Optimal Measurement and Control in Quantum Dynamical Systems.

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Abstract

A Markovian model for a quantum automata, i.e. an open quantum dynamical system with input and output channels and a feedback is described. A multi-stage version of the theory of quantum measurement and statistical decisions applied to the optimal control problem for quantum dynamical discrete-time objects is developed. Quantum analogies of Stratonovich non-stationary filtering and Bellman quantum dynamical programming for the time being discrete are obtained.

The Gaussian case of quantum one-dimensional linear Markovian dynamical system with a quantum linear transmission line is studied. The optimal quantum multi-stage decision rule consisting of the classical linear optimal control strategy and quantum optimal filtering procedure is found. The latter contains the optimal quantum coherent measurement on the output of the line and the recursive processing by Kalman–Busy filter.

All the results are illustrated by an example of the optimal control problem for a quantum open oscillator at the input of a quantum wave transmission line.

1 Introduction

High perspective of applying quantum coherent electromagnetic generators of optical and infra-red frequency band for communication and control of quantum dynamical objects stimulates an increase of the interest in theoretical investigations of potential possibilities of information systems containing quantum channels.

Due to fundamental limitations of quantum-mechanical measurement a specific problem of optimal nondemolition measurement on the input and the output of quantum channels arises in such investigations. Here we shall consider such a problem for the channels with a feedback, corresponding to the optimal control in quantum open systems. It is essential in quantum theory that systems under the observation should be open, i.e. matched with channels, in order not to demolish them, by letting out an information.

This paper gives the positive answers in a mathematically constructive way to the following fundamental questions of quantum systems theory: Is it possible at all to observe and control a quantum dynamical system in the real time without not destroying it? If yes, what are the optimal strategies of that observation and control? How the dynamics of a quantum system is to be changed under the obtained information and its use as a feedback? What is are the fundamental limitations of quantum observability and controllability? Is there any possibility to obtain a time continuous limit of such observation and control in a quantum system?

The non-dynamical problem of quantum measurement optimization formulated primarily for detection and estimation in the static quantum communication systems by K. Helstrom [1] was studied intensively by several authors [2–7] within the framework of the single-stage (static) quantum-statistical decision theory. The dynamic problem of quantum nondemolition measurement for communication and control has been studied in details by author even in continuous time [8,9] since the pioneering paper [10]. However the earlier paper [11] on the solution of the discrete-time problem of optimal measurement has never been published in full, in spite of the practical importance of this case for the digital communication and control in quantum channels with a feedback. The novelty of this paper was such, that only a few people working in the newly open area of quantum stochastic processes could appreciate it at that time, and it was too far yet from applications. Recently, however, in view of the new possibilities of quantum computations, the interest to quantum theory of communication and control has been renewed, and the time development of the discrete models of quantum open systems for communication and control became actual. Moreover, after the development of time-continuous theory of quantum nondemolition measurement and filtering within the quantum stochastic calculus approach [12], these models can be considered as discrete time analogous and approximations of this theory. The discrete time case is mathematically simpler, it doesn't need the theory of quantum stochastic integration, and might be considered on its own as a dynamical programming for quantum computations, or multi-stage variant of optimal quantum-statistical decision theory.

The quantum dynamical programming for multi-stage optimal measurement problem can be considerably simplified due to assumption that not only the processing of the measurement results but also the quantum measurement itself may depend on all previous measurement results. It corresponds to the assumption that we can choose a quantum measurement apparatus on the basis of the previous measurement data separately at every instant in time. Though in reality it is possible to imagine such a situation only for a finite number of stages and a finite set of measurement results (time and measuring scale being discrete), this extension of admissible measurement and decision procedures is mathematically very convenient and from the physical point of view is not contradictory. The choice of the measuring apparatus and of the observed data processing according to all previous measurement results on the whole defines the strategy in multi-stage quantum decision theory described here. Within the framework of such an approach the problem of quantum filtering of random signal sequences was reduced in [13] to the well-studied problem of the static optimal quantum measurement on every fixed stage with conditional a priori distribution depending on the previous observed data.

Here we describe the multi-stage quantum statistical decision theory applied to the problem of optimal control of a quantum Markovian discrete time system with a matched quantum channel. This theory may be considered as a quantum (operational) analogue of the stochastic control theory, based on Stratonovich theory of conditional Markovian processes [14], and Bellman dynamic programming [15].

The optimal filtering and the control strategy are found here in case of one-dimensional quantum linear Markovian system with quantum Gaussian noises and the mean-square loss function both in the discrete and continuous time.

In order to pose the problem of measurement and control correctly from the physical point of view , let us consider the following motivating example.

2 Controlled quantum open oscillator with quantum transmission line

We are going to give a Markovian model of the simplest quantum system with a communication channel: the quantum open oscillator matched with a transmission line. It is an excellent mathematical model of a single-mode antenna for quantum radiophysics and optical control and communication.

Let x be an operator of complex amplitude of a quantum oscillator with Hamiltonian $\Omega x^* x$, which satisfies the canonical commutation relations with x^* being an adjoint operator

$$[x, x^*] = xx^* - x^*x = \hbar \mathbf{1} \tag{2.1}$$

where $\mathbf{1}$ is the unit operator, and $\hbar > 0$ is the Planck constant.

Assume that in general case this oscillator is controlled by the complex amplitude u by means of a quantum-mechanical transmission line with wave resistance $\gamma/2$, where the operator of the wave $y\left(t - \frac{s}{c}\right)$ travelling from the oscillator into the line is measured. In the simplest case of ideal conjugation between the line and the measuring apparatus, when there is no reflection of the wave travelling from the oscillator, i.e. in case of the matched line, $x\left(t\right)$ and $y\left(t\right)$ are described by the pair of linear equations [16]

$$dx\left(t\right)/dt + \alpha x(t) = \gamma u(t) + v\left(t\right), \quad x\left(0\right) = x, \tag{2.2}$$

$$y\left(t\right) = \bar{\alpha} x\left(t\right) - dx\left(t\right)/dt = \gamma\left(x\left(t\right) - u\left(t\right)\right) - v\left(t\right), \tag{2.3}$$

where, generally speaking, α is a complex number with fixed real part, $\alpha + \bar{\alpha} = \gamma$, and with arbitrary imaginary part depending on the choice of the representation, $v\left(t + \frac{s}{c}\right)$ the amplitude operator of the wave travelling out of line towards the oscillator, this operator is responsible for the commutator preservation. Under natural for super-high and optical frequencies assumption of narrowness of the frequency band which we deal with the commutators for $v\left(t\right)$ in the representation of "rotating waves" have delta-function form [17]:

$$\left[v\left(t\right), v\left(t'\right)\right] = 0, \quad \left[v\left(t\right), v\left(t'\right)^*\right] = \gamma\hbar\mathbf{1}\delta\left(t - t'\right). \tag{2.4}$$

Integrating equation (2.2) and taking into account that $v\left(t\right)$ does not depend on $x(t')$ when $t > t'$, it is easy to verify that the commutator $\left[x\left(t\right), x\left(t\right)^*\right]$ is constant, moreover, $x\left(t\right)$ commutes both with $y\left(t'\right)$ and $y\left(t'\right)^*$ when $t > t'$, and the commutators for $y(t), y(t'), y(t')^*$ coincide with (2.4). The latter means that considering von Neumann reduction which appears as a result of some quantum measurement of $y(t)$ at previous instants of time $t' < t$ does not affect the future behaviour of $x(t^1), y(t^1), t^1 > t$, so that equations (2.2), (2.3) remain unchanged. This fact together with the Markovianity hypothesis of the quantum process $x(t)$ which hold for quantum thermal equilibrium states of the wave $v(t)$ in case of narrow band approximation [17] simplifies to the large extent optimal measurement and control problems for the simplest quantum dynamical system mentioned above.

Let us assume, that the initial state x is Gaussian with the mathematical expectation $\langle x \rangle = z$ and

$$\langle (x-z)(x-z) \rangle = 0, \quad \langle (x-z)^* (x-z) \rangle = \hbar\Sigma,$$

$v(t)$ is the quantum Gaussian white noise, which is described by the following correlations

$$\langle v(t)\ v(t') \rangle = 0, \quad \langle v(t)^* v(t') \rangle = \hbar\sigma\delta(t-t'),$$

with $\sigma = \gamma(\exp(\hbar\gamma/kT) - 1)^{-1}$for the equilibrium state with the temperature T, where $k > 0$ is the Boltzmann constant.

As an example, let us try to choose the optimal measurement of the controlled quantum oscillator (2.2) with transmission line (2.3), so that to minimize its energy $\Omega \langle x^*(t)\ x(t) \rangle$ at the final instant of time $t = \tau$ by means of the control strategy the norm $\int_0^\tau \mid u(t) \mid^2 dt$ of which should not be too great. If we want also to force the quantum amplitude $x(t)$ to follow the classical process $u(t)$, this problem can be characterized by the quality criterion

$$\Omega \langle x(\tau)^* x(\tau) \rangle + \int_0^\tau \langle \theta u(t)^* u(t) + \omega (x(t) - u(t))^* (x(t) - u(t)) \rangle\, dt. \tag{2.5}$$

Here $\theta, \omega \geq 0$ are parameters responsible for the measurement quality: when $\theta = \Omega = 0$ (2.5) corresponds to the problem of pure filtration, when $\omega = 0$, $\theta \neq 0$, it corresponds to the pure control problem.

It will be shown below (see §5) that the optimal measurement minimising criterion (2.5) is statistically equivalent to the measurement of the stochastic process $z(t) = \widehat{x}(t) + x^\circ(t)$ described by Kalman–Bucy filter:

$$d\hat{x}(t)/dt + \alpha\hat{x}(t) = \gamma u(t) + \kappa(t)(y(t) - \gamma(\hat{x}(t) - u(t))). \tag{2.6}$$

Here $\hat{x}(0) = z, \quad \kappa(t) = (\gamma\Sigma(t) - \sigma)/(\mu + \sigma), \Sigma(t)$ is the solution of the equation

$$d\Sigma(t)/dt = (\sigma - \gamma\Sigma(t))(\mu + \gamma\Sigma(t))/(\mu + \sigma), \quad \Sigma(0) = \Sigma,$$

$$dx^\circ(t)/dt + \alpha x^\circ(t) = \kappa(t)(v^\circ(t) - \gamma x^\circ(t)), \quad x^\circ(0) = 0, \tag{2.7}$$

where $v^\circ(t)$ is the amplitude operator with commutators

$$[v^\circ(t), v^\circ(t')] = 0, \quad \left[v^\circ(t), v^\circ(t')^*\right] = -\hbar\gamma\delta(t-t')$$

which change the quantum process $\hat{x}(t)$ into the classical (commutative) diffusion complex process, and with correlations of vacuum noise of the intensity $\mu = 0$ if $\gamma \leq 0$ and $\mu = \gamma$ if $\gamma > 0$:

$$\langle v^\circ(t)\, v^\circ(t') \rangle = 0,\ \langle v^\circ(t)^*\, v^\circ(t') \rangle = \hbar\mu\delta(t-t'). \tag{2.8}$$

For instance, such measurement takes place by the heterodyning [7] where $v^{\circ}(t)$ stands for a standard wave. In this case the optimal control strategy $u^{o}(t)$ coincides with the classical one: $u^{o}(t) = -\lambda(t)\, z(t)$, where $\lambda(t) = (\gamma\Omega(t) - \omega) / (\theta + \omega)$, and $\Omega(t)$ is a solution of the equation:

$$-d\Omega(t)/dt = (\omega - \gamma\Omega(t))\,(\theta + \gamma\Omega(t)) / (\theta + \omega), \quad \Omega(\tau) = \Omega, \tag{2.9}$$

which together with (2.7) defines the minimum quantity of losses (2.5):

$$\hbar\left(\Omega(0)\Sigma + \int_0^{\tau}\left(\Omega(t)\sigma + (\gamma\Omega(t) - \omega)^2\Sigma(t) / (\theta + \omega)\right)dt\right) + \Omega(0) \mid z \mid^2 .$$

By setting $\sigma = 0$, $\omega = 0$, we obtain in particular the solution of the terminal control problem for an oscillator with thermal noise equal to zero. But in this case unlike the classical one the optimal measurement remains indirect and the equation (2.7) remains regular corresponding to the white noise in the channel of intensity $\mid\gamma\mid\hbar$. Thus to consider the quantum measurement postulates is statistically equivalent to the adding of white noise into the channel of intensity $\mid\gamma\mid\hbar$ what excludes the singular case of pure measurement of the amplitude $\hat{x}$.

It is interesting to note that, in the stationary case $\Sigma = \sigma/\gamma$, e.g. in the case of thermal equilibrium when $\gamma > 0$, $T > 0$ and $\Sigma = (\exp\{\hbar\gamma/kT\} - 1)^{-1}$, the optimal amplification coefficient $\kappa(t)$ equals to zero which means the possibility of optimal control of the quantum oscillator without measurement. It also holds when $\omega = \gamma\Omega$, the solution of equation (2.9) is stationary and optimal feed-back coefficient $\lambda(t)$ equals to zero. But in the contrary case $\gamma < 0$, $T < 0$ which corresponds to the active medium of the oscillator (laser) the optimal coefficients $\kappa(t)$, $\lambda(t)$ are strictly negative and non-zero even for the stationary solution $\Sigma(t) = 0$, $\Omega(t) = \theta/\mid\gamma\mid$ of equations (2.7), (2.9).

3 Quantum dynamical filtering

Now let us give a rigorous setting of the quantum dynamical observation problem for the optimal control of a quantum-mechanical object when time is discrete $t \in \{t_k\}_{k=0,1,\ldots}$. Let $\mathcal{A}_k$ be von Neumann algebras on a Hilbert space $\mathcal{H}$, each is generated by one or a few dynamical variables (operators) $x_k = (x_k^i)^{i\in I}$ in $\mathcal{H}$. One can consider a quantum-mechanical object in the Heisenberg picture at the instant of time $t_{k+1} > t_k > 0$ with $x_k = x(t_k)$, such that all algebras $\mathcal{A}_k$ are equivalent to the initial algebra $\mathcal{A}_0 = \mathcal{A}$, generated by the positions and momentums $x = (q, p)$ at $t = 0$. Let $\mathcal{B}_k, k = 1, 2, \ldots$ be von Neumann algebras of observables generated in $\mathcal{H}$ by output dynamical variables $y_k = \left(y_k^j\right)^{j\in J}$, by means of which this object can be observed in a nondemolition way say, on the time intervals $(t_{k-1}, t_k]$. As it has been shown above on the example of the matched transmission line, the output observables $b_k \in \mathcal{B}_k$ in the matched channels should commute with all present and future operators $a_{k^1} \in \mathcal{A}_{k^1}$, $k^1 \geq k$ of the dynamical system, but not necessarily with the past ones $a_{k'} \in \mathcal{A}_{k'}$, $k' < k$. This commutativity condition together with the commutativity $b'_k b_{k'} = b_{k'} b'_k$ for all $b'_k \in \mathcal{B}_k$, $b_{k'} \in \mathcal{B}_{k'}$ $\forall k' \neq k$ will be referred as the nondemolition condition.

Let us denote $\mathcal{P}_k$, $\mathcal{R}_k$ the dual spaces to $\mathcal{A}_k, \mathcal{B}_k$ with respect to some standard pairings $< .,. >$, say the subspaces of trace class operators $\pi_k \in \mathcal{A}_k$, $\rho_k \in \mathcal{B}_k$ which are dual to the simple algebras of all bounded operators $a_k \in \mathcal{A}_k$, $b_k \in \mathcal{B}_k$ on the corresponding Hilbert spaces with respect to the bilinear trace-forms

$$< \pi_k, a_k >= \mathrm{tr}\left[\pi_k a_k\right], < \rho_k, b_k >= \mathrm{tr}\left[\rho_k b_k\right],$$

and denote $\mathcal{S}_k$ the corresponding subspace dual to the von Neumann algebra $\mathcal{B}_k \vee \mathcal{A}_k$ generated by the commutating $\mathcal{B}_k$ and $\mathcal{A}_k$. We shall use the operational terminology, briefly summarised in the Appendix. Thus we shall call the positive normalized elements $\pi_k \in \mathcal{P}_k$, $\rho_k \in \mathcal{R}_k$ and $\sigma_k \in \mathcal{S}_k$, which are usually described by the statistical density operators, the statistical states of the quantum object at the instants of time t_k, the states of the channel on the interval $(t_{k-1}, t_k]$, and the joint state of the object and channel at the moment t_k respectively, or simply the states on $\mathcal{A}_k$, $\mathcal{B}_k$ and $\mathcal{B}_k \vee \mathcal{A}_k \subseteq \mathcal{B}_k \otimes \mathcal{A}_k$.

Now we adopt the hypothesis of Markovianity of the Heisenberg dynamics, restricted to the described quantum object and output channel in $\mathcal{H}$, with respect to a given state of the whole system ω. Let all the induced states $\sigma_k = \omega | \left(\mathcal{B}_k \vee \mathcal{A}_k\right), k = 1, 2\ldots$ and their restrictions ρ_k, π_k on $\mathcal{B}_k$, $\mathcal{A}_k$ be defined by the initial state $\pi_0 = \pi$ on $\mathcal{A}_0 = \mathcal{A}$ and by a family $\{M_k\}_{k-1,2\ldots}$ of statistical morphisms $\pi_{k-1} \mapsto \sigma_k = \pi_{k-1} M_k$. These transition maps $\mathcal{P}_{k-1} \rightarrow \mathcal{S}_k$ can be described as the pre-dual to positive normalized superoperators $M_k : \mathcal{B}_k \otimes \mathcal{A}_k \rightarrow \mathcal{A}_{k-1}$ having for the simple algebras the form

$$M_k c_k = \mathrm{tr}_{\mathcal{B}_k^\circ}\left[\rho_k^\circ c_k\right], \quad \forall c_k \in \mathcal{B}_k \otimes \mathcal{A}_k.$$

Here ρ_k°, $k = 1, 2, \ldots$ are states on some algebras $\mathcal{B}_k^\circ$, for which the simple algebras $\mathcal{B}_k \otimes \mathcal{A}_k$ are isomorphic to the von Neumann tensor products $\mathcal{A}_{k-1} \otimes \mathcal{B}_k^\circ$, and $\mathrm{tr}_{\mathcal{B}^\circ}$ is the partial trace on $\mathcal{B}^\circ$ such that $M_k\left[a_{k-1} \otimes b_k^\circ\right] =< \rho_k^\circ, b_k^\circ > a_{k-1}$ for all $a_{k-1} \in \mathcal{A}_{k-1}$, $b_k^\circ \in \mathcal{B}_k^\circ$. This assumption corresponds to the requirement that the channel should be matched with the object and implies the semigroup dynamics [20] $\pi_{k-1} \mapsto \pi_k = \mathrm{tr}_{\mathcal{B}_k}\left\{\pi_k M_k\right\}$ of the quantum-mechanical object with discrete time. Furthermore, we shall suppose that every morphism M_k may depend on the results $\zeta^k = \{\zeta_{k'}\}_{k'<k}$ of previous measurement data $\zeta_{k'} \in Z, k' < k$, say via dependence of some controlled parameters $u \in U$ of the sequence $\{\zeta_{k'}\}_{k'<k}$ due to a feedback $\zeta^k \mapsto u$.

The nondemolition measurements during the time intervals $(t_{k-1}, t_k]$ are described by positive operator-valued measures $b_k\left(d\zeta\right) \in \mathcal{B}_k$, $k = 1, 2, \ldots$ on the data space $Z \ni \zeta$ with a given Borel structure of the measurable subsets $dz \subseteq Z$ such that $b_k\left(Z\right) = \mathbf{1}$ is the identity operator of $\mathcal{B}_k$. We shall assume that every Z-measurement $b_k\left(d\zeta\right)$ also may depend on all preceding measurement results $\zeta_1, \ldots, \zeta_{k-1}$, and not only due to a dependance on $u \in U$ and the feedback, but directly, being adaptive in time. The functions $\zeta^k \mapsto (M_k\left(\zeta^k\right), b_k\left(\zeta^k, d\zeta\right))$ are supposed to be weakly measurable in the sense that for all $\pi_{k-1} \in \mathcal{P}_{k-1}$ and $a_k \in \mathcal{A}_k$ and all Borel subsets $d\zeta \subseteq Z$ the complex functions

$$\zeta^k \mapsto < \pi_{k-1} M_k\left(\zeta^k\right), b_k\left(\zeta^k, d\zeta\right) a_k >$$

are Borel functions on $Z^k = \prod_{k'<k} Z_{k'}$, where $Z_k = Z, Z_0 = U$. We shall call every sequence $\left\{b_k\left(\zeta^k, d\zeta\right)\right\}_{k=1,2,\ldots}$ of such "conditional", or adaptive measurements the measurement strategy.

Let us denote $B_k\left(\zeta^k, d\zeta\right)$ the conditional transition measures $\mathcal{P}_{k-1} \to \mathcal{P}_k$, that is the operational-valued conditional measures on Z, defined as the predual to superoperator values $B_k\left(\zeta^k, d\zeta\right) : \mathcal{A}_k \to \mathcal{A}_{k-1}$ by the formula

$$a_k \mapsto B_k\left(\zeta^k, d\zeta\right) a_k = M_k\left(\zeta^k\right)\left[b_k\left(\zeta^k, d\zeta\right) \otimes a_k\right], \tag{3.1}$$

and denote $\pi_k\left(d\zeta^{k+1}\right)$ the $\mathcal{P}_k$ - valued measures on Z^{k+1} obtained for $k = 1, 2, \dots$ by the recurrency

$$\pi_k\left(d\zeta^k \times d\zeta\right) = \pi_{k-1}\left(d\zeta^k\right) B_k\left(\zeta^k, d\zeta\right) \tag{3.2}$$

with the initial condition $\pi_0\left(d\zeta^1\right) = \pi\delta\left(u_0, d\zeta^1\right)$ if $Z^1 = U$.

Lemma 1 *All the measures* $\pi_k\left(d\zeta^{k+1}\right)$ *are positive in the sense that*

$$\int < \pi_k\left(d\zeta^{k+1}\right), a_k\left(\zeta^{k+1}\right) > \geq 0$$

for all $\mathcal{A}_k$ *- valued positive measurable functions* $a_k\left(\zeta^{k+1}\right) \geq 0$ *and are normalized,* $< \pi_k\left(Z^{k+1}\right), \mathbf{1} >= 1$*, where* $\mathbf{1}$ *is the identity operator of* $\mathcal{A}_k$*.*

Proof As the superoperator-valued measures $B_k\left(\zeta^k, d\zeta\right)$ are positive and normalized in the sense that $\int B_k\left(\zeta^k, d\zeta\right)\left[a_k\left(\zeta^k, \zeta\right)\right] \geq 0$ for all $a_k\left(\zeta^{k+1}\right) \geq 0$ and $B_k\left(\zeta^k, Z\right)\mathbf{1} = \mathbf{1}$, the lemma can be easily proved by induction, using the positivity and normalization of π_0. Thus the measure $\pi_k\left(d\zeta^{k+1}\right)$, obtained by the recurrency (3.2), describes the total statistical state on the algebra $\mathcal{A}_k$ and on the expanding space $Z^{k+1} = Z^k \times Z$ ∎

Let us define a posteriori state of the object at time t_k as $\mathcal{P}_k$ –valued Radon-Nikodim derivative

$$\pi_{k-1}\left(\zeta^k\right) = \pi_{k-1}\left(d\zeta^k\right) / < \pi_{k-1}\left(d\zeta^k\right), \mathbf{1} > \tag{3.3}$$

which exists in the weak sense due to absolute continuity of π_{k-1} with respect to $< \pi_{k-1}, \mathbf{1} >$.

Theorem 2 *The a posteriori states* $\pi_k\left(\zeta^{k+1}\right)$, $k = 1, 2, \dots$ *can be obtained by the nonlinear recurrency*

$$\pi_k\left(\zeta^k, \zeta\right) = \pi_{k-1}\left(\zeta^k\right) T_k\left(\zeta^k, \zeta, \pi_{k-1}\left(\zeta^k\right)\right), \quad \pi_0\left(\zeta^1\right) = \pi,$$

where $T_k\left(\zeta^{k+1}, \pi_{k-1}\right)$ *is the* $(\mathcal{P}_{k-1} \to \mathcal{P}_k)$*-valued Radon-Nikodim derivative*

$$T_k\left(\zeta^k, \zeta, \pi_{k-1}\right) = B_k\left(\zeta^k, d\zeta\right) / < \pi_{k-1} B_k\left(\zeta^k, d\zeta\right), \mathbf{1} > .$$

Proof The nonlinear transition operations T_k are defined in the weak sense almost everywhere by the Radon-Nikodim derivatives

$$< \pi_{k-1} T_k\left(\zeta^{k+1}, \pi_{k-1}\right), a_k >=< \pi_{k-1} B_k\left(\zeta^k, d\zeta\right), a_k > / < \pi_{k-1} B_k\left(\zeta^k, d\zeta\right), \mathbf{1} > .$$

The proof of the theorem follows immediately by induction due to the Bayes formula

$$< \pi_k \left(d\zeta^k \times d\zeta\right), \mathbf{1} > / < \pi_{k-1}\left(d\zeta^k\right), \mathbf{1} >=< \pi_{k-1}\left(\zeta^k\right) B_k\left(\zeta^k, d\zeta\right), \mathbf{1} >,$$

from the definitions (3.2), (3.3) ∎

Note that the equation (3.4), describing the conditional Markovian evolution of a posteriori state of a quantum-mechanical object, can be regarded as a quantum generalization of Stratonovich nonlinear filter equation with discrete time. A semi-quantum case when a partially observed object is described by a classical Markovian process $\{x_k\}_{k=0,1,\ldots}$ and the channel is non-classical, was considered in [12].

4 Quantum dynamical programming

Let us consider the problem of optimization of the observation strategy $\left\{b_k\left(\zeta^k, d\zeta\right)\right\}$ on the fixed discrete time interval $[0, K]$. The optimal strategy $\{b_k^o\}_{k\in[0,K)}$ is defined as a strategy, which minimizes the average cost

$$\alpha =< \pi_K, a_K > + \sum_{k=1}^{K} \int < \pi_{k-1}\left(d\zeta^k\right), c_{k-1}\left(\zeta^k\right) >, \tag{4.1}$$

given by a self-adjoint semi-bounded operator $a_K \in \mathcal{A}_K$ of final losses, and by similar operator-valued functions $\zeta^{k+1} \mapsto c_k\left(\zeta^{k+1}\right) \in \mathcal{A}_k$, $k = 0, \ldots, K-1$. (In the case of unbounded a_K and $c_k\left(\zeta^{k+1}\right)$ only their spectral measures should belong to $\mathcal{A}_K$ and $\mathcal{A}_k$.) Let us remark that the cost (4.1) does not depend on the last measurement $b_K\left(\zeta^K, d\zeta\right)$ which can be chosen arbitrarily, and $\pi_K = \pi_K\left(Z^{K+1}\right)$. As it follows from definitions (3.1), (3.2) the $\sum_{k'=1}^{k}$ in (4.1) for any $k = 1, \ldots, K$ is independent of the measures $b_{k^1}\left(\zeta^{k^1}, d\zeta\right)$ for $k^1 \geq k$. Hence in order to find the optimal Z–measurement $b_k\left(\zeta^k, d\zeta\right)$ from some $k < K$ it is enough to vary the future average observation cost functional

$$\alpha_k =< \pi_K, a_K > + \sum_{k'=k+1}^{K} \int < \pi_{k'-1}\left(d\zeta^{k'}\right), c_{k'-1}\left(\zeta^{k'}\right) > . \tag{4.2}$$

Lemma 3 *The explicit dependence of α_k on $b_k\left(\zeta^k, d\zeta\right)$ is affine*

$$\alpha_k = \int_{Z^k}\int_Z < \rho_k\left(d\zeta^k, \zeta\right), b_k\left(\zeta^k, d\zeta\right) >, \tag{4.3}$$

where $\rho_k\left(d\zeta^k, \zeta\right) = \pi_{k-1}\left(d\zeta^k\right) A_k\left(\zeta^k, \zeta\right)$. Here $A_k\left(\zeta^{k+1}\right)$ is a $(\mathcal{P}_{k-1} \to \mathcal{R}_k)$ –valued function on ζ^{k+1} which is defined as predual to the superoperators

$$b_k \mapsto A_k\left(\zeta^{k+1}\right) b_k = M_k\left(\zeta^k\right)\left[b_k \otimes a_k\left(\zeta^{k+1}\right)\right], \quad \forall b_k \in \mathcal{B}_k, \tag{4.4}$$

where $a_k\left(\zeta^{k+1}\right)$ is an operator-valued function on Z^k satisfying the linear inverse-time recurrency

$$a_{k-1}\left(\zeta^k\right) = \int B_k\left(\zeta^k, d\zeta\right) a_k\left(\zeta^k, \zeta\right) + c_{k-1}\left(\zeta^k\right), \tag{4.5}$$

$k = 1, \ldots, K$ with the boundary condition $\alpha_K\left(\zeta^{K+1}\right) = a_K$.

Proof First let us prove that the future losses (4.2) can be represented as

$$\alpha_k = \int_{Z^k} \int_Z < \pi_k \left(d\zeta^k \times d\zeta \right), a_k \left(\zeta^k, \zeta \right) >,$$

where $a_k \left(\zeta^{k+1} \right) \in \mathcal{A}_k$ is the solution to the equation (4.5). It is obviously valid for $k = K$, and if it is true for a $k < K$, then substituting (3.2) into this representation of α_k, we obtain

$$\begin{aligned} & \int < \pi_k \left(d\zeta^{k+1} \right), a_k \left(\zeta^{k+1} \right) > + \int < \pi_{k-1} \left(d\zeta^k \right), c_{k-1} \left(\zeta^k \right) > \\ = & \int_{Z^k} < \pi_{k-1} \left(d\zeta^k \right), \int_Z B_k \left(\zeta^k, d\zeta \right) a_k \left(\zeta^k, \zeta \right) > + c_{k-1} \left(\zeta^k \right). \end{aligned}$$

So this is also valid for α_{k-1} with a_{k-1} given in (4.5), and by using the inverse-time induction, it is valid for any $k \in [0, K)$. Now we can obtain (4.3) by

$$< \pi_k \left(d\zeta^k \times d\zeta \right), a_k \left(\zeta^k, \zeta \right) > = < \pi_{k-1} \left(d\zeta^k \right) B_k \left(\zeta^k, d\zeta \right), a_k \left(\zeta^k, \zeta \right) >$$

$$= < \pi_{k-1} \left(d\zeta^k \right) M_k \left(\zeta^k \right), b_k \left(\zeta^k, d\zeta \right) \otimes a_k \left(\zeta^k, \zeta \right) >$$

$$= < \pi_{k-1} \left(d\zeta^k \right) A_k \left(\zeta^k \right), b_k \left(\zeta^k, d\zeta \right) > = < \rho_k \left(d\zeta^k, \zeta \right), b_k \left(\zeta^k, d\zeta \right) >,$$

where we used the definitions (3.1) and (4.4) for the operations B_k and A_k ■

Theorem 4 *If the strategy* $\left\{ b_k^o \left(\zeta^k, d\zeta \right) \right\}_{k \in [1,K)}$ *is optimal for the cost functional (4.1), it satisfies the following system of equations*

$$\left(\rho_k \left(d\zeta^k, \zeta \right) - \lambda_k \left(d\zeta^k \right) \right) b_k^o \left(\zeta^k, d\zeta \right) = 0, \tag{4.6}$$

$k \in [1, K)$*, where*

$$\lambda_k \left(d\zeta^k \right) = \int_Z \rho_k \left(d\zeta^k, \zeta \right) b_k^o \left(\zeta^k, d\zeta \right).$$

These equations together with the system of inequalities

$$\rho_k \left(d\zeta^k, \zeta \right) \geq \lambda_k \left(d\zeta^k \right), \quad k \in [1, K) \tag{4.7}$$

give the necessary and sufficient conditions of the optimality for quantum measurement strategy b_k^o*,* $k = 1, ...K - 1$ *corresponding to the minimal values*

$$\alpha_k^o = \int < \lambda_k \left(d\zeta^k \right), \mathbf{1} > \tag{4.8}$$

of the future average costs (4.2).

Proof As the variables $b_k \left(\zeta^k, d\zeta \right)$, $k = 1, 2, \ldots, K$ of the functional (4.2) are independent, the optimal measure $b_k^o \left(\zeta^k, d\zeta \right)$ minimizes the affine functional separately for every fixed family $\left\{ b_{k^1} \left(\zeta^{k^1}, d\zeta \right) \right\}_{k^1 > k}$. The necessary and sufficient conditions (4.6), (4.7) of optimality for $b_k^o\left(\zeta^k, d\zeta\right)$, minimizing the affine functional (4.3) with a fixed k, follow immediately by the linear programming method, as it was noted in the single-stage theory of optimal quantum measurements [2, 4–7] ■

Note that the minimal value α^o of the total average cost (4.1) is given by the solution $a^o = a_0^o$ of the recurrency (4.5) with $B_k = B_k^o$ at $k = 0$ as $\alpha^o =< \pi, a^o >$.

Let us note that with the help of the a posteriori states $\pi_k\left(\zeta^k\right)$, one can write conditions (4.6), (4.7) in the following form

$$\left(\rho_k\left(\zeta^k, \zeta\right) - \lambda_k\left(\zeta^k\right)\right) b_k^o\left(\zeta^k, d\zeta\right) = 0, \tag{4.9}$$

$$\rho_k\left(\zeta^{k+1}\right) \geq \lambda_k\left(\zeta^k\right), \quad k \in [1, K), \tag{4.10}$$

where $\rho_k\left(\zeta^{k+1}\right) = \pi_{k-1}\left(\zeta^k\right) A_k\left(\zeta^{k+1}\right)$. According to the Bellman dynamical programming method [15] the verification of the optimality condition formulated above can be carried out sequentially in inverse time $k = K-1, \ldots, 1$ applying the recurrence (4.5) for the superoperator $A_k\left(\zeta^{k+1}\right)$ after solving the filtering recurrent equation (3.4).

The optimal control of Markovian partially observed quantum-mechanical object can be reduced to the optimal measurement problem investigated above as follows. Let $M_k\left(u_{k-1}\right) : \mathcal{P}_{k-1} \to \mathcal{R}_k \otimes \mathcal{P}_k$ be the quantum statistical morphisms (transitions) controlled by some parameters $u_k \in U$, $k = 0, \ldots, K-1$. A control strategy $\{\gamma_k\}_{k<K}$ is given by a choice of the feedback, defined by a measurable dependence γ_k of each u_k on all measurement data $\eta_{k'} \in Y$, $k' \leq k$, and also on the preceding controls $u_{k'}$, $k' < k$. The optimal control for a fixed measurement strategy is supposed to minimize the average cost defined by a final operator a_K and operator-valued cost functions $c_k\left(u_k\right)$, $k = 0, \ldots, K-1$. Denoting $\zeta^1 = u_0, \zeta^k = (u_0, \eta_1, u_1, \ldots, \eta_{k-1}, u_{k-1})$, $\zeta = (\eta, u)$, the average cost functional even with random control strategies can be represented in the form (4.1), given by the quantum measurement strategy $\left\{b_k\left(\zeta^k, d\zeta\right)\right\}$ on $Z = Y \times U$ of the form

$$b_k^o\left(\zeta^k, d\eta \times du\right) = b_k^o\left(\zeta^k, d\eta\right) \delta\left(\gamma_k^o\left(\zeta^k, \eta\right), du\right) \tag{4.11}$$

and $c_0\left(\zeta^1\right) = c_0\left(u_0\right)$, $c_k\left(\zeta^{k+1}\right) = c_k\left(u_k\right)$. The quantum optimal control problem can be formulated then as one of searching for the optimal $Y \times U$ –measurements $b_k^o\left(\zeta^k, d\zeta\right)$, $k \in (1, k)$, and an optimal initial control u^o corresponding to the minimal value

$$\alpha^o = \inf_u < \lambda_1(u), \mathbf{1} > + < \pi_0, c_0(u) >$$

of average cost (4.1). In general, the optimal measurement strategy may not be in the product form (4.11), but if there exists a non-randomized strategy $u_k^o = \gamma_k^o\left(\zeta^k, \eta\right)$, $k \in [1, K)$ for some Y-measurements $b_k^o\left(\zeta^k, d\eta\right)$ for which the $Y \times U$ - measurements are optimal, where $\delta(\cdot, \cdot)$ is the Dirac δ- measure, then the data spaces Y may be called the sufficient spaces. The optimal measurements $b_k^o\left(\zeta^k, d\eta\right)$ on sufficient data spaces Y satisfy obviously the equations

$$\left(\rho_k\left(\zeta^k, \eta, \gamma_k^o\left(\zeta^k, \eta\right)\right) - \lambda_k\left(\zeta^k\right)\right) b_k^o\left(\zeta^k, d\eta\right) = 0, \quad k \in [1, K),$$

where

$$\lambda_k\left(\zeta^k\right) = \int \rho_k\left(\zeta^k, \eta, \gamma_k^o\left(\zeta^k, \eta\right)\right) b_k^o\left(\zeta^k, d\eta\right),$$

which together with the inequalities (4.10) are necessary and sufficient for the non-randomized control strategy $\{\gamma_k^o\}$.

5 Quantum filtering in Boson linear Markovian system in a Gaussian state

We examine a Markovian one-dimensional quantum dynamical system, described at discrete instants $t_k = k\Delta$ by the algebras $\mathcal{A}_k$ and $\mathcal{B}_k$, which are generated by the non-selfadjoint operators $x_k \neq x_k^*$ and $y_k \neq y_k^*$ respectively, satisfying the canonical commutation relations. Let us suppose that they act in the same Hilbert space $\mathcal{H}$, where they satisfy the linear quantum stochastic equations

$$x_k = \phi x_{k-1} + \beta u_{k-1} + v_k \tag{5.1}$$

$$y_k = \gamma x_{k-1} + \delta u_{k-1} + w_k. \tag{5.2}$$

Here $\phi, \beta, \gamma, \delta$ are some, in general complex parameters, the controls u_k can also accept complex values, $x_0 = x$ is the initial operator in $\mathcal{H}$, generating the algebra $\mathcal{A}$, and v_k, w_k are some operators in $\mathcal{H}$, generating the algebras $\mathcal{B}_k^\circ$. To obtain the Markov dynamics, we need to assume the independence of x and all the pairs (v_k, w_k) such that the algebras $\mathcal{A}$ and $\mathcal{B}_k^\circ$, corresponding to different instants of time t_k, commutate, and the joint state ω is the product of the states on $\mathcal{A}$ and all $\mathcal{B}_k^\circ, k = 1, 2 \ldots$. We shall define the canonical commutation relations for the generating operators x, v_k, w_k with their adjoints as following:

$$[x, x^*] = \hbar\mathbf{1} \qquad [v_k, v_k^*] = \left(1- \mid \phi \mid^2\right)\hbar\mathbf{1},$$

$$[w_k, w_k^*] = \left(\varepsilon- \mid \gamma \mid^2\right)\hbar\mathbf{1}, \quad [w_k, v_k^*] = -\bar{\phi}\gamma\hbar\mathbf{1}, \tag{5.3}$$

where $\hbar > 0$ and $\mathbf{1}$ is the identity in $\mathcal{H}$ (other, unwritten commutators, including all those corresponding to different instants of time to be equal to zero.) Here the choice of the commutator $[w_k, v_k^*]$, responsible for the commutativity $[y_k, x_k^*] = 0$ is essential, the other nonzero commutators are chosen so that the commutators

$$[x_k, x_k^*] = \hbar\mathbf{1}, \quad [y_k, y_k^*] = \varepsilon\hbar\mathbf{1}$$

should be constant. The described system we shall call the discrete linear Markovian quantum open oscillator.

Let us describe the states $\pi_k \in \mathcal{P}_k$ by the Glauber [21] distributions $p_k(\xi)$, $\xi \in \mathbf{C}$, normalized on the complex plane $\mathbf{C}$ with respect to the Lebesgue measure $d\xi = d\mathrm{Re}\xi d\mathrm{Im}\xi/\pi\hbar$. In the representation described in the Appendix, the Markovian morphisms $\mathcal{P}_{k-1} \to \mathcal{P}_k$, corresponding to the linear equations (5.1), (5.2), transform the distributions $p_{k-1}(\xi)$ into the two-dimensional distributions

$$g_k(\xi, \eta) = \int q_k\left(\xi - \phi\xi^1 - \beta u, \eta - \gamma\xi^1 - \delta u\right) p_{k-1}\left(\xi^1\right) d\xi^1, \tag{5.4}$$

where $q_k(\xi,\eta)$ are some other (not necessarily Glauber) distributions on $\mathbf{C}^2$, which describe the independent states ρ_k° on algebras $\mathcal{B}_k^\circ$.

When $\varepsilon = 0$, the operators y_k, y_k^* are simultaneously measurable, and the a posteriori states on $\mathcal{A}_k$ under the fixed spectral values $y_{k'} = \eta_{k'}$ and $u_{k'}$, $k' < k$ are defined recurrently by the a posteriori Glauber distribution $p_k\left(\xi \mid \zeta^{k-1}\right)$ according the Bayes formula

$$p_k\left(\xi \mid \zeta^{k-1}\right) = g_k\left(\xi, \eta_k \mid \zeta^k\right)/r_k\left(\eta_k \mid \zeta^k\right).$$

Here $g_k\left(\xi,\eta \mid \zeta^k\right)$are the distributions obtained by substitution of $p_{k-1}\left(\xi \mid \zeta^k\right)$ into (5.4) instead of $p_{k-1}(\xi)$, and

$$r_k\left(\eta \mid \zeta^k\right) = \int g_k\left(\xi,\eta \mid \zeta^k\right) d\xi$$

are the probability distributions, describing the complex values η_k, which arise as the results of the direct measurements of y_k under the fixed $\zeta^k = (u_0, \eta_1, u_1, \ldots, \eta_{k-1}, u_{k-1})$.

When $\varepsilon \neq 0$, only indirect measurement of y_k are possible which are described, for instance, by the $\mathcal{B}_k$ –valued measures

$$b_k(d\eta) = \# m_k(\eta - y_k) \# d\eta, \tag{5.5}$$

represented by some distributions $m_k(\eta)$ on $\mathbf{C}$ as it is described in the Appendix (A.3). In this case in order to calculate a posteriori Glauber distribution one should change $q_k(\xi,\eta)$ in formula (5.4) for the distribution

$$q_k^1(\xi,\eta) = \int m_k\left(\eta - \eta^1\right) q_k\left(\xi,\eta^1\right) d\eta^1. \tag{5.6}$$

Theorem 5 *Let the initial state π of the quantum oscillator be described by the Glauber distribution of Gaussian type*

$$p(\xi) = \frac{1}{\Sigma}\exp\left\{-\mid \xi - z \mid^2 /\hbar\Sigma\right\}, \tag{5.7}$$

the distributions $q_k(\xi,\eta)$, describing the transitions (5.4), be also Gaussian:

$$q_k(\xi,\eta) = \frac{\exp\left\{-\left(\nu \mid \xi \mid^2 + 2\mathrm{Re}\upsilon\xi\bar{\eta} + \sigma \mid \eta \mid^2\right)/\hbar\left(\sigma\nu - \mid \upsilon \mid^2\right)\right\}}{\sigma\nu - \mid \upsilon \mid^2}, \tag{5.8}$$

and the measures b_k are described as in (5.5), by the Gaussian distributions

$$m_k(\eta) = \frac{1}{\mu}\exp\left\{-\mid \eta \mid^2 /\hbar\mu\right\}. \tag{5.9}$$

Then a posteriori states (3.3) at each instant $k = 1, 2, \ldots,$ are given by the conditional Glauber distributions of Gaussian form

$$p_k\left(\xi \mid \zeta^{k+1}\right) = \frac{1}{\Sigma_k}\exp\left\{-\mid \xi - z_k \mid^2 /\hbar\Sigma_k\right\}, \tag{5.10}$$

where z_k, Σ_k are defined by the recurrent equations of the complex Kalman filter:

$$z_k = \phi z_{k-1} + \beta u_{k-1} + \kappa_k\left(\eta_k - \gamma z_{k-1} - \delta u_{k-1}\right), \quad z_0 = z, \tag{5.11}$$

$$\Sigma_k =| \phi |^2 \Sigma_{k-1} + \sigma - | \kappa_k |^2 \Psi_k, \quad \Sigma_0 = \Sigma, \tag{5.12}$$

where

$$\kappa_k = \left(\phi\bar{\gamma}\Sigma_{k-1} - \upsilon\right)/\Psi_k \quad \Psi_k =| \gamma |^2 \Sigma_{k-1} + \nu^1, \quad \nu^1 = \nu + \mu.$$

Proof Due to the chosen representation, the proof is similar to the derivation of the classical one-dimensional Kalman filter for the complex Gaussian process x_k given by (5.1) and $y_k^1 = y_k + w_k^\circ$, where w_k° are independent Gaussian variables with zero mean values and the covariances $\mu \geq \varepsilon$. (For this proof see, for instance, [22].) One should only take into account that distributions (5.6) are also Gaussian of the type (5.8) with the parameter $\nu^1 = \nu + \mu$ instead of ν. Substituting $q(\xi, \eta)$ in (5.4) by $q^1(\xi, \eta)$ and $p_{k-1}(\xi)$ by the conditional distribution $p_{k-1}\left(\xi \mid \zeta^k\right)$ of type (5.10), we obtain

$$g_k^1\left(\xi, \eta \mid \zeta^k\right) = p_k\left(\xi \mid \zeta^{k-1}\right) r_k^1\left(\eta \mid \zeta^k\right),$$

where $p_k\left(\xi \mid \zeta^{k-1}\right)$ is the Gaussian distribution (5.10) with the parameters (5.11), (5.12), and

$$r_k^1\left(\eta \mid \zeta^k\right) = \frac{1}{\Psi_k} \exp\left\{- \mid \eta - \gamma z_{k-1} \mid^2 / \hbar\Psi_k\right\}. \tag{5.13}$$

Thus the quantum Gaussian filtering is controlled by the classical Kalman filter for the complex amplitude in the Glauber representation ∎

Note, that in distinction from the classical case, the covariance matrix of distributions (5.8), (5.9) should not only be non–negative definite but should also satisfy the Heisenberg uncertainty principle

$$\begin{pmatrix} \sigma & -\upsilon \\ -\bar{\upsilon} & \nu \end{pmatrix} \geq \begin{pmatrix} | \phi |^2 - 1 & \phi\bar{\gamma} \\ \gamma\bar{\phi} & | \gamma |^2 - \varepsilon \end{pmatrix}, \quad \mu \geq \varepsilon, \tag{5.14}$$

as it follows from inequality (A.5). In particular it excludes the case $\mu = 0$ of the direct observation of y_k when $\varepsilon > 0$.

As shown in the next paragraph, a posteriori mathematical expectations z_k with $\mu = \max(0, \varepsilon)$ appear to be the optimal estimates $u_k^o = z_k$ of the operators x_k with respect to the square quality criterion $c_k(u_k) =: | x_k - u_k |^2$: with the minimal mean square error $\hbar\Sigma_k$. In the commutative case $[x_k, x_k^*] = 0$ this optimality was proved in [11].

Note, that instead of calculating z_k by means of the recurrent formula (5.11) using the results $(\eta_1, ..., \eta_k)$ of the indirect measurement (5.5) one may regard z_k itself as a results of the measurement described by the $\mathcal{B}_k$–valued measure:

$$b_k\left(\zeta^k, dz\right) = \# n_k(z - \hat{x}_k) \# dz, \tag{5.15}$$

where

$$n_k(z) = \frac{1}{| \kappa_k |^2} m_k(z / | \kappa_k |),$$

and

$$\hat{x}_k = \phi z_{k-1} + \beta u_{k-1} + \kappa_k(y_k - \gamma z_{k-1}) \tag{5.16}$$

is an operator, depending on the values z_{k-1}, u_k, and independent of the preceding measurement and control results.

It is interesting to consider the time continuous limit, when the quantum oscillator (5.1), (5.2) is described by the quantum stochastic differential equations

$$dx(t) + \alpha x(t)\,dt = \beta u(t)\,dt + v(dt), \tag{5.17}$$

$$y(dt) = \gamma x(t)\,dt + \delta u(t)\,dt + w(dt), \tag{5.18}$$

i.e. by equations (5.1), (5.2) with $x(t_k) = x_k,\quad y(\Delta t_k) = y_k,\quad \phi \simeq 1 - \alpha\Delta,\quad \beta \simeq \beta\Delta,\quad \gamma \simeq \gamma\Delta,\quad \varepsilon \simeq \varepsilon\Delta$, where $(\Delta t_k) = t_k - t_{k-1} = \Delta$ tends to zero. In addition to that the commutation relations (5.3) change in the following way

$$[x, x^*] = \hbar\mathbf{1}, \quad [v(dt), v(dt)^*] = (\alpha + \bar{\alpha})\,\hbar dt\mathbf{1},$$

$$[w(dt), w(dt)^*] = \varepsilon\hbar dt\mathbf{1}, \quad [w(dt), \nu(dt)^*] = -\gamma\hbar dt\mathbf{1},$$

and the other commutators including those corresponding to the different instants of time are equal to zero. By passing to the limit as $\Delta \longrightarrow 0$ when $\sigma \simeq \sigma\Delta,\quad \upsilon \simeq \upsilon{\cdot}\Delta,\quad \nu \simeq \nu{\cdot}\Delta$, it is easy to obtain under the assumptions of the Theorem 3 that a posteriori state $\pi(t, \zeta^t)$ is described by the Glauber distribution $p(t, \xi \mid \zeta^t)$ of Gaussian type (5.10) with the parameters $z(t), \Sigma(t)$ which correspond to the Kalman–Busy filter

$$dz(t) + \alpha z(t)\,dt = \beta u(t)\,dt + \kappa(t)\left(\eta(dt) - \left(\gamma z(t) - \delta u(t)\right)dt\right). \tag{5.19}$$

Here $\kappa(t) = \left(\bar{\gamma}\Sigma(t) - \upsilon\right)/\nu^1,\quad \nu^1 = \upsilon + \mu,\quad z(0) = z,\quad \Sigma(0) = \Sigma,$

$$d\Sigma(t)/dt + (\alpha + \bar{\alpha})\,\Sigma(t) = \sigma - \mid \kappa(t) \mid^2 \nu^1,$$

and $\eta(dt)$ are the results of the corresponding indirect measurement of $y(dt)$ which are realized by the measurement of the sum $y(dt) + w^\circ(dt)$, where $w^\circ(dt)$ is an independent quantum white noise, defined by the coefficients ε, μ :

$$[w^\circ(dt), w^\circ(dt)^*] = -\varepsilon\hbar dt\mathbf{1}, \quad \langle w^\circ(dt)^* w^\circ(dt)\rangle = \mu\hbar dt.$$

As shown at the end of the next paragraph, such “continuous” measurement appears to be also optimal in the Gaussian case when $\mu = \max(0, \varepsilon)$.

6 Optimal measurement and control in quantum open linear system

In the following theorem it is not required that the distributions p_0, q_k and m_k should be Gaussian and it is assumed only that they should have the zero mathematical expectations, and the covariations should coinside with the covariances $\Sigma, \delta, \upsilon, \nu, \mu$ of the distributions (5.7) – (5.9) respectively, not necessary being of the form (5.11).

Theorem 6 *Let the operator of final losses be quadratic:* $a_K = \Omega x_K^* x_K$*, where* $\Omega \geq 0$*, and*

$$c_k(u_k) = \omega x_k^* x_k - \vartheta\bar{u}_k x_k - \bar{\vartheta} u_k x_k^* + \vartheta^1 \mid u_k \mid^2, \quad \omega \geq 0, \quad \vartheta^1 > 0 \tag{6.1}$$

be quadratic loss operators for all $k \in [0, K)$. Suppose $u_k = -\lambda_k z_k, k \in [0, K)$ is a linear control strategy, where z_k are the linear estimates (5.11) based on the results η_k of the indirect measurement (5.5), and

$$\lambda_k = \left(\phi\bar{\beta}\Omega_{k+1} - \vartheta\right) / \Upsilon_k, \tag{6.2}$$

with $\Upsilon_k =| \beta |^2 \Omega_{k+1} + \vartheta^1$ and Ω_k satisfying the following equation

$$\Omega_k =| \phi |^2 \Omega_{k+1} + \omega - | \lambda_k |^2 \Upsilon_k, \quad \Omega_K = \Omega. \tag{6.3}$$

Then the operators of future losses (4.5) are also quadratic:

$$a_k\left(\zeta^{k+1}\right) = d_k\mathbf{1} + \Upsilon_k \mid u_k + \lambda_k z_k \mid^2 + \Omega_k x_k^* x_k$$
$$+\Gamma_k \left(z_k - x_k\right)^* \left(z_k - x_k\right) - 2\mathrm{Re}\Lambda_k \left(u_k + \lambda_k z_k\right)^* \left(z_k - x_k\right), \tag{6.4}$$

where

$$d_k = \hbar \sum_{i=k+1}^{K} \left(\Omega_i\sigma + \Gamma_i\left(\sigma + 2\mathrm{Re}x_i\bar{\upsilon} + \nu^1 \mid \kappa_i \mid^2\right)\right),$$
$$\Gamma_k =| \lambda_k |^2 \Upsilon_k + | \phi - \kappa_{k+1}\gamma |^2 \Gamma_{k+1}, \quad \Gamma_k = 0, \tag{6.5}$$

and

$$\Lambda_k = \phi\bar{\beta}\Omega_{k+1} - \vartheta.$$

Proof In the representation

$$a_K =: \alpha_K\left(x_K\right):, \quad c_k\left(u_k\right) =: \sigma\left(x_k, u_k\right):, \quad a_k\left(\zeta^{k+1}\right) =: \alpha_{k+1}\left(x_k, \zeta^{k+1}\right):$$

the recurrent equation (4.6) has the form

$$\alpha_k\left(\xi_k, \zeta^{k+1}\right) = \int \alpha_{k+1}\left(\xi, \zeta^{k+1}, \eta, u\right) q^1\left(\xi - \phi\xi_k - \beta u_k, \eta - \gamma\xi_k - \delta u_k\right) d\xi d\eta + \sigma\left(\xi_k, u_k\right) \tag{6.6}$$

where

$$u = -\lambda_{k+1} z, \quad z = \phi z_k + \beta u_k + \kappa_{k+1}\left(\eta - \gamma z_k\right).$$

Let us assume that the function $\alpha_k(\xi)$ has the quadratic form (6.4); in particular, it has this form at $k = K$, namely $\alpha_K(\xi) = \Omega \mid \xi \mid^2$. Inserting the latter into (6.6) and integrating, we obtain, that the function α_{k-1} is of the same form with $\Upsilon_{k-1} = \vartheta^1 + \mid \beta \mid^2 \Omega_k$ and $\Omega_{k-1}, \Gamma_{k-1}$ given by (6.3) and (6.5), and

$$d_{k-1} = d_k + \hbar\left(\Omega_k\sigma + \Gamma_k\left(\sigma + 2\mathrm{Re}\kappa_k\bar{\upsilon} + \nu^1 \mid \kappa_k \mid^2\right)\right).$$

Summing $\Sigma_{i=k}^K \left(d_{i-1} - d_i\right)$ and taking into account that $d_K = 0$ and $\Gamma_K = 0$, we obtain (6.4) also for $k - 1$ ∎

Lemma 7 *Let us assume that starting from the instant $k+1$, the controls u_k are chosen to be linear $u_{k^1} = -\lambda_{k^1} z_{k^1}$ with the coefficients (6.2), where $z_{k^1}, k^1 > k$ depend linearly on the results of the subsequent indirect measurement $\eta_{k+1}, \ldots, \eta_{K-1}$ by virtue of the formula (5.11) with the initial condition $z_k = z$. Let also the indirect measurements be described by the Gaussian distributions (5.9) up to the k. Then the operator $\rho_k\left(\zeta^k, \zeta\right)$, defined in (4.9), has the following normal form*

$$\rho_k\left(\zeta^k, \zeta\right) = \lambda_k\left(\zeta^k\right) +$$

$$+ : \left(\Upsilon_k \mid u + \lambda_k \hat{x}_k \mid^2 + \mid \phi - \kappa_{k+1}\gamma \mid^2 \Gamma_{k+1} \mid z - \hat{x}_k \mid^2\right) r_k^o\left(y_k \mid \zeta^k\right) :, \tag{6.7}$$

where

$$\lambda_k\left(\zeta^k\right) =: \left(\Omega_k \mid \hat{x}_k \mid^2 + \left(\hbar\left(\Omega_k + \Gamma_k\right)\Sigma_k + d_k\right)\mathbf{1}\right) r_k^o\left(y_k \mid \zeta^k\right) :,$$

the operator $\hat{x}_k$, defined in (5.16), is linear with respect to y_k, and $r_k^o\left(\eta \mid \zeta^k\right)$ is the distribution (5.14) with the parameters $\nu^o = \nu + \mu^o$, where $\mu^o = \max(0, \varepsilon)$.

Proof Indeed, the operator $\rho_k\left(\zeta^{k+1}\right)$ similar to the density operator $\rho \in \mathcal{R}_k$ is defined by the distribution $r_k\left(\eta, \zeta^{k+1}\right) =$

$$\int\int \alpha_k\left(\xi, \zeta^{k+1}\right) q_k\left(\xi - \phi\xi_{k-1} - \beta u_{k-1}, \eta - \gamma\xi_{k-1} - \delta u_{k-1}\right) p_{k-1}\left(\xi_{k-1} \mid \zeta^k\right) d\xi d\xi_{k-1}. \tag{6.8}$$

It is a symbol of the contrary order (see the Appendix), which is normal when $\varepsilon < 0$ and antinormal when $\varepsilon > 0$. In the former case, inserting the operator symbol (6.4) into (6.9) and integrating with respect to the Gaussian type of the distribution $p_{k-1}\left(\xi_{k-1} \mid \zeta^k\right)$, we obtain (6.7), where $r_k^o\left(\eta \mid \zeta^k\right)$coincides with the distribution $r_k\left(\eta \mid \zeta^k\right)$of the Gaussian type (5.13) with the parameter $v^1 = \nu$. In the latter case $\varepsilon > 0$, the normal symbol of the operator $\rho_k\left(\zeta^{k+1}\right)$ is obtained from (6.9) by means of the convolution of type (A.2) with the distribution (5.9) with $\mu = \varepsilon$, and in the result of the parameter ν increases for ε. In this case $r_k^o\left(\eta|\zeta^k\right)$ is also the normal symbol of the conditional density operator $\rho_k\left(\zeta^k\right)$ on $\mathcal{B}_k$ ■

Theorem 8 *Let the quantum oscillator (5.1), (5.2) be described by the Gaussian initial and transitional distributions of the Gaussian form (5.7), (5.8), and the quality criterion (4.2) be defined by the quadratic final and transitional operators $\alpha_K = \Omega x_K^* x_K$ and $c_k(u_k)$ of form (6.1) respectively. Then the optimal strategy is linear: $u_k = -\lambda_k z_k$, where λ_k is defined by (6.2), and z_k are optimal linear estimates (5.11) based on the results $\{\eta_i\}_{i \leq k}$ of the coherent measurements (5.5) which are described by the distributions (5.9) with the minimal value of the parameter $\mu = \mu^o$.*

Proof We should verify the necessary and sufficient optimality conditions (4.10), (4.11) for the operator (6.7) and the mentioned above measurement at each instant k. As $\Upsilon_k, \Gamma_k \geq 0$, and the density operator $: r_k\left(y_k \mid \zeta^k\right) :$ is non–negative definite, the differences $\rho_k\left(\zeta^{k+1}\right) - \lambda_k\left(\zeta^k\right)$ are non-negative definite operators as well. It remains to verify the equations (4.13) for the optimal strategy $\gamma_k^o\left(\zeta^k,\eta\right) = -\lambda_k z_k$ of the coherent measurements (5.5) or, what is the same, of the measurements (5.15) with the Gaussian distributions $n_k^o(z)$, corresponding to the case $\mu = \mu^o$. Inserting $u = -\lambda_k z$ into (6.7) and taking into account (6.5), we obtain

$$\rho_k\left(\zeta^k,\eta,\gamma_k^o\left(\zeta^k,\eta\right)\right) - \lambda_k\left(\zeta^k\right) = \Gamma_k :\mid z - \hat{x}_k \mid^2 r_k\left(y_k \mid \zeta^k\right): .$$

Thus, equations (4.13) with $\varepsilon > 0$ can be written in the form

$$(z - \hat{x}_k)\,\#n_k^o\,(z - \hat{x}_k)\,\# = 0, \tag{6.9}$$

and the adjoint ones can be written for $\varepsilon < 0$ also as

$$\#n^o\,(z - \hat{x}_k)\,\#\,(z - \hat{x}_k) = 0. \tag{6.10}$$

The operators $\#n_k^o\,(z - \hat{x}_k)\,\#$ described by the Gaussian distributions $n_k^o(z)$, which realize the lower bound of the Heisenberg inequality, are well known as proportional to coherent projectors [8]. The operators $\hat{x}_k$ when $\varepsilon > 0$, are proportional to the annihilation operators, and when $\varepsilon < 0$, they are proportional to the creation operators, for which the coherent projectors are the right and the left eigen-projectors respectively. Hence, the equations (6.9) is satisfied in the case $\varepsilon > 0$, and the equation (6.10) is satisfied if $\varepsilon < 0$. Note that, in the antinormal case when the coherent projectors are described by the Dirac distributions $\delta(z)$ on $\mathbf{C}$, these equations are written as the identities

$$(z - \hat{x}_k)\,\#\delta\,(z - \hat{x}_k)\,\# = \#\,(z - \hat{x}_k)\,\delta\,(z - \hat{x}_k)\,\# = 0, \quad \varepsilon > 0,$$

$$\#\delta\,(z - \hat{x}_k)\,\#\,(z - \hat{x}_k) = \#\delta\,(z - \hat{x}_k)\,(z - \hat{x}_k)\,\# = 0, \quad \varepsilon < 0.$$

The minimal losses, corresponding to the optimal quantum strategy are defined by the following formula

$$\alpha^o = \Omega_0 \mid z \mid^2 + \hbar\left(\Omega_0\Sigma + \sum_{k=1}^{K}\left(\Omega_k\sigma + \bar{\lambda}_{k-1}\left(\phi\bar{\beta}\Omega_k - \vartheta\right)\Sigma_{k-1}\right)\right), \tag{6.11}$$

where λ_k, Ω_k are defined by (6.2), (6.3), and κ_k, Σ_k by (5.11), (5.12) with $\mu = \max(0,\varepsilon)$ ∎

Let us also obtain the solution to the corresponding time-continuous optimal control problem for the quantum open system, described by the linear stochastic differential equations (5.17), (5.18) and the quadratic integral criterion

$$\Omega\left(:\mid x(\tau)\mid^2:\right) + \int_0^\tau \left(\omega :\mid x(t)\mid^2: -2\mathrm{Re}\vartheta\bar{u}(t)\,x(t) + \vartheta^1 \mid u(t)\mid^2\right)dt.$$

This criterion is obtained by setting $\omega \simeq \omega\Delta, \quad \vartheta \simeq \vartheta\Delta, \quad \vartheta^1 \simeq \vartheta^1\Delta$ in the conditions of the Theorem 6.1, and passing to the limit as $\Delta \longrightarrow 0$. So, the solution to the quantum optimal control problem for the time continuous quantum open system (5.17), (5.18) with quantum white noises v, w is defined as the limit of the solution to the discrete problem at $\Delta \longrightarrow 0$.

The optimal strategy, obtained in this limit, is obviously linear with respect to the optimal estimate $z(t)$ of $x(t)$ as in the classical case [20]: $u(t) = -\lambda(t) z(t)$, where $\lambda(t) = \left(\bar{\beta}\Omega(t) - \vartheta\right)/\vartheta^1$, $\quad \Omega(\tau) = \Omega$, and $\Omega(t)$ satisfies the equation:

$$-d\Omega(t)/dt + (\alpha + \bar{\alpha})\Omega(t) = \omega - \mid \lambda(t) \mid^2 \vartheta^1.$$

The optimal estimate $z(t)$ is obtained by coherent measurements, corresponding to the case $\mu = \max(0, \varepsilon)$ in the time-continuous Kalman filter, and the minimal mean square losses are defined by the integral

$$\alpha^o = \Omega_0 \mid z \mid^2 + \hbar\left(\Omega_0\Sigma + \int_0^\tau \left(\Omega(t)\sigma + \bar{\lambda}(t)\left(\bar{\beta}\Omega(t) - \vartheta\right)\Sigma(t)\,dt\right)\right).$$

In particular, when $\beta = \gamma = \varepsilon = \alpha + \bar{\alpha} > 0$, $\quad \nu = \upsilon = \sigma$, $\quad \vartheta^1 = \vartheta + \theta$, $\quad \vartheta = \omega$, we obtain the solution to the optimal control problem for the quantum open oscillator matched with the transmission line (2.3) of the wave resistance $\gamma/2$ which was considered as the motivating example in §2. In this case the equations (5.17), (5.18) are reduced to (2.2), (2.3), where the generalized derivatives $\upsilon(t) = \upsilon(dt)/dt$, $\quad y(t) = y(dt)/dt$ represent the direct and reverse waves on the input of the open oscillator.

A APPENDIX

Let $\mathcal{A}$, $\mathcal{B}$ be von–Neumann algebras, i.e. selfadjoint weakly closed subalgebras of operators in a complex Hilbert space $\mathcal{H}$ including the identity operator $\mathbf{1}$, and $\mathcal{P}$, $\mathcal{R}$ be predual spaces of ultra weakly continuous functionals on $\mathcal{A}$ and $\mathcal{B}$, respectively. The elements $\pi \in \mathcal{P}$ and $\rho \in \mathcal{R}$ are called states on $\mathcal{A}$ and $\mathcal{B}$ respectively if $<\pi, a> \geq 0$, $<\rho, b> \geq 0 \quad \forall a \geq 0, b \geq 0$ ($a, b \geq 0$ means the non-negative definiteness of the operators $a \in \mathcal{A}$ and $b \in \mathcal{B}$), and if $<\pi, \mathbf{1}> = 1$, $<\rho, \mathbf{1}> = 1$. Linear operators transforming operators $b \in \mathcal{B}$ into operators $a \in \mathcal{A}$ are called superoperators, and the predual linear maps $\mathcal{P} \to \mathcal{R}$ are called operations. The typical example of a superoperator gives a representation $b \mapsto u^* b u$, where u is a unitary operator. An operation $M : \pi \mapsto \pi M \in \mathcal{R}$ is called the (statistical) morphism if the dual superoperator $b \mapsto Mb \in \mathcal{A}$ is positive[1] $Mb \geq 0$, $\quad \forall b \geq 0$ and $M\mathbf{1} = \mathbf{1}$ (it is convenient to denote the morphisms and dual superoperators by the same symbol with the right and the left action respectively: $<\pi M, b> = <\pi, Mb>$.)

A $\mathcal{B}$–valued measure $b(d\zeta)$ on some Borel space $Z \ni \zeta$ is called Z–measurement, if $b(d\zeta) \geq 0$ for any Borel $d\zeta \subseteq Z$ and $\int b(d\zeta) = \mathbf{1}$ in the same sense. If $M : \mathcal{P} \to \mathcal{R}$ is a morphism describing a quantum channel, π–the state on its input and $b(d\zeta)$ –the measurement on its output, then the probability distribution on Z is calculated by any of the formulas

$$P(d\zeta) = <\pi M, b(d\zeta)> = <\pi, Mb(d\zeta)>. \tag{A.1}$$

Let, for instance, the subalgebras $\mathcal{A}$ and $\mathcal{B}$ be generated by the operators x and y respectively with the canonical commutation relations

$$[x, y] = 0, \quad [x, x^*] = \hbar\mathbf{1}, \quad [y, x^*] = \gamma\hbar\mathbf{1}, \quad [y, y^*] = \varepsilon\hbar\mathbf{1},$$

[1]For a physical realization of the statistical morphisms by conditional expectations of the representations a stronger condition of complete positivity $[Mb_{ik}]_{i,k=1\ldots n} \geq 0, \forall n$, where $[b_{ik}]_{i,k=1\ldots n} \geq 0$ is any non-negative definite operator-matrix with the elements $b_{ik} \in \mathcal{B}$, should be imposed on the morphisms.

where $\gamma \in \mathbf{C}, \varepsilon \in \mathbf{R}$ and $\hbar > 0$ is a constant.

It may be assumed that $y = \gamma x + v$ holds, where v is an operator in $\mathcal{H}$ commuting with x and x^*, but not commuting with the adjoint one: $[v, v^*] = (\varepsilon - \mid \gamma \mid^2) \hbar \mathbf{1}$, and the algebra generated by the pair x, y can be represented in the form of the tensor product $\mathcal{A} \otimes \mathcal{B}^\circ$, where $\mathcal{B}^\circ$ is the von–Neumann algebra generated by the operator v.

We shall write the operators, generated by the operators x and v in the form $\#\varphi(x, v)\#$, where $\varphi(\xi, \eta)$ are complex–valued functions of $\xi, \eta \in \mathbf{C}$, called symbols, and the notation $\# \cdot \#$ indicates such order of action for the operators between them, that first act the operators x, v, and then their conjugate. For instance, $\# \mid x \mid^2 \# = x^* x$. In a sufficiently wide class of symbols any operator from $\mathcal{A} \otimes \mathcal{B}^\circ$ can be represented in such a form, and this representation is single-valued and injective. In the case $y = \gamma x + v$ the operators $a \in \mathcal{A}$ are described by the symbols $\varphi(\xi, \eta) = \alpha(\xi)$ and the operators $b \in \mathcal{B}$ by the symbols $\varphi(\xi, \eta) = \beta(\gamma\xi - \eta)$as in the classical commutative case $\hbar = 0$. The states in this quasi-classical representation are described by distributions $q(\xi, \eta)$, generalizing the probability densities and representing the density operators as the symbols of the contrary order, which are dual to the order for the symbols $\varphi(\xi, \eta)$. Due to $\hbar > 0$, $x^*/\sqrt{\hbar}$ is the standard creation operator, and $x/\sqrt{\hbar}$ is the standard annihilation operator, so that the representation $a = \#\alpha(x)\#$ of operators $a \in \mathcal{A}$ is normal [19], described by the holomorphic symbols $\alpha(\xi)$ with respect to both $\xi, \bar{\xi}$. The corresponding symbols $p(\xi)$ of the states π on $\mathcal{A}$ are described by the Glauber distributions $p(\xi)$, which are defined as the linear functionals

$$< \pi, a >= \int p(\xi) \alpha(\xi) d\xi \quad (d\xi = d\mathrm{Re}\xi d\mathrm{Im}\xi / \pi\hbar),$$

describing the symbols of the density operator π, appropriate to the antinormal order. The normal order is denoted by the parentheses $: \quad :$, so we have $\#\alpha(x)\# =: \alpha(x):$ when $[x, x^*] \geq 0$. Note, that the antinormal symbol $p(\xi)$ of the density operator π and the normal symbol $p^o(\xi)$ are connected by the convolution [21]

$$p^o(\xi) = \int \exp\left\{- \mid \xi - \xi^1 \mid^2 / \hbar\right\} p\left(\xi^1\right) d\xi^1. \tag{A.2}$$

The appropriate representation of the algebra $\mathcal{B}^\circ$, and hence $\mathcal{A} \otimes \mathcal{B}^\circ$, is normal only if $\varepsilon > \mid \gamma \mid^2$, when $[v, v^*] > 0$. If $m(\eta)$ is a distribution which defines a state on $\mathcal{B}^\circ$ and there is no statistical dependence, a state on $\mathcal{A} \otimes \mathcal{B}^\circ$ is described by the product $p(\xi) m(\eta)$ and a state on the sub-algebra $\mathcal{B}$ by the convolution

$$r(\eta) = \int m(\eta - \gamma\xi) p(\xi) d\xi. \tag{A.3}$$

A superoperator $\mathcal{B} \to \mathcal{A}$, which is dual to a morphism (A.3), is described by the symbol transformation

$$\alpha(\xi) = \int \beta(\eta) m(\eta - \gamma\xi) d\eta.$$

For the normality of the appropriate representation $b = \#\beta(y)\#$ of the operators $b \in \mathcal{B}$ with the distribution (A.3) being Glauber, it is sufficient, that $\varepsilon > 0$. When $\varepsilon < 0$, the distribution $r(\eta)$ is the normal symbol of the appropriate density operator $\rho = \#r(y)\#$.

Let us consider the complex measurements, described by the measurements of the sum $\kappa y + w = z$, where w is an operator in $\mathcal{H}$, which commutes with y and y^*, but does not commute with the adjoint one w^*:

$$[w, w^*] = -\varepsilon \mid \kappa \mid^2 \hbar \mathbf{1},$$

so that $[z, z^*] = 0$ (it is assumed that the space $\mathcal{H}$ is chosen sufficiently wide, otherwise such an operator in $\mathcal{H}$ may not exist.)

If $\eta(\zeta)$ is a distribution describing a state on the algebra $\mathcal{B}^1$ generated by the operator w, then the probability distribution of the results of such a measurement on the output of the channel is described by the normed with respect to the Lebesgue measure $d\zeta$ density

$$s(\zeta) = \int\int n(\zeta - \kappa\eta)\, m(\eta - \gamma\xi)\, p(\xi)\, d\eta d\xi.$$

In accordance with formula (A.1) such a measurement is described by the $\mathcal{B}$ –valued measure

$$b(d\zeta) = \# n(\zeta - \kappa y)\, \# d\zeta, \tag{A.4}$$

and the distribution $n(\zeta)$ satisfies the condition

$$\int \mid \zeta \mid^2 n(\zeta)\, d\zeta \geq \max\left\{\varepsilon \mid \kappa \mid^2 \hbar, 0\right\} \tag{A.5}$$

in accordance with the inequality $w^* w \geq \{[w^*, w], 0\}$. When $\varepsilon > 0$ and representation (A.4) is normal, inequality (A.5) prohibits, in particular, distributions of Dirac δ –form.

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