.$$

The operators  $\langle y|S : \mathcal{G} \rightarrow \mathcal{H}$  correspond to the adjoint ones  $S^\dagger|y) : \eta \mapsto S^\dagger(\eta \otimes |y))$ ,

$$(2.4) \quad \langle \eta, \langle y|S(\xi \otimes \varphi) \rangle = \langle S^\dagger(\eta \otimes |y)), \xi \otimes \varphi \rangle$$

defining the (generalized) vector-functions  $S^\dagger|y)\eta$  by

$$\int S^\dagger|y)\eta \varphi_0(y) d\nu = S^\dagger(\eta \otimes \varphi_0) \quad \forall \eta, \varphi.$$

The operator  $(R_0\chi)(y) = R\chi(y)$  commutes with  $Q = \kappa I \otimes \hat{q}$  as well as with any other operator  $C_0 = C \otimes \hat{1}$  representing an object variable  $C : \mathcal{H} \rightarrow \mathcal{H}$  in  $\mathcal{H} \otimes \mathcal{F}$  as the constant function  $Z(y) = C$ . This is because the general operator  $Z$  in  $\mathcal{H} \otimes \mathcal{F}$  commuting with  $Q$  corresponds to an operator-valued function  $Z(y) : \mathcal{H} \rightarrow \mathcal{H}$ , which is defined by the operator  $Z$  as

$$(2.5) \quad \langle y|Z\psi = Z(y)\langle y|\psi, \quad \forall \psi \in \mathcal{G}, \quad y \in \Lambda$$

in the case  $Z = Q$  it corresponds to  $Z(y) = \kappa y I$ :  $\langle y|Q\psi = \kappa y \langle y|\psi$ . It is trivial in this case that the Heisenberg operators  $X = S^\dagger Z S$  satisfy the nondemolition condition  $[X, Y] = 0$  with respect to the output observable  $Y = S^\dagger Q S$ , but not the initial operators  $Z : [Z, Y] \neq 0$  if  $[Z(y), R] \neq 0$ . This makes it possible to condition, by the observation of  $Y$ , the future measurements of any dynamical variable of the quantum object, but not the potential measurements of  $Z$  in the past with respect to the present observation of  $Y$  if they have not been done initially.

Indeed, let  $P_\Delta = S^\dagger I_\Delta S$  be the spectral orthoprojector of  $Y$ , given for a measurable  $\Delta \subseteq \Lambda$  by  $I_\Delta = I \otimes \hat{1}_\Delta$  as

$$(2.6) \quad \langle y|I_\Delta \chi = 1_\Delta(y)\chi(y) = 1_\Delta(y)\langle y|\chi, \quad 1_\Delta(y) = \begin{cases} 1 & y \in \Delta \\ 0 & y \neq \Delta \end{cases}$$

and  $p_\Delta = \langle \eta \otimes \varphi, P_\Delta(\eta \otimes \varphi) \rangle \neq 0$ . Then the formula

$$(2.7) \quad \varepsilon_\Delta[X] = \langle \eta, \omega[X P_\Delta] \eta \rangle / \langle \eta, \omega[P_\Delta] \eta \rangle$$

where  $\langle \eta, \omega[X] \eta \rangle = \langle \eta \otimes \varphi, X(\eta \otimes \varphi) \rangle$ ,  $\forall \eta \in \mathcal{H}$ , defines the conditional expectation of  $X = S^\dagger Z S$  with respect to  $Y$ . It gives the conditional probability  $\varepsilon_\Delta[X] \in [0, 1]$  for any orthoprojector  $X = O$ , while  $\varepsilon_\Delta[Z]$  defined by the same formula for  $Z = \{Z(y)\}$  may not be the conditional expectation due to the lack of positivity  $\omega[E P_\Delta] \geq 0$ , for all  $\varphi \in \mathcal{F}$  if the orthoprojector  $Z = E$  does not commute with  $P_\Delta$ . The necessity of the nondemolition principle for the existence of the conditional probabilities is the consequence of the main filtering theorem consistent with the causality principle according to which the conditioning with respect to the current observation has the sense of preparation for future measurements but not for past ones.

This theorem proved in the general algebraic form in Ref. [30] reads in the simplest formulation as

MAIN MEASUREMENT THEOREM. Let  $O$  be an orthoprojector in  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ . Then for each state vector  $\psi = \xi \otimes \varphi$  there exists the conditional probability  $\varepsilon_\Delta[O] \in [0, 1]$ , defined by the compatibility condition

$$(2.8) \quad \varepsilon_\Delta[O] \langle \xi \otimes \varphi, P_\Delta(\xi \otimes \varphi) \rangle = \langle \xi \otimes \varphi, OP_\Delta(\xi \otimes \varphi) \rangle$$

if and only if  $OP_\Delta = P_\Delta O$ . It is uniquely defined for any measurable  $\Delta \subset \Lambda$  with respect to  $P_\Delta = S^\dagger I_\Delta S$ ,  $\varphi = \varphi_0$  as

$$(2.9) \quad \varepsilon_\Delta[O] = \frac{1}{\mu_\Delta} \int_\Delta \langle \chi_y, E(y) \chi_y \rangle d\mu$$

Here  $E(y) : \mathcal{H} \rightarrow \mathcal{H}$  is the orthoprojector valued function, describing  $O$ , commuting with all  $P_\Delta$  in the Schrödinger picture as  $O = S^\dagger E S$ ,  $\mu_\Delta = \int_\Delta g_\xi(y) d\nu$  is the absolutely continuous with respect to  $\nu$  probability distribution of  $y \in \Lambda$ ,  $g_\xi(y) = \|\chi(y)\|^2$ ,  $\chi(y) = \langle y | S(\xi \otimes \varphi_0) \rangle$ , and  $y \mapsto \chi_y$  is the random state vector  $\chi_y \in \mathcal{H}$  of the object uniquely defined for almost all  $y : g_\xi(y) \neq 0$  up to the random phase  $\theta(y) = \arg c_\xi(y)$  by the normalization

$$(2.10) \quad \chi_y = \chi(y)/c_\xi(y), \quad |c_\xi(y)|^2 = g_\xi(y)$$

### 3. THE GENERALIZED *a posteriori* REDUCTION

It follows immediately from the main theorem that the input state vector  $\xi : \|\xi\| = 1$  of the object of measurement has to be changed for  $\chi_y \in \mathcal{H}$  due to the preparation  $\xi \mapsto \{\chi(y) : y \in \Lambda\}$  of the *a priori* state vector  $\chi = S(\eta \otimes \varphi_0)$  of the meter and the object after the objectification  $\hat{q} = y$ . The former is given by the dynamical interaction in the pointer representation  $\chi(y) = \langle y | \chi$  due to the choice of the measurement apparatus and the output observables, and the latter is caused by statistical filtering  $\chi \mapsto \chi(y)$  due to the registration of the measurement result  $y \in \Lambda$  and the normalization  $\chi_y = \chi(y)/\|\chi(y)\|$ .

While the process of preparation described by a unitary operator applied to a fixed initial state of the meter encounters no objection among physicists, the process of objectification encounters objection because of the nonunitarity of the filtering and nonlinearity of the normalization. But the main theorem shows clearly that there is nothing mysterious in the objectification. It is not a physical process but only a mathematical operation to evaluate the *conditional state*

$$(3.1) \quad \pi_y[Z] = \varepsilon_y[S^\dagger Z S] = \langle \chi_y, Z(y) \chi_y \rangle$$

which are defined by the conditional expectations  $\varepsilon_y[X] = \lim_{\Delta \downarrow y} \varepsilon_\Delta[X]$  of the Heisenberg operators  $X$  for  $Z = \{Z(y)\}$ . The linear random operator

$$(3.2) \quad G(y) : \xi \in \mathcal{H} \mapsto \langle y | S(\xi \otimes \varphi_0), \quad y \in \Lambda$$

defines the reduction transformations  $G(y)$  as the partial matrix elements  $\langle y | S \varphi_0$  of the unitary operator  $S$ . They map the normalized vectors  $\xi \in \mathcal{H}$  into the *a posteriori* ones  $\chi(y) = G(y)\xi$ , renormalized to the probability density

$$g_\xi(y) = \|G(y)\xi\|^2 = \langle \xi, E(y)\xi \rangle, \quad E = G^\dagger G.$$

If the condition (2.2) holds, then the only eigen vectors  $|x\rangle$  of  $R$  remain unchanged up to a phase multiplier:

$$(3.3) \quad G(y)|x\rangle = |x\rangle \varphi_x(y), \quad \varphi_x(y) = \langle y | \hat{s}_x \varphi_0 = \langle y | \varphi_x$$

and hence  $\chi_y = e^{i\theta_x(y)}|x\rangle$ , where  $\theta_x(y) = \arg \varphi_x(y)$ . The superpositions  $\xi = \int |x\rangle\xi(x)d\lambda$  change their amplitudes  $\xi(x) = \langle x|\xi$  for  $\chi_y(x) = \langle x|\chi_y$

$$(3.4) \quad \langle x|\chi_y = c_\xi^{-1}(y)\chi(x, y), \quad \chi(x, y) = \langle x|G(y)\xi = \chi_x(y)\xi(x)$$

where  $c_\xi(y) = (\int |\varphi_x(y)|^2 h(x)d\lambda)^{1/2}$ ,  $h(x) = |\xi(x)|^2$ .

In the case of a purely continuous spectrum of  $R$  there are no invariant state vectors at all because the generalized eigenvectors cannot be considered as input ones due to  $|x\rangle \notin \mathcal{H}$  as  $\langle x|x\rangle = \infty$  in that case.

The generalized reduction (3.1) of the state-vector corresponds to the limit case  $\Delta \downarrow y$  when the accuracy of the instrument  $\Delta \ni y$  tends to the single-point subset  $\{y\} \subset \Lambda$ . It is not even the mathematical idealization of the real physical experiment if the observable  $\hat{q}$  has the discrete spectrum  $\Lambda = \{y_i\}$ .

Prior to discussing why the generalized reduction does not contradict the main postulates of the quantum theory, let us show how to derive the von Neumann projection postulate in this way, corresponding to the orthogonal transformations  $G(y_i) = F_i$  given by a partition  $\sum A_i = \mathbb{R}$  of the spectrum of  $R$  as  $F_i = E_{A_i}$ . Here  $A \mapsto E_A$ ,  $E_A^\dagger E_{A'} = E_{A \cap A'}$ ,  $\sum E_{A_i} = I$  is the spectral measure of  $R = \int x dE$  which might be either of discrete or of continuous type as in the cases

$$E_A = \sum_{x \in A} |x\rangle\langle x|, \quad E_A = \int_A |x\rangle\langle x| dx,$$

corresponding to the nondegenerate spectrum of  $R : dE = |x\rangle\langle x| d\lambda$ .

Considering the indices  $i$  of  $y_i$  in  $\mathbf{Z}$  it is always possible to find the unitary interaction in the Hilbert space  $\mathcal{G} = \mathcal{H} \otimes l^2(\mathbf{Z})$  of the two-sided sequences  $\psi = \{\eta^k | k = 0, \pm 1, \pm 2, \dots\}$  with  $\eta^k \in \mathcal{H}$  such that  $\|\psi\|^2 = \sum_{-\infty}^{\infty} \langle \eta^k, \eta^k \rangle < \infty$ . Indeed, we can define the interaction as the block-matrix  $S^\dagger = [W_k^i]$  acting in  $\mathcal{G}$  as  $W^i \psi = \sum_{k=-\infty}^{\infty} W_k^i \eta^k$ , by  $W_k^i = F_{k-i}$ , where  $F_k = 0$  if there is no point  $y_k$  in  $\Lambda$  numbered by a  $k \in \mathbf{Z}$ . It is the unitary one because  $S = [F_{i-k}]$  is inverse to  $S^\dagger = [F_{k-i}]$  as

$$\sum_{j=-\infty}^{\infty} F_{i-j} F_{k-j} = \delta_{k-i}^j \sum_{j=-\infty}^{\infty} F_{-j} = \delta_k^i \sum F_i = \delta_k^i I$$

due to the orthogonality  $F_i F_k = 0$ ,  $i \neq k$ , and completeness  $\sum F_i = I$  of  $\{F_i\}$ .

Let us fix the initial sequence  $\varphi_0 \in l^2(\mathbf{Z})$  as the eigenstate  $\varphi_0 = \{\delta_0^k\} = |0\rangle$  of the input observable  $\hat{k}$  in  $l^2(\mathbf{Z})$  as the counting operator

$$(3.5) \quad \hat{k} = \sum_{k=-\infty}^{\infty} k |k\rangle\langle k|, \quad |i\rangle = \{\delta_i^k\} \in l^2(\mathbf{Z})$$

with the spectrum  $\mathbf{Z}$ . Then we obtain the conditional states (3.1) defined as

$$\pi_i[Z] = \frac{1}{p_i} \langle F_i \eta, Z_i F_i \eta \rangle = \langle \eta_i, Z_i \eta_i \rangle, \quad \eta_i = F_i \eta / p_i^{1/2}$$

up to the normalizations  $p_i = \langle F_i \eta, F_i \eta \rangle \neq 0$  by the linear operations  $\sigma \mapsto W_i^0 \sigma W_i^0$ ,

$$(3.6) \quad W_i^0 \eta = \langle i | S(\varphi_0 \otimes \eta) = \sum_{k=-\infty}^{\infty} F_{i-k} \delta_0^k \eta = F_i \eta.$$

It is only in that case that the *a posteriori* state always remains unchanged under the repetitions of the measurement. Such an interaction satisfies the condition (2.2)

with  $\varphi_x = \hat{s}_x \varphi_0$  given by the sequences  $\varphi_x = \{\delta_{i(x)}^k\} = |i(x)\rangle$  because

$$F_{i-k}|x\rangle = |x\rangle \delta_{i-k}^{i(x)} = W_i^k|x\rangle \quad (= |x\rangle), \quad \forall x \in A_{i-k},$$

where  $i(x) = i$  if  $x \in A_i$  is the index map of the coarse-graining  $\{A_i\}$  of the spectrum of  $R$ . Hence in the  $x$ -representation  $\psi = \int |x\rangle \psi(x) d\lambda$ ,  $\psi(x) = \langle x|\psi$  it can be described by the shifts  $\hat{s}_x^\dagger = [\delta_{k-i}^{i(x)}]$  in  $l^2(\mathbf{Z})$

$$(3.7) \quad \hat{s}_x^\dagger : \psi(x) = \{\eta^k(x)\} \mapsto \{\langle x|\eta^{i(x)+k}\rangle\} \quad \eta^k(x) = \langle x|\eta^k$$

replacing the initial state  $\varphi_0 = |0\rangle$  of the meter for each  $x$  by another eigenstate  $|i(x)\rangle = \hat{s}_x|0\rangle$  if  $x \notin A_0$ .

This realizes the coarse-grained measurement of  $R$  by means of the nondemolition observation of the output

$$(3.8) \quad Y = S^\dagger(I \otimes \hat{k})S = i(R) \otimes \hat{1} + I \otimes \hat{k},$$

where  $i(R) = \int i(x) dE = \sum i F_i$ . If  $q(R) = \hbar i(R)$  is the quantized operator  $R$  given, say, by the integer  $i(x) = \lfloor x/\hbar \rfloor$ , then the rescaled model  $\hat{y}_x = \hbar \hat{s}_x^\dagger \hat{k} \hat{s}_x = q(x) \hat{1} + \hbar \hat{k}$  of the nondemolition measurement has the classical limit  $\lim \hat{y}_x = x \hat{1}$  if  $\hbar \rightarrow 0$ , corresponding to the direct observation of a continuous variable  $R$  by means of  $\lim \hbar Y = R \otimes \hat{1}$ .

Note that the observable  $Y$  commutes with the arbitrary Heisenberg operator  $A = S^\dagger(C \otimes \hat{1})S$  of the object, but not with the initial operators  $C_0 = C \otimes \hat{1}$  if  $[C, i(R)] \neq 0$ .

The unitary operator  $S^\dagger$  is given by the interaction potential  $q(R) \otimes \hat{p}$  as  $S^\dagger = \exp\{(i/\hbar)q(R) \otimes \hat{p}\}$ , where  $i = \sqrt{-1}$ , and  $\hat{p} = [ \langle i|\hat{p}|k\rangle ]$ ,  $\langle i|\hat{p}|k\rangle = (1/2\pi) \int_{-\pi}^{\pi} p e^{-i(i-k)p} dp$  is the matrix of the momentum operator in  $l^2(\mathbf{Z})$ , generating the shifts  $\hat{s}_x^\dagger = [ \langle i|\hat{s}_x^\dagger|k\rangle ]$  as  $\hat{s}_x^\dagger = e^{i(x)i\hat{p}}$ :

$$\langle i|\hat{s}_x^\dagger|k\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x)ip} e^{-i(i-k)p} dp = \delta_{i-k}^{i(x)}.$$

The nondemolition observation reproduces the statistics of the ‘‘demolition’’ measurement of  $R$  by the direct observation of  $q(R)$  because the output observable  $Y$  has the same characteristic function with respect to the state vector  $\xi \otimes \varphi_0$  as  $i(R)$  with respect to  $\xi$ :

$$\begin{aligned} \langle \xi \otimes \varphi_0, \exp\{ipY\}(\xi \otimes \varphi_0) \rangle &= \langle S(\xi \otimes \varphi_0), e^{ipQ} S(\xi \otimes \varphi_0) \rangle \\ &= \sum \langle F_i \xi, e^{ip} F_i \xi \rangle = \langle \xi, \exp\{ipi(R)\} \xi \rangle. \end{aligned}$$

Here  $p$  is the parameter of the characteristic function,  $Q = I \otimes \hat{k}$ , and  $F_i = \langle i|S\varphi_0 = F_i^\dagger$  are the orthoprojectors, such that  $\sum_i F_i^\dagger F_i = \int i(x) dE = i(R)$ . If the observable  $R$  is discrete, then the nondemolition observation (3.8) realizes the precise measurement of  $R$ , if the partition  $\{A_i\}$  separates all the eigenvalues  $\{x_i\}$  as in the case  $x_i \in A_i, \forall i$ , corresponding to  $x_i = \hbar i$ ,  $A_i = [\hbar i, \hbar(i+1)[$ ,  $i = 0, 1, 2, \dots$

The nondemolition principle helps not only to derive the projection postulate as a reduced description of the shift interaction in the enlarged Hilbert space  $\mathcal{G}$  with respect to the initial eigenvector  $\varphi_0 = |0\rangle$  of the discrete meter  $\hat{q}$ , but also extends it to the generalized reductions under the unsharp measurements with arbitrary spectrum  $\Lambda$ , corresponding to the nonrepeatable instruments [1,2]

$$(3.9) \quad \Pi_\Delta[C] = \int_\Delta \Psi[C](y) d\nu, \quad \Psi[C](y) = G(y)^\dagger C G(y).$$

The density  $\Psi(y)$  of the instrument defines completely positive but not necessarily orthoprojective operations  $E(y) = \Psi[I](y)$ , called the effects for the probability densities  $g(y) = \sigma[E(y)]$ , and also the nonlinear operation  $\sigma \mapsto \sigma \circ \Psi(y)/\sigma[E(y)]$  of the generalized reduction, mapping the pure input states  $\sigma_\xi[C] = \langle \xi|C|\xi \rangle$  into the *a posteriori* ones

$$(3.10) \quad \rho_y[C] = \frac{1}{g_\xi(y)} \rho[C](y) = \pi_y[C_0], \quad \rho[C](y) = \langle \chi(y), C\chi(y) \rangle.$$

They are also pure because of the completeness of the nondemolition measurement, i.e., nondegeneracy of the spectrum of the observable  $\hat{q}$  in  $\mathcal{F}$ . Thus, the reduction of the state-vector is simply the way of representing in the form (3.1) the *a posteriori* pure states (3.10) given at the limit  $\Delta \rightarrow 0$  by the usual (in the statistics) Bayesian formula (2.7) for  $X = S^\dagger C_0 S = A$ , which is applicable due to the commutativity of  $A$  and  $P_\Delta$ .

The reduction  $\sigma_1 \rightarrow \rho_y$  of the prepared state  $\sigma_1 = \sigma \circ \Psi$  for the object measurement is given as the evaluation of the conditional expectations which are the standard attributes of any statistical theory. All the attempts to derive the reduction as a result of deterministic interaction only are essentially the doomed attempts to derive the probabilistic interpretation of quantum theory. There is no physical explanation of the stochasticity of the measurement process as there is no adequate explanation of the randomness of an observable in a pure quantum state.

It is not a dynamical but a purely statistical effect because the input and output state-vectors of this process are not the observables of the individual object of the statistical ensemble but only the means for calculating the *a priori* and the *a posteriori* probabilities of the observables of this object. Hence there is no observation involving just a single quantum object which can confirm the reduction of its state. The reduction of the state-vector can be treated as an observable process only for an infinite ensemble of similar object plus meter systems. But the measurements for the corresponding collective observables also involves preparation and objectification procedures, this time for the ensemble, i.e., for a second quantized compound system. So the desirable treatment of all the reductions as some objective stochastic process can never be reached in this way. They are secondary stochastic since they are dependent on the random information that has been gained up to a given time instant  $t$ .

The reduction of the state-vector is not at variance with the coherent superposition principle, because a vector  $\eta \in \mathcal{H}$  is not yet a pure quantum state but defines it rather up to a constant  $c \in \mathbb{C}$  as the one-dimensional subspace  $\{c\eta|c \in \mathbb{C}\} \subset \mathcal{H}$  which is a point of the projective space over  $\mathcal{H}$ . For every reduced state-vector  $\chi_y$  there exists an equivalent one, namely  $\chi(y) = \sqrt{g_\xi(y)}\chi_y$ , defining the same quantum pure state, given by the linear transformation  $G(y) : \xi \mapsto \chi(y)$ , so that the superposition principle holds:  $\chi(y) = \sum c_i \chi^i(y)$  if  $\xi = \sum c_i \xi^i$ . The pure state transformation  $G(y)$  does not need to be unitary, but as an operator  $G : \mathcal{H} \rightarrow \mathcal{G}$  with

$$G^\dagger G = \int G(y)^\dagger G(y) d\nu = \int \varphi_0^\dagger S^\dagger |y\rangle \langle y| S \varphi_0 d\nu = \varphi_0^\dagger S^\dagger S \varphi_0 = I$$

it preserves the total probability by mapping the normalized  $\xi \in \mathcal{H}$  into the  $\chi(y) = G(y)\xi$ , normalized to the probability density  $g_\xi(y)$ .

According to the nondemolition principle it makes sense to apply the vector  $\chi = \{\chi(y)\}$  of the system after the measurement preparation only against the

reduced observables  $Z = \{Z(y)\}$  which commute with  $Q = \kappa I \otimes \hat{q}$ . Otherwise according to the main theorem the conditional probabilities of the future observations may not exist for an initial state-vector  $\chi_0 = \eta \otimes \varphi$  and a given result  $y \in \Lambda$  of the measurement. It is against the physical causality to consider the unreduced operators as the observables for the future measurements since the causality means that the future observations must be statistically predictable from the data of a measurement and such prediction can be given only by the conditional probabilities (2.9). Once the output observables are selected as a part of a preparation, the algebra of the actual observables is reduced and there is no way to measure an observable  $Z$  which is not compatible with  $Q$ . It could be measured in the past if another preparation had been made but the irreversibility of the time arrow does not give this possibility. Thus, the quantum measurement theory implies a kind of time-dependent superselection rule for algebras such as those of the observables  $Z$  chosen as the actual observable at the moment  $t$ . But it does not prevent one from considering other operators as the virtual observables defining super operators, i.e., the subsidiary operators for the description of some meaningful operations, although an evaluation of their expectations does not make any sense as it does for the differential operators in the classical theory.

The *a priori* states are the induced ones

$$\sigma_1(C) = \int \langle \chi_y, C \chi_y \rangle d\mu = \langle \chi, C_0 \chi \rangle, \quad C_0 = C \otimes \hat{1}$$

on the algebra generated by the operators in  $\mathcal{H}$  of the object only. They are given as the statistical mixtures of the *a posteriori* pure states (3.10) of the object even if the initial state  $\sigma$  was pure. But it does not contradict quantum mechanics because the prepared state  $\phi(Z) = \langle \chi, Z \chi \rangle$  of the quantum system after the measurement is reduced to the object plus pointer but is still given uniquely by the state-vector  $\chi \in \mathcal{G}$ , up to a random phase. Namely, the vector  $\chi$  is a coherent superposition

$$\chi = \sum \chi_i \otimes |y_i\rangle c_i, \quad \chi_i = \chi(y_i)/c_i, \quad |c_i|^2 = p_i$$

of the *a posteriori* states  $\chi_i \otimes |y_i\rangle$  of the system, if  $\hat{q}$  has the spectral decomposition  $\hat{q} = \sum y_i |y_i\rangle \langle y_i|$  and  $p_i$  are the probabilities of  $y_i$ .

This uniqueness does not hold for the density-matrix representations  $\phi[Z] = \text{Tr}\{\hat{\phi}Z\}$ ; among the equivalent density matrices  $\hat{\phi} \geq 0$  there exists always the projector  $\hat{\phi} = |\chi\rangle \langle \chi|$ , but there are also mixtures such as the diagonal one

$$\hat{\phi}_1 = \sum p_i |\eta_i\rangle \langle \eta_i|, \quad |\eta_i\rangle = \eta_i \otimes |y_i\rangle$$

in the discrete case  $\Lambda = \{y_i\}$ . Hence, the diagonalization  $\hat{\phi} \mapsto \hat{\phi}_1$  of the density matrix due to the measurement of  $\hat{q}$  is only the rule to choose the most mixed one  $\hat{\phi}_1$  which is equivalent to the coherent choice  $\hat{\phi}$  due to

$$\text{Tr}\{\hat{\phi}Z\} = \sum p_i \langle \eta_i, Z \eta_i \rangle = \text{Tr}\{\hat{\phi}_1 Z\}$$

for all reduced operators  $Z = \sum Z_i \otimes |y_i\rangle \langle y_i|$ . There is no special need to fix such a choice, which is even impossible in the continuous spectrum case. This is because the continuous observable  $\hat{q}$  has no ordinary eigenvectors,  $\langle y|y\rangle = \infty$  and hence  $\chi_y \otimes |y\rangle \notin \mathcal{G}$ , but there exist the eigenstates  $\omega_y[\hat{z}] = z(y)$  on the algebra of complex functions  $z(y)$ , defining the conditional expectations  $\varepsilon_y[X]$  for  $X = S^\dagger Z S$  as

$$\varepsilon_y[X] = \pi_y[S X S^\dagger], \quad \pi_y = \rho_y \otimes \omega_y, \quad \forall y \in \Lambda.$$

Thus, the nondemolition principle abandons the collapse problem, reducing it to the evaluation of the *a posteriori* state. The decrease of the observable algebra is the only reason for the irreversibility of the linear transformation  $\phi_0 \mapsto \phi$  of the initial states  $\phi_0(X) = \langle \chi_0, X\chi_0 \rangle$ , which are pure on the algebra of all operators  $X$  into the prepared (mixed) ones on the algebra of the reduced operators  $Z$ .

#### 4. THE MAIN MEASUREMENT PROBLEM

As was shown using an instantaneous measurement as an example, the nondemolition principle leads to the notion of the instrument, described by the operational-valued measure (3.9), and gives rise to the generalized reduction (3.10) of the quantum statistical states. In the operational approach [1,2] one starts from the instrumental description  $\sigma \mapsto \sigma \circ \Phi(y) = \rho(y)$  of the measurement, which is equivalent to postulating the generalized reduction (20) given up to the probabilistic normalization  $g(y) = \rho[I](y)$  by the linear map  $\sigma \mapsto \sigma \circ \Psi(y)$  due to  $\Psi_y(\sigma) = (1/g(y))\sigma \circ \Psi(y) = \rho_y$ .

The main measurement problem is the reconstruction of an interaction representation of the quantum measurement, that is, finding a proper dilation  $\mathcal{G}$  of the Hilbert space  $\mathcal{H}$  and the output process  $Y$ , satisfying the nondemolition (and self-nondemolition) condition (2.1) with respect to the Heisenberg operators  $X$  of the object of measurement in order to derive the same reduction as the result of conditional expectation.

The minimal dilation giving, in principle, the solution of this problem even for non-Markovian relativistic cases was constructed in [35], but it is worth finding also more realistic, nonminimal dilations defining the object of measurement as a quantum stochastic process in the strong sense for the particular Markovian cases.

In the case of a single instantaneous measurement described by an instrument  $\Pi_\Delta$ , this can be formulated as the problem of finding the unitary dilation  $U\varphi_0 : \eta \in \mathcal{H} \mapsto U(\eta \otimes \varphi_0)$  in a tensor product  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$  and an observable  $\hat{y} = \int y d\hat{\Gamma}$  in  $\mathcal{F}$ , giving  $\Pi_\Delta$  as the conditional expectation

$$\Pi_\Delta[C] = \omega_0[AE_\Delta], \quad \langle \eta, \omega_0[X]\eta \rangle = \langle \eta \otimes \varphi_0, X\eta \otimes \varphi_0 \rangle$$

of  $AE_\Delta = U^\dagger(C \otimes \hat{\Gamma}_\Delta)U$ . In principle, such a quadruple  $(\mathcal{F}, \varphi_0, \hat{y}, U)$  was constructed in [36] and [37] for the normal completely positive  $\Pi_\Delta$ , giving a justification of the general reduction postulate as described above for the case of the projective  $\Pi_\Delta$ . For the continuous observation this problem was solved [39] on the infinitesimal level in terms of the quantum stochastic unitary dilation of a differential evolution equation for characteristic operations

$$\tilde{\Psi}(t, q) = \int e^{iqy} \Psi(t, y) d\nu, \quad \Psi(t, y) = \lim_{\Delta \downarrow y} \frac{1}{\nu_\Delta} \Pi_\Delta^t,$$

where  $d\pi$  is a standard probability measure of  $dy \subset \Lambda$ . This corresponds to the stationary Markovian evolution of the convolutional instrumental semigroups  $\{\Pi_\Delta^t | t \geq 0\}$  giving the reduced description of the continuous measurement, with the data  $y(t)$  having the values in an additive group.

Unfortunately the characteristic operational description of the quantum measurement is not relevant to the sample-paths representation. It is not suitable for the conditioning of the quantum evolution under the given data of the observations and hence does not allow one to obtain explicitly the corresponding dynamical reduction. Moreover, the continuous measurements have the data  $y$  not necessary in a

group, and in the nonstationary cases they cannot be described by the convolution instrumental semigroups and the corresponding evolution equations.

Recently a new differential description of continual nondemolition measurements was developed within the noncommutative stochastic calculus method [13,14,31]. A general stochastic filtering equation was derived for the infinitesimal sample-paths representation of the quantum conditional expectations, giving the continuous generalized reduction of the *a posteriori* states [25,26,29].

Simultaneously, some particular cases of the filtering equation for the stochastic state-vector  $\varphi(t, \omega) = \chi_{y^t}(\omega)$ , corresponding to the functional spectrum  $\Lambda^t$  of the diffusion trajectories  $y^t(\omega) = \{y(s, \omega) | s \leq t\}$ , were discovered within the phenomenological theories of the dynamical reduction and spontaneous localization [16–18]. As was shown in [21,27] and [29], the nonlinearity of such equations is related only to the normalization  $\|\varphi(t, \omega)\| = 1$  and after the proper renormalization  $\chi_t(\omega) = \sqrt{g_t(\omega)}\varphi(t, \omega)$ , where  $g_t(\omega)$  is the probability density of the process

$$y(s, \omega) = \frac{1}{s} \int_0^s \langle \varphi(t, \omega), R\varphi(t, \omega) \rangle dt + s^{-1}w_s, \quad s \in [0, t]$$

generated by the standard Wiener process  $\omega = \{w_t\}$  with respect to the Wiener probability measure  $d\pi$  on the continuous trajectories  $\omega \in \Omega$ , they become the linear ones

$$(4.1) \quad d\chi_t + \left( \frac{i}{\hbar}H + \frac{1}{2}L^\dagger L \right) \chi_t dt = L\chi_t dw.$$

Here  $H$  is the Hamiltonian of the object,  $L$  is an arbitrary operator in  $\mathcal{H}$  defining the variable  $R = \sqrt{\hbar}(L + L^\dagger)$ , under the continuous measurement, and  $dw = w_{t+dt} - w_t$  is the forward increment, such that the stochastic equation (4.1) has to be solved in the Itô sense. This solution can be explicitly written as

$$(4.2) \quad \chi_t(\omega) = T_t(\omega)\xi, \quad T_t(\omega) = \exp\{w_t L - tL^2\}$$

in the case  $L = \sqrt{\pi/2\hbar}R$ , ( $\hbar = 2\pi\hbar$ ),  $H = 0$ , corresponding to the unsharp measurement of the self-adjoint operator  $R$  during the time interval  $[0, t)$  with the trivial free Hamiltonian evolution of the object. In the case  $H \neq 0$  this can be used for the approximate solution of (4.1) with  $L^\dagger = L$ ,  $\chi(0) = \eta$  as  $\chi_t(\omega) \simeq T_t(\omega)\xi(t)$ , where  $\xi(t) = V(t)\eta$  is the unitary evolution  $V(t) = \exp\{-\frac{i}{\hbar}Ht\}$  without the measurement.

The stochastic transformation (4.2) defines the operational density

$$\Theta_t[C](\omega) = T_t^\dagger(\omega)CT_t(\omega)$$

of an instrument as in (3.9) with respect to the standard Wiener probability measure  $d\pi$  on  $\omega^t = \{w_s\}_{s \leq t} \in \Omega^t$  having the Gaussian marginal distribution of  $q_t = \sqrt{\hbar}w_t$

$$d\nu := \int_{q_t \in dq} d\pi = (ht)^{-1/2} \exp[-\pi q^2/ht] dq.$$

Hence  $\Psi(t, q)d\nu := \int_{q_t \in dq} \Theta_t(\omega)d\pi = \Phi(t, y)dy$ , where  $y = \frac{1}{t}q$ ,

$$(4.3) \quad \Phi[C](t, y) = \sqrt{\frac{t}{\hbar}} \exp\left\{-\frac{\pi t}{2\hbar}(y - R)^2\right\} C \exp\left\{-\frac{\pi t}{2\hbar}(y - R)^2\right\},$$

because  $\Theta_t(\omega)$  depends only on  $w_t$ :  $\Theta_t(\omega) = \Psi(t, \sqrt{\hbar}w_t)$ , and

$$\Psi[C](t, q) = G(t, q)^\dagger C G(t, q), \quad G(t, q) = \exp \left\{ -\frac{\pi}{\hbar} \left( qR - \frac{t}{2} R^2 \right) \right\}.$$

The operator  $E(t, y) = \Phi[I](t, y) = f_R(t, y)$ ,

$$f_R(t, y) = \sqrt{\frac{t}{\hbar}} \exp \left\{ -\pi \frac{t}{\hbar} (y - R)^2 \right\} = F(t, y)^\dagger F(t, y)$$

defines the probability density of the unsharp measurement of  $R$  with respect to the ordinary Lebesgue measure  $dy$  as the convolution

$$g_\xi(t, y) = \int \sqrt{\frac{t}{\hbar}} \exp \left\{ -\pi \frac{t}{\hbar} (y - x)^2 \right\} h_\xi(x) d\lambda = (f_0 * h_\xi)(y),$$

where  $h_\xi(x) = |\xi(x)|^2$ ,  $\xi(x) = \langle x | \xi \rangle$ ,  $d\lambda = \sum \delta(x - x_i) dx$  in the case of discrete spectrum  $\{x_i\}$  of  $R$ , and  $d\lambda = dx$  in the case of purely continuous spectrum of  $R$ .

This means that the continuous unsharp measurement of  $R$  can be described by the observation model  $y_x(t) = x + (1/t)q_t$  of signal  $x$  plus Gaussian error  $e(t) = (1/t)q_t$  with independent increments as

$$(4.4) \quad y_R(t) = R + e(t)I, \quad e(t) = \frac{\sqrt{\hbar}}{t} w_t.$$

The noise  $e(t)$  with the mean value  $\langle e(t) \rangle = 0$  gives a decreasing unsharpness  $\langle e(t)^2 \rangle = \hbar/t$  of the measurement from infinity to zero that is inversely proportional to the duration of the observation interval  $t > 0$ .

In general, such a model can be realized [21]–[25] as the nondemolition observation within the quantum stochastic theory of unitary evolution of the compound system on the product  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$  with the Fock space  $\mathcal{F}$  over the one-particle space  $L^2(\mathbb{R}_+)$  for a one-dimensional bosonic field, modeling the measurement apparatus of the continuous observation.

Let us illustrate this general construction for our particular case  $H = 0$ ,  $L = L^\dagger$ . The unitary interaction  $S(t)$  in  $\mathcal{G}$ , defining the transformations (4.2) as (3.2) with respect to the vacuum state-vector  $\varphi_0 \in \mathcal{F}$ , is generated by the field momenta operators

$$(4.5) \quad \hat{p}_s = \frac{i}{2} \sqrt{\hbar} (\hat{a}_s^\dagger - \hat{a}_s), \quad s \in \mathbb{R}_+$$

as  $S(t) = \exp \left\{ -\frac{i}{\hbar} R \otimes \hat{p}_t \right\}$ .

Here  $\hat{a}_s$  and  $\hat{a}_s^\dagger$  are the canonical annihilation and creation operators in  $\mathcal{F}$ , localized on the intervals  $[0, s]$  according to the commutation relations

$$[\hat{a}_r, \hat{a}_s] = 0, \quad [\hat{a}_r, \hat{a}_s^\dagger] = s\hat{1}, \quad \forall r \geq s,$$

The pointer of the apparatus for the measurement of  $R$  is defined by the field coordinate observables

$$(4.6) \quad \hat{q}_s = \sqrt{\hbar} (\hat{a}_s + \hat{a}_s^\dagger), \quad s \in \mathbb{R}_+$$

which are compatible with  $[\hat{q}_r, \hat{q}_s] = 0$  as well as with  $[\hat{p}_r, \hat{p}_s] = 0$ , but incompatible with (4.5):

$$[\hat{p}_r, \hat{q}_s] = s \frac{\hbar}{i} \hat{1}, \quad \forall r \geq s.$$

The operators  $S^\dagger(t)$  satisfy the condition (2.2):  $\langle x|S(t) = \hat{s}_x(t)\langle x|$ , where the unitary operators  $\hat{s}_x^\dagger(t) : \mathcal{F} \rightarrow \mathcal{F}$  can be described by the shifts

$$(4.7) \quad \hat{s}_x^\dagger(t) : |q, t\rangle \mapsto |xt + q, t\rangle, \quad \forall x, q, t$$

similarly to (3.7). Here  $|q, t\rangle$  is the (generalized) marginal eigenvector of the self-adjoint operator

$$\hat{e}(t) = t^{-1}\hat{q}_t, \quad \hat{q}_t|q, t\rangle = q|q, t\rangle,$$

uniquely defined up to the phase by an eigenvalue  $q \in \mathbb{R}$  as the Dirac  $\delta_q$  in the  $\hat{q}_t$ -representation  $L^2(\mathbb{R})$  of the Hilbert subspace  $\mathcal{A}(t)\varphi_0 \subset \mathcal{F}$ , where  $\mathcal{A}(t)$  is the Abelian algebra generated by  $\hat{q}_t$ , and  $\varphi_0 \in \mathcal{F}$  is the vacuum-vector of the Fock space  $\mathcal{F}$ . Due to this,

$$\hat{g}_x(t) = \hat{s}_x^\dagger(t)\hat{e}(t)\hat{s}_x(t) = x\hat{1} + \hat{e}(t),$$

which gives the quantum stochastic realization of the observation model (4.4) in terms of the output nondemolition process  $\hat{y}_R(t) = \frac{1}{t}Y(t)$ ,

$$(4.8) \quad Y(t) = S^\dagger(t)(I \otimes \hat{q}_t)S(t) = tR \otimes \hat{1} + I \otimes \hat{q}_t$$

similarly to (3.8) with  $\hat{q}_t$  represented by the operator  $\sqrt{\hbar}(\hat{a}_t + \hat{a}_t^\dagger)$ . Indeed, the classical noise  $q_t = \sqrt{\hbar}w_t$  is statistically equivalent to the quantum one  $\hat{q}_t = \sqrt{\hbar}(\hat{a}_t + \hat{a}_t^\dagger)$  with respect to the vacuum state, as can be seen by a comparison of their characteristic functionals:

$$\begin{aligned} \langle e^{i \int f(s) dq} \rangle & : = \int \exp\{i\sqrt{\hbar} \int_0^\infty f(s)dw\}d\pi = \exp\left\{-\frac{\hbar}{2} \int_0^\infty f(s)^2 ds\right\} \\ & = \langle \varphi_0, e^{i \int f(s) d\hat{a}_t^\dagger} e^{-\frac{\hbar}{2} \int_0^\infty f(s)^2 ds} e^{i \int f(s) d\hat{a}_t} \varphi_0 \rangle = \langle \varphi_0, e^{i \int f(s) d\hat{q}_t} \varphi_0 \rangle. \end{aligned}$$

Here we used the annihilation property  $\hat{a}_s\varphi_0 = 0$  and the Wick ordering formula

$$(4.9) \quad \exp\{z'\hat{a}_s + \hat{a}_s^\dagger z\} = e^{z\hat{a}_s^\dagger} \exp\left\{z'\frac{s}{2}z\right\} e^{z'\hat{a}_s}.$$

The observable process (4.8) satisfies the nondemolition condition (2.1) (and self-nondemolition) with respect to any quantum process  $X(t) = (S^\dagger Z S)(t)$  given by the operators  $Z(t)$ , commuting with all  $Q(s) = I \otimes \hat{q}(s)$ ,  $s \leq t$ , because

$$Y(s) = S^\dagger(t)(I \otimes \hat{q}(s))S(t), \quad \forall s \leq t,$$

as follows from the commutation relations

$$\hat{s}_x^\dagger(t)\hat{q}_s = (sx\hat{1} + \hat{q}_s)\hat{s}_x^\dagger(t), \quad \forall s \leq t$$

for  $\hat{s}_x^\dagger(t) = \exp\left\{\frac{i}{\hbar}x\hat{p}_t\right\}$ . Indeed, due to this

$$[X(t), Y(s)] = W(t)[Z(t), Q(s)]W^\dagger(t) = 0,$$

if  $t > s$  and  $[Z(t), Q(s)] = 0$ , as in the cases  $Z(t) = C \otimes \hat{1}$  and  $Z(t) = Q(t)$ , where  $Q(t) = I \otimes \hat{q}_t$ .

Now we can find the transform

$$\langle q, t|S\varphi_0 = G(t, q)\varphi_0(t, q) = \frac{1}{\sqrt{t}}T\left(t, \frac{1}{t}q\right),$$

where  $\varphi_0(t, q) = \langle q, t|\varphi_0$  is the vacuum-vector  $\varphi_0 \in \mathcal{F}$  in the marginal  $\hat{q}_t = q$  representation

$$\varphi_0(t, q) = (ht)^{-1/4} \exp\{-\pi q^2/2ht\}, \quad q \in \mathbb{R}$$

normalized with respect to the Lebesgue measure  $dq$  on  $\mathbb{R}$ . To this end, let us apply the formula (4.9) to  $S^\dagger(t) = \exp\left\{\frac{i}{\hbar}R \otimes \hat{p}_t\right\}$ :

$$\exp\{-L \otimes \hat{a}_t + L \otimes \hat{a}_t^\dagger\} = e^{L \otimes \hat{a}_t^\dagger} \exp\left\{-\frac{t}{2}L^2\right\} e^{-L \otimes \hat{a}_t},$$

where  $L = R/2\sqrt{\hbar}$ . Using the annihilation property  $\exp\{\pm L \otimes \hat{a}_t\}\varphi_0 = \varphi_0$ , we obtain

$$\begin{aligned} W(t)^\dagger \varphi_0 &= e^{L \otimes \hat{a}_t^\dagger} \exp\left\{-\frac{t}{2}L^2\right\} e^{-L \otimes \hat{a}_t} \varphi_0 \\ &= e^{L \otimes \hat{a}_t^\dagger} \exp\left\{-\frac{t}{2}L^2\right\} e^{L \otimes \hat{a}_t} \varphi_0 = e^{L \otimes \hat{w}_t - tL^2} \varphi_0. \end{aligned}$$

This is equivalent to (4.2) because of the Segal isometry of the vectors  $\exp\{x\hat{w}_t\}\varphi_0 \in \mathcal{F}$ , where  $x \in \mathbb{R}$ ,  $\hat{w}_t = \hat{a}_t + \hat{a}_t^\dagger$ , and the stochastic functions  $\exp\{xw_t\} \in L_\pi^2(\Omega)$  in the Hilbert space of the Wiener measure  $\pi$  on  $\Omega$ . Hence the transform  $F\left(t, \frac{1}{t}q\right) = \sqrt{t}G(t, q)\varphi_0(t, q)$  defining the density  $\Phi(t, y) = F(t, y)^\dagger[\cdot]F(t, y)$  of the instrument (3.9) with respect to  $dy$  has the same form, as in (4.3):

$$(4.10) \quad F(t, y) = (t/\hbar)^{1/4} \exp\left\{-\frac{\pi t}{2\hbar}(y - R)^2\right\}.$$

## 5. A HAMILTONIAN MODEL FOR CONTINUOUS REDUCTION

As we have shown in the previous section the continuous reduction equation (4.1) for the non-normalized stochastic state-vector  $\chi(t, \omega)$  can be obtained from an interaction model of the object of measurement with a bosonic field. This can be done by conditioning with respect to a nondemolition continuous observation of field coordinate observables (4.6) in the vacuum state.

The unitary evolution  $\psi(t) = U(t)\psi_0$  in the tensor product  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$  with the Fock space  $\mathcal{F}$  corresponding to (4.1) can be written as the generalized Schrödinger equation

$$(5.1) \quad d\psi(t) + K_0\psi(t)dt = (L \otimes d\hat{a}_t^\dagger - L^\dagger \otimes d\hat{a}_t)\psi(t)$$

in terms of the annihilation and creation canonical field operators  $\hat{a}_s, \hat{a}_s^\dagger$ . This is a singular differential equation which has to be treated as a quantum stochastic one [29] in terms of the forward increments  $d\psi(t) = \psi(t+dt) - \psi(t)$  with  $K_0 = K \otimes \hat{1}$ ,  $K = (i/\hbar)H + \frac{1}{2}L^\dagger L$ . In the particular case  $L = R/2\sqrt{\hbar} = L^\dagger$  of interest, eq. (5.1) can be written simply as a classical stochastic one,  $d\psi + K\psi dt = (i/\hbar)Rdp$ , in Itô sense with respect to a Wiener process  $p_t$  of the same intensity  $(dp_t)^2 = \hbar dt/4$  as the field momenta operators (4.5) with respect to the vacuum state. But the standard Wiener process  $v_t = 2p_t/\sqrt{\hbar}$  cannot be identified with the Wiener process  $w_t$  in the reduction equation (4.1) because of the nondemolition principle. Moreover, there is no way to get the nondemolition property (2.1) for

$$X(t) = U(t)^\dagger X_0 U(t), \quad Y(s) = U(s)^\dagger Y_0(s) U(s)$$

with the independent or if only commuting  $v_t$  and  $w_t$ , as one can see in the simplest case  $H = 0$ ,  $X_0 = \frac{\hbar}{i} \frac{d}{dx} \otimes \hat{1}$ ,  $R = x$ ,  $Y_0(s) = I \otimes \hat{q}_s$ .

Indeed, the error process  $q_t = \sqrt{\hbar}w_t$  is appearing in (4.4) as a classical representation of the field coordinate observables (4.6) which do not commute with (4.5).

In this case, eq. (5.1) gives the unitary operator  $U(t) = \exp\{-\frac{i}{\hbar}x \otimes \hat{p}_t\}$  and the Heisenberg operators

$$X(t) = \frac{\hbar}{i} \frac{d}{dx} \otimes \hat{1} - I \otimes \hat{p}_t, \quad Y(s) = sx \otimes \hat{1} + I \otimes \hat{q}_s$$

commute for all  $t \geq s$  only because

$$\left[ \frac{\hbar}{i} \frac{d}{dx}, sx \right] \otimes \hat{1} = s \frac{\hbar}{i} I \otimes \hat{1} = [\hat{p}_t, \hat{q}_s], \quad \forall t \geq s.$$

Hence, there is no way to obtain (2.1) for the classical stochastic processes  $p_t, q_s$  by replacing simultaneously  $\hat{p}_t$  and  $\hat{q}_s$  for commuting  $\sqrt{\hbar}v_t/2$  and  $\sqrt{\hbar}w_t$  even though  $p_t$  is statistically identical to  $\hat{p}_t$  and separately  $q_s$  to  $\hat{q}_s$ .

Let us show now how one can get a completely different type of the reduction equation than postulated in [16]–[20] simply by fixing an another nondemolition process for the same interaction, corresponding to the Schrödinger stochastic equation (5.1) with  $L = L^\dagger$  and  $H = 0$ .

We fix the discrete pointer of the measurement apparatus, which is described by the observable  $\hat{n}_s = \frac{1}{s} \hat{a}_s^\dagger \hat{a}_s$ , by counting the quanta of the Bosonic field in the mode  $1_s(r) = 1$  if  $r \in [0, s)$  and  $1_s(r) = 0$  if  $r \notin [0, s)$ . The operators  $\hat{n}_t$  have the integer eigenvalues  $0, 1, 2, \dots$  corresponding to the eigen-vectors

$$|n, t\rangle = e^{t/2} (\hat{a}_t^\dagger / t)^n \varphi_0, \quad \hat{a}_t \varphi_0 = 0$$

which we have normalized with respect to the standard Poissonian distribution

$$(5.2) \quad \nu_n = e^{-t} t^n / n!, \quad n = 0, 1, \dots$$

as  $\langle n, t | n, t \rangle = 1/\nu_n$ . Let us find the matrix elements

$$\langle n, t | S(t) \varphi_0 = G(t, n)$$

for the unitary evolution operators

$$(5.3) \quad S(t) = \exp\{-L \otimes \hat{a}_t + L \otimes \hat{a}_t^\dagger\},$$

by resolving eq. (5.1) in the considered case. This can be done again by representing  $S(t)$  in the form (4.9) for  $z' = L$ ,  $z = -L$  and the commutation rule

$$(I \otimes \hat{a}_t) e^{L \otimes \hat{a}_t^\dagger} = e^{L \otimes \hat{a}_t^\dagger} (tL \otimes \hat{1} + I \otimes \hat{a}_t).$$

Due to the annihilation property, this gives

$$(5.4) \quad \varphi_0^\dagger (\hat{a}_t / t)^n e^{L \otimes \hat{a}_t^\dagger} \exp\left\{\frac{t}{2}(1 - L^2)\right\} e^{-L \otimes \hat{a}_t} \varphi_0 = L^n \exp\left\{\frac{t}{2}(1 - L^2)\right\} = G(t, n).$$

The obtained reduction transformations are not unitary and not projective for any  $n = 0, 1, 2, \dots$ , but they define the nonorthogonal identity resolution

$$\sum_{n=0}^{\infty} G(t, n)^\dagger G(t, n) e^{-t} t^n / n! = I$$

corresponding to the operational density

$$(5.5) \quad \Psi[C](t, n) = e^t L^n e^{-L^2/2} C L^n e^{-L^2/2}$$

with respect to the measure (5.2). Now we can easily obtain the stochastic reduction equation for  $\chi(t, \omega) = T(t, \omega)\eta$  if we replace the eigenvalue  $n$  of  $\hat{n}_t$  by the standard Poissonian process  $n_t(\omega)$  with the marginal distributions (5.2). Such a process  $n_t$

describes the trajectories  $t \mapsto n_t(\omega)$  that spontaneously increase by  $dn_t(\omega) = 1$  at random time instants  $\omega = \{t_1 < t_2 < \dots\}$  as the spectral functions  $\{n_t(\omega)\}$  for the commutative family  $\{\hat{n}_t\}$ . The corresponding equation for the stochastic state-vector  $\chi(t, \omega) = \chi(t, n_t(\omega))$  can be written in the Itô sense as

$$(5.6) \quad d\chi(t) + \frac{1}{2}(L^2 - I)\chi(t)dt = (L - 1)\chi(t)dn_t.$$

Obviously Eq. (5.6) has the unique solution  $\chi(t) = F(t)\eta$  written for a given  $\eta \in \mathcal{H}$  as

$$(5.7) \quad \chi(t) = L^{n_t} \exp \left\{ \frac{t}{2}(1 - L^2) \right\} \eta = G(t, n_t)\eta$$

because of  $d\chi(t) = (L - 1)\chi(t)$  when  $dn_t = 1$ , otherwise  $d\chi(t) = \frac{1}{2}(1 - L^2)\chi(t)dt$  in terms of the forward differential  $d\chi(t) = \chi(t + dt) - \chi(t)$ .

Such an equation was derived in [26]–[30] also for the general quantum stochastic equation (5.1) on the basis of quantum stochastic calculus and filtering theory [31]. Moreover, it was proved that any other stochastic reduction equation can be obtained as a mixture of Eq. (4.1) and (5.4) which are of fundamentally different types.

Finally let us write down a Hamiltonian interaction model corresponding to the quantum stochastic Schrödinger equation (5.1). Using the notion of chronologically ordered exponential

$$(5.8) \quad U(t) = \exp^{\leftarrow} \left\{ -\frac{i}{\hbar} \int_0^t H(r)dr \right\}$$

one can extend its solutions  $\psi(t) = \exp \left\{ -\frac{i}{\hbar} R \otimes \hat{p}_t \right\} \psi_0$  also to the general case,  $H \neq 0$ ,  $L^\dagger \neq L$  in terms of the generalized Hamiltonian

$$H(t) = H_0 + \frac{\hbar}{i}(L^\dagger \otimes \hat{a}(t) - L \otimes \hat{a}(t)^\dagger),$$

where  $\hat{a}(t) = d\hat{a}_t/dt$ ,  $\hat{a}^\dagger(t) = d\hat{a}_t^\dagger/dt$ ,  $H_0 = H \otimes \hat{1}$ . The time-dependent Hamiltonian  $H(t)$  can be treated as the object interaction Hamiltonian

$$H(t) = H_0 + \frac{\hbar}{i} e^{\frac{i}{\hbar} H_1 t} (L^\dagger \otimes \hat{a}(0) - L \otimes \hat{a}(0)^\dagger) e^{-\frac{i}{\hbar} H_1 t}$$

for a special free evolution Hamiltonian  $H_1 = I \otimes \hat{h}$  of the quantum bosonic field  $\hat{a}(r)$ ,  $r \in \mathbb{R}$  described by the canonical commutation relations

$$[\hat{a}(r), \hat{a}(s)] = 0, \quad [\hat{a}(r), \hat{a}(s)^\dagger] = \delta(r - s)\hat{1}, \quad \forall r, s \in \mathbb{R}.$$

This free evolution in the Fock space  $\mathcal{F}$  over one particle space  $L^2(\mathbb{R})$  is simply given by the shifts

$$e^{\frac{i}{\hbar} \hat{h} t} \hat{a}(r) e^{-\frac{i}{\hbar} \hat{h} t} = \hat{a}(r + t), \quad \forall r, t \in \mathbb{R},$$

corresponding to the second quantization  $\hat{h} = \hat{a}^\dagger \hat{\varepsilon} \hat{a}$  of the one-particle Hamiltonian  $\hat{\varepsilon} = \frac{\hbar}{i} \frac{\partial}{\partial r}$  in  $L^2(\mathbb{R})$ . Hence, the total Hamiltonian of the system “object plus measurement apparatus” can be written as

$$(5.9) \quad H_s = H \otimes \hat{1} + \frac{\hbar}{i}(L^\dagger \otimes \hat{a}(0) - L \otimes \hat{a}(0)^\dagger + I \otimes \hat{a}^\dagger \hat{a}'),$$

where  $a^\dagger a' = \int_{-\infty}^{\infty} \hat{a}(r)^\dagger \hat{a}(r)' dr$ ,  $\hat{a}(r)' = d\hat{a}(r)/dr$ . Of course, the free field Hamiltonian  $\hat{h} = \hbar \hat{a}^\dagger \hat{a}' / i$  is rather unusual as with respect to the single-particle energy  $\varepsilon(p) = p$  in the momentum representation giving the unbounded (from below) spectrum of  $\hat{\varepsilon}$ .

But one can consider such an energy as an approximation

$$(5.10) \quad \varepsilon(p) = \lim_{p_0 \rightarrow \infty} c \left( \sqrt{(p+p_0)^2 + (m_0 c)^2} - \sqrt{p_0^2 + (m_0 c)^2} \right) = v_0 p$$

in the velocity units  $v_0 = c/\sqrt{1 + (m_0 c/p_0)^2} = 1$  for the shift  $\varepsilon_0(p) - \varepsilon_0(0)$  of the standard relativistic energy  $\varepsilon_0(p) = c\sqrt{(p+p_0)^2 + (m_0 c)^2}$  as the function of small deviations  $|p| \ll p_0$  from the initially fixed momentum  $p_0 > 0$ . This corresponds to the treatment of the measurement apparatus as a beam of bosons with mean momentum  $p_0 \rightarrow \infty$  given in an initial coherent state by a plane wave

$$f_0(r) = c \exp\{i p_0 r / \hbar\}.$$

This input beam of bosons illuminate the position  $R = \sqrt{\hbar}(L + L^\dagger)$  of the object of measurement via the observation of the commuting position operators  $Y(t)$ ,  $t \in \mathbb{R}$  of the output field given by the generalized Heisenberg operator-process,

$$\begin{aligned} \dot{Y}(t) &= e^{\frac{i}{\hbar} H_s t} (I \otimes \hat{q}(0)) e^{-\frac{i}{\hbar} H_s t} \\ &= U(t)^\dagger (I \otimes \hat{q}(t)) U(t) = R(t) + I \otimes \hat{q}(t) \end{aligned}$$

This is the simplest quantum Hamiltonian model for the continuous nondemolition measurement of the physical quantity  $R$  of a quantum object.

Thus the unitary evolution group  $U_s(t) = e^{\frac{-i}{\hbar} H_s t}$  of the compound system is defined on the product  $\mathcal{H} \otimes \mathcal{F}$  with the two-sided Fock space  $\mathcal{F} = \Gamma(L^2(\mathbb{R}))$  by  $U_s(t) = V_1(t)U(t)$ , where  $V_1 = I \otimes \hat{v}$  is the free evolution group  $\hat{v}(t) = e^{\frac{-i}{\hbar} \hat{h} t}$  of the field, corresponding to the shifts

$$f \in L^2(\mathbb{R}) \mapsto f^t(s) = f(s - t)$$

of the one-particle space  $L^2(\mathbb{R})$ . To obtain such an evolution from a realistic Hamiltonian of a system of atoms interacting with an electromagnetic field one has to use a Markovian approximation, corresponding to the weak-coupling or low density limits [39].

Thus, the problem of unitary dilation of the continuous reduction and spontaneous collapse was solved in [25] even for infinite-dimensional Wiener noise in a stochastic equation of type (4.2).

## CONCLUSION

Analysis [1] of the quantum measurement notion shows that it is a complex process, consisting of the stage of preparation [15] and the stage of registration, i.e., fixing of the pointer and its output state and the objectification [40].

The dynamical process of the interaction is properly treated within the quantum theory of singular coupling to get the nontrivial models of continuous nondemolition observation while the statistical process of the objectification is properly treated within the quantum theory of stochastic filtering to get the nonlinear models of continuous spontaneous localization [21–31].

The nondemolition principle plays the role of superselection for the observable processes provided the quantum dynamics is given and restricts the dynamics provided the observation is given. It is a necessary and sufficient condition for the statistical interpretation of quantum causality, giving rise to the quantum noise environment but not to the classical noise environment of the phenomenological continuous reduction and spontaneous localization theories [16–20].

The axiomatic quantum measurement theory based on the nondemolition principle abandons the projection postulate as the redundancy given by a unitary interaction with a meter in the initial eigen-state. It treats the reduction of the wave packet not as a real dynamical process but as the statistical evaluation of the *a posteriori* states for the prediction of the probabilities of the future measurements conditioned by the past observation.

There is no need to postulate a nonstandard, nonunitary, and nonlinear evolution for the continuous state-vector reduction in the phenomenological quantum theories of spontaneous localization, and there is no universal reduction modification of the fundamental Schrödinger equation. The nonunitary stochastic evolution giving the continuous reduction and the spontaneous localization of the state-vector can be and has been rigorously derived within the quantum stochastic theory of unitary evolution of the corresponding compound system, the object of the measurement and an input Bose field in the vacuum state.

The statistical treatment of the quantum measurement as nondemolition observation is possible only in the framework of open systems theory in the spirit of the modern astrophysical theory of the spreading universe. The open systems theory assumes the possibility of producing for each quantum object an arbitrary time series of its copies and enlarges these objects into an environment, a quantum field, innovating the measurement apparatus by means of a singular interaction for a continuous observation.

It is nonsense to consider seriously a complete observation in the closed universe; there is no universal quantum observation, no universal reduction and spontaneous localization for the wave function of the world. Nobody can prepare an *a priori* state compatible with a complete world observation and reduce the *a posteriori* state, except God. But acceptance of God as an external subject of the physical world is at variance with the closeness assumption of the universe. Thus, the world state-vector has no statistical interpretation, and the humanitarian validity of these interpretations would, in any case, be zero. The probabilistic interpretation of the state-vector is relevant to only the induced states of the quantum open objects being prepared by experimentalists in an appropriate compound system for the nondemolition observation to produce the reduced states after the registration.

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#### REFERENCES

- : [1] G. LUDWIG, *Math. Phys.*, 4:331 (1967); 9, 1 (1968).
- : [2] E.B. DAVIES, J. LEWIS, *Commun. Math. Phys.*, 17:239–260, (1970).
- : [3] L.E. BALLENTINE, *Rev. Mod. Phys.*, 42:358–381, (1970).

- : [4] A. SHIMONY, *Phys. Rev. D*, 9:2321–2323, (1974).
- : [5] V.P. BELAVKIN, Optimal linear random filtration of quantum Boson signals. *Problems of Control and Inform. Theory*, 3:47–62, (1974).
- : [6] V.P. BELAVKIN, Optimal quantum filtration of Markovian signals. *Problems of Control and Inform. Theory*, 7(5):345–360, (1978).
- : [7] V.P. BELAVKIN, Optimal filtering of Markov signals with quantum noise, *Radio Eng. Electron. Physics*, 25:1445–1453, (1980).
- : [8] A. BARCHIELLI, L. LANZ, G.M. PROSPERI, *Nuovo Cimento*, 72B:79, (1982).
- : [9] V.P. BELAVKIN, Theory of Control of Observable Quantum Systems, *Automatica and Remote Control*, 44(2):178–188, (1983).
- : [10] A. PERES, *Am. J. Phys.*, 52:644, (1984).
- : [11] R.L. STRATONOVICH, Conditional Markov processes and their applications to optimal control, *MGU*, Moscow 1966.
- : [12] R.E. KALMAN, R.S. BUCY, New results in linear filtering theory and prediction problems, *J. Basic Engineering, Trans. ASME*, 83:95–108, (1961).
- : [13] V.P. BELAVKIN, Nondemolition measurement and control in quantum dynamical systems. In: Proc. of CISM seminar on "*Inform. Compl. and Control in Quantum Physics*", A. Blaquiere, ed., Udine 1985, 311–239, Springer–Verlag, Wien, 1987.
- : [14] V.P. BELAVKIN, Nondemolition measurements, nonlinear filtering and dynamical programming of quantum stochastic processes. In: Proc. of Bellmann Continuum Workshop "*Modelling and Control of Systems*", A. Blaquiere, ed., Sophia–Antipolis 1988, 245–265, *Lect. Not. Cont. Inf. Sci.*, 121, Springer–Verlag, Berlin, 1988.
- : [15] L.E. BALENTINE, *Int. J. Theor. Phys.*, 27:211–218, (1987).
- : [16] P. PEARLE, *Phys. Rev.*, D29:235, (1984).
- : [17] N. GISEN, *J. Phys. A.: Math. Gen.*, 19:205–210, (1986).
- : [18] L. DIOSI, *Phys. Rev.*, A40:1165–1174, (1988).
- : [19] G.C. GHIRARDI, A. RIMINI, T. WEBER, *Phys. Rev.* D34(2):470–491, (1986).
- : [20] G.C. GHIRARDI, P. PEARLE, A. RIMINI, *Phys. Rev.*, A42:478–89, (1990).
- : [21] V.P. BELAVKIN, A new wave equation for a continuous non–demolition measurement", *Phys. Lett.*, A140:355–358, (1989).
- : [22] V.P. BELAVKIN, P. STASZEWSKI, A quantum particle undergoing continuous observation", *Phys. Lett.*, A140:359–362, (1989).
- : [23] V.P. BELAVKIN, A posterior Schrödinger equation for continuous non–demolition measurement, *J. Math. Phys.*, 31(12):2930–2934, (1990).
- : [24] V.P. BELAVKIN, P. STASZEWSKI, Nondemolition observation of a free quantum particle, *Phys. Rev. A.*, 45(3):1347–1356, (1992).
- : [25] V.P. BELAVKIN, Quantum continual measurements and a posteriori collapse on CCR, *Commun. Math. Phys.*, 146, 611–635, (1992).
- : [26] V.P. BELAVKIN, A continuous counting observation and posterior quantum dynamics, *J. Phys. A, Math. Gen.*, 22: L 1109–1114, (1989).
- : [27] V.P. BELAVKIN, A stochastic posterior Schrödinger equation for counting non–demolition measurement, *Letters in Math. Phys.*, 20:85–89, (1990).

- : [28] V.P. BELAVKIN, P. STASZEWSKI, *Rep. Math. Phys.*, 29:213–225, (1991).
- : [29] V.P. BELAVKIN, Stochastic posterior equations for quantum nonlinear filtering. *Probab., Theory and Math. Stat.*, ed. B. Grigelionis, 1:91–109, VSP/Mokslas 1990.
- : [30] A. BARCHIELLI, V.P. BELAVKIN, Measurements continuous in time and a posteriori states in quantum mechanics, *J. Phys. A, Math. Gen.*, 24:1495–1514, (1991).
- : [31] V.P. BELAVKIN, Quantum stochastic calculus and quantum nonlinear filtering, *J. of Multivar. Analysis*, 42(2):171–201, (1992).
- : [32] V.B. BRAGINSKI, Y.I. VORONTZOV, F.J. HALILI, *Sov. Phys.-JETP*, 46(2):765, 4:171–201, (1977).
- : [33] K.S. THORNE, R.W.P. DREVER, C.M. CAVES, M. ZIMMERMANN, V.D. SANDBERG, *Phys. Rev. Lett.*, 40:667, (1978).
- : [34] A.S. HOLEVO, Quantum estimation. In *Advances in statistical signal processing*, 1:157–202, (1987).
- : [35] V.P. BELAVKIN, Reconstruction theorem for quantum stochastic processes, *Theoret. Math. Phys.*, 3:409–431, (1985).
- : [36] K. KRAUS, *States, Effects and operations*, Springer–Verlag, Berlin 1983.
- : [37] E.B. OZAWA, *J. Math. Phys.*, 25:79–87, (1984).
- : [38] A. BARCHIELLI, G. LUPIERI, *J. Math. Phys.*, 26:2222–2230, (1985).
- : [39] L. ACCARDI, R. ALICKI, A. FRIGERIO, Y.G. LU, An invitation to weak coupling and low density limits, *Quantum probability and re. topics VI*, ed. L. Accardi, 3–62, World Scientific, Singapore 1991.
- : [40] P. BUSCH, P.J. LAHTI, P. MITTELSTAEDT, *The quantum theory of measurement, Lecture Notes in Physics*, Springer–Verlag, Berlin 1991.

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