A Radon-Nikodym Theorem for Completely Positive Maps

V P Belavkin
School of Mathematical Sciences, University of Nottingham,
Nottingham NG7 2RD
E-mail: vpb@maths.nott.ac.uk
and
P Staszewski
Institute of Physics, Nicholas Copernicus University, Toruń, Poland


Abstract

The aim of this paper is to generalize a noncommutative Radon-Nikodym theorem to the case of completely positive (CP) map. By only assuming absolute continuity with respect to another CP map the existence of a Hermitian-positive density as the unique “Radon-Nikodym derivative” is proved in the commutant of the Steinspring representation of the reference CP map.

1 Preliminaries and definitions

Let $\mathcal{A}$ be a C*-normed algebra, and let $B(\mathfrak{h})$ denote the algebra of all bounded operators on a Hilbert space $\mathfrak{h}$. In this paper we will obtain a positive self-adjoint density operator $\rho$ for a completely positive map $\kappa$ from $\mathcal{A}$ into $B(\mathfrak{h})$ strongly absolutely continuous with respect to another such map $\phi$ given, say, by a faithful weight or trace $\varphi$ as $\phi = 1_{\varphi}$. It will be uniquely defined as a noncommutative generalization of Radon-Nikodym derivative $\kappa_{\phi}$ in the Hilbert space $\mathcal{H}$ of Steinspring representation of $\phi$.

To this end, we first recall the definition of compete positivity. If $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras, $M(n)$ ($n \geq 1$) the algebra of $n \times n$ complex matrices and $\kappa$ is a linear map from $\mathcal{A}$ to $\mathcal{B}$, we shall say that $\kappa$ is $n$-positive if the map

$$\kappa_n : \mathcal{A} \otimes M(n) \rightarrow \mathcal{B} \otimes M(n),$$

$$\kappa_n(a \otimes m) = \kappa(a) \otimes m, \quad a \in \mathcal{A}, m \in M(n),$$

is positive. The map $\kappa$ is called completely positive if it is $n$-positive for all integers $n$. 
The completely positive maps play an important role in the description of quantum channels and time evolutions of open quantum systems [2].

Let us consider two quantum systems described in terms of C*-algebras $\mathcal{A}$ and $\mathcal{B}$. It can be easily shown that if the Heisenberg dynamics of the compound system is described by a $*$-endomorphism $\gamma$ of $\mathcal{A} \otimes \mathcal{B}$, then the reduced dynamics as conditional expectation $\epsilon$ of $\gamma$ corresponding to an independent state on $\mathcal{B}$ is described by a completely positive identity preserving maps $\mu : \mathcal{A} \to \mathcal{A}$ (such $\mu = \epsilon \circ \gamma$ is usually called a dynamical map on $\mathcal{A}$). The complete positivity of a reduced dynamics was first pointed out by Kraus [4] in the context of state changes produced by quantum measurements.

If a C*-algebra $\mathcal{A}$ describes an open physical system subject to completely positive dynamics, then any dynamical map of this system, considered in a representation $\iota$, is a completely positive map of norm one $\mathcal{A} \to \mathcal{B}(\mathfrak{h})$, where $\kappa = \iota \circ \mu$.

Let us recall that the condition of complete positivity of $\kappa$ can be written [5] in the form

$$\sum_{i,k=1}^{n} \langle \eta_i | \kappa(a_i^* a_k) \eta_k \rangle \geq 0, \quad \forall \eta_j \in \mathfrak{h}, \forall a_j \in \mathcal{A}, j = 1, \ldots, n, \forall n \in \mathbb{N}.$$ 

The condition of normalization of $\kappa$ can be expressed in the form $\kappa(1) = 1$ if $1 \in \mathcal{A}$ and 1 stand for identities in $\mathcal{A}$ and $\mathcal{B}(\mathfrak{h})$, respectively.

According to the famous results of Stinespring [5] any (normalized) completely positive map $\kappa : \mathcal{A} \to \mathcal{B}(\mathfrak{h})$ can be represented in the form

$$\kappa(a) = F_{\kappa}^* \pi_{\kappa}(a) F_{\kappa},$$

where $\pi_{\kappa} : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\kappa)$ is a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\kappa$ and $F_{\kappa}$ is a bounded (isometric) linear operator from $\mathfrak{h}$ into $\mathcal{H}_\kappa$. Such a representation of a completely positive map will be called spatial. The normalization condition for a dynamical map implies the isometricity $F_{\kappa}^* F_{\kappa} = 1$.

Let $\phi$ and $\kappa$ denote completely positive maps from $\mathcal{A}$ into $\mathcal{B}(\mathfrak{h})$ and let $\{(a_{jm})_m, j = 1, \ldots, n\}$ be a family of sequences in $\mathcal{A}$. Such a family will be called a $(\phi, \kappa)$ family of sequences if for any $n \in \mathbb{N}$

$$\lim_{m \to \infty} \sum_{i,k=1}^{n} \langle \eta_i | \phi(a_{im}^* a_{km}) \eta_k \rangle = \lim_{m,r \to \infty} \sum_{i,k=1}^{n} \langle \eta_j | \kappa((a_{im} - a_{ir})^*(a_{km} - a_{kr})) \eta_k \rangle = 0 \quad (1.1)$$

$$\forall \eta_j \in \mathfrak{h}, \quad j = 1, \ldots, n.$$ 

Now we generalize various forms and strengthened forms of the concept of absolute continuity [3] in the case of completely positive maps.

**Definition 1** A completely positive map $\kappa$ is called
(1) completely absolutely continuous with respect to a completely positive map \( \phi \) if for any \( n \in \mathbb{N} \)

\[
\inf_m \sum_{i,k=1}^n \langle \eta_i | \phi(a_{im}^* a_{km}) \eta_k \rangle = 0
\]

for any increasing family \( \{ A_m \} \) of matrices \( A_m = [a_{im}^* a_{km}] \) implies

\[
\inf_m \sum_{i,k=1}^n \langle \eta_i | \kappa(a_{im}^* a_{km}) \eta_k \rangle = 0, \quad \forall \eta_j \in \mathfrak{h}, \; j = 1, \ldots, n,
\]

(2) strongly completely absolutely continuous with respect to \( \phi \) if for any \( (\phi, \kappa) \)
family of sequences \( \{(a_{jm})_m, j = 1, \ldots, n\} \) we have for any \( n \in \mathbb{N} \)

\[
\lim_{m \to \infty} \sum_{i,k=1}^n \langle \eta_i | \kappa(a_{im}^* a_{km}) \eta_k \rangle = 0, \quad \forall \eta_j \in \mathfrak{h}, \; j = 1, \ldots, n,
\]

(3) completely dominated by \( \phi \) if there exists a \( \lambda > 0 \) such that for any \( n \in \mathbb{N} \)

\[
\sum_{i,k=1}^n \langle \eta_i | \kappa(a_{ik}^* a_{ik}) \eta_k \rangle \leq \lambda \sum_{i,k=1}^n \langle \eta_i | \phi(a_{ik}^* a_{ik}) \eta_k \rangle,
\]

\[
\forall \eta_j \in \mathfrak{h}, \quad \forall a_j \in \mathcal{A}, \quad j = 1, \ldots, n.
\]

It is rather obvious that (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1).

In the particular case \( \phi(a) = \varphi(a)1 \), where \( \varphi : \mathcal{A} \to \mathbb{C} \) denotes a positive functional on \( \mathcal{A} \) (e.g. a reference state, or trace), we shall say that \( \kappa \) is completely absolutely continuous or strongly completely absolutely continuous or completely dominated by the functional \( \varphi \). If completely positive maps are of the form \( \phi(a) = \varphi(a)1, \; \kappa(a) = \varphi(a)1 \), where \( \varphi, \varphi \) are positive functionals on \( \mathcal{A} \), then one can easily verify that our forms of absolute continuity (1)-(3) imply that \( \varphi \) is (1') \( \varphi \)-absolutely continuous, (2') strongly \( \varphi \)-absolutely continuous, (3') \( \varphi \)-dominated, respectively in the sense of Gudder [3].

2 A Radon-Nikodym theorem for completely positive maps

**Theorem 2** Let \( \phi \) and \( \kappa \) be a bounded completely positive maps form \( \mathcal{A} \) into \( B(\mathfrak{h}) \) and let \( \mathcal{H} \) be a Hilbert space of a representation \( \pi : \mathcal{A} \to B(\mathcal{H}) \) in which \( \phi \) is spatial, that is

\[
\phi(a) = F^* \pi(a) F, \quad \forall a \in \mathcal{A}, \tag{2.1}
\]

where \( F \) is assumed to be bounded operator \( \mathfrak{h} \to \mathcal{H} \). Then
(a) $\kappa$ is completely absolutely continuous with respect to $\phi$ if and only if it has a spatial representation $\kappa(a) = K^*\pi(a)K$ with $\pi(a)K = \vartheta\pi(a)F$, where $\vartheta$ is a densely defined operator in the minimal $\mathcal{H}$, commuting with $\pi(A) = \{\pi(a), a \in A\}$ on the linear $\mathcal{D} = \left\{\sum_j \pi(a_j)F\eta_j\right\}$.

(b) $\kappa$ is strongly completely absolutely continuous with respect to $\phi$ if and only if $\kappa$ is spatial in $(\pi, \mathcal{H})$ and there exists a positive self-adjoint operator $\varphi$, uniquely defined on $\mathcal{D}$, affiliated with the commutant $\pi(A)'$ and such that

$$\kappa(a) = F^*\varphi\pi(a)F = (\varphi^{1/2}F)^*\pi(a)(\varphi^{1/2}F), \quad \forall a \in A, \quad (2.2)$$

(c) $\kappa$ is completely dominated by $\phi$ if and only if $(2.2)$ holds and $\varphi$ is bounded.

**Proof.** Let us first sketch the prove the part (a) given in [1].

The condition of absolute continuity means that $\kappa$ is normal in the minimal spatial representation of $\phi$ with the support orthoprojector $P_\kappa$ majorised by the support $P_\phi$ of $\phi$. Therefore it is spatial, with the operator $K : \mathfrak{h} \to \mathcal{H}$ uniquely defining the operator $\vartheta = \pi'(K)$ on $\mathcal{D}$ by

$$\pi'(K)\pi(a)F\eta = \pi(a)K\eta, \quad \forall A \in \mathcal{A}, \eta \in \mathfrak{h}.$$

such that it commutes with $\pi(A)$. The reverse is obvious.

Let us now prove the part (b) of our theorem.

$(\Rightarrow)$ Let $\pi_\kappa$ be a representation of a C*-algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}_\kappa$ generated by the algebraic tensor product $\mathcal{A} \otimes \mathfrak{h}$ with respect to a positive Hermitian bilinear form

$$\left\langle \sum_i a_i \otimes \eta_i, \sum_k a_k \otimes \eta_k \right\rangle_\kappa = \sum_{i,k=1}^n \left\langle \eta_i | \kappa(a_i^*a_k)\eta_k \right\rangle_\kappa$$

(2.3)

defined by the equality

$$\pi_\kappa(a)\left| \sum_j a_j \otimes \eta_j \right\rangle_\kappa = \left| \sum_j a a_j \otimes \eta_j \right\rangle_\kappa$$

(2.4)

(for details see [5]).

Let us denote by $F_\kappa$ the bounded operator $\mathcal{H}_\kappa \to \mathfrak{h}$,

$$F_\kappa : \eta \mapsto \left| 1 \otimes \eta \right\rangle_\kappa,$$

(2.5)

(a canonical isometry $\mathfrak{h} \to \mathcal{H}_\kappa$ if $\kappa$ is normalized). Then we have [5]

$$\kappa(a) = F_\kappa^*\pi_\kappa(a)F_\kappa.$$ 

(2.6)

Define an operator $I_\kappa$ in $\mathcal{H}$ into $\mathcal{H}_\kappa$ by the formula

$$I_\kappa : \sum_j \pi(a_j)F\eta_j \mapsto \left| \sum_j a_j \otimes \eta_j \right\rangle_\kappa = \sum_j \pi_\kappa(a_j)F_\kappa\eta_j.$$ 

(2.7)
This is a consistent definition of a linear operator on the space \( D \subseteq \mathcal{H} \) because condition (2) implies (1) from which, taking into account (2.1), (2.5) and (2.6), we obtain the condition

\[
\left( \sum_k \pi(a_k) F \eta_k \right) \left( \sum_j \pi(a_j) F \eta_j \right) = 0 \Rightarrow \left( \sum_k a_k \otimes \eta_k \right) \left( \sum_j a_j \otimes \eta_j \right)_\kappa = 0.
\]

Obviously, we have \( F_\kappa = I_\kappa F \).

To prove that \( I_\kappa \) is closable let us first note that any sequence of elements of \( \left\{ \sum_j \pi(a_j) F \eta_j \right\} \) can be expressed in the form \( \left( \sum_j \pi(a_{jm}) F \eta_j \right)_m \) be any sequence such that \( \left( \left\| \sum_j \pi(a_{jm}) F \eta_j \right\| \right)_m \to 0 \) and \( \left( \left\| \sum_j a_{jm} \otimes \eta_j \right\|_\kappa \right)_m \) is convergent. Then for any set \( \eta_j, j = 1, \ldots, n \)

\[
\lim_{m \to \infty} \sum_{i,k} \langle \eta_i | \phi(a_{im}^* a_{km}) \eta_k \rangle = \lim_{m \to \infty} \sum_{i,k} \langle \eta_i | F^* \pi(a_{im})^* \pi(a_{km}) F \eta_k \rangle = \lim_{m \to \infty} \sum_{i,k} \langle \pi(a_{im}) F \eta_i | \pi(a_{km}) F \eta_k \rangle = \lim_{m \to \infty} \left\| \sum_i \pi(a_{im}) F \eta_i \right\|^2 = 0.
\]

Moreover,

\[
\lim_{m,r \to \infty} \sum_{i,k} \langle \eta_i | \kappa ((a_{im} - a_{ir})^* (a_{km} - a_{kr})) \eta_k \rangle = \lim_{m,r \to \infty} \left\langle \sum_i (a_{im} - a_{ir}) \otimes \eta_i \left| \sum_k (a_{km} - a_{kr}) \otimes \eta_k \right\rangle_\kappa = \lim_{m,r \to \infty} \left\| \sum_i (a_{im} - a_{ir}) \otimes \eta_i \right\|_\kappa^2 = 0,
\]

hence \( \left( \sum a_{im} \otimes \eta_i \right)_\kappa \) is Cauchy by assumption. Hence \( \left\{ (a_{jm})_m, j = 1, \ldots, n \right\} \) form a \( (\phi, \kappa) \) family of sequences. Then from the strong complete absolute continuity of \( \kappa \) with respect to \( \phi \) we have for any \( n \in \mathbb{N} \)

\[
0 = \lim_{m \to \infty} \sum_{i,k} \langle \eta_i | \kappa (a_{im}^* a_{km}) \eta_k \rangle = \lim_{m \to \infty} \left\langle \sum_i a_{im} \otimes \eta_i \left| \sum_k a_{km} \otimes \eta_k \right\rangle_\kappa = \lim_{m \to \infty} \left\| \sum_i a_{im} \otimes \eta_i \right\|_\kappa^2.
\]

This proves that \( I_\kappa \) is closable.
Denote by \( \tilde{I}_\kappa \) its closure. Then there exists an adjoint operator \( I_\kappa^* \) defined on the lineal \( \{ \| \sum_j a_j \otimes \eta_j \|_\kappa \} \), dense in \( \mathcal{H}_\kappa \), by the equality
\[
\left\langle \sum_j a_j \otimes \eta_j, \sum_k a_k \otimes \eta_k \right\rangle_\kappa = \left\langle \sum_j \pi(a_j) F \eta_j, I_\kappa^* \sum_k a_k \otimes \eta_k \right\rangle.
\]  
(2.8)

The positive self-adjoint operator \( \varrho = I_\kappa^* \tilde{I}_\kappa \) on the lineal \( \{ \| \pi(a_j) F \eta_j \| \} \) is affiliated with \( \pi(\mathcal{A})' \) because on the domains of \( I_\kappa \) and \( I_\kappa^* \) we have
\[
\pi_\kappa(a) I_\kappa = I_\kappa \pi(a), \quad I_\kappa^* \pi_\kappa(a) = \pi(a) I_\kappa^*.
\]
(2.9)

Let us verify the first of the equalities (2.9). Taking into account (2.7) and (2.4) we have
\[
\pi_\kappa(a) I_\kappa \left\| \sum_j \pi(a_j) F \eta_j \right\|_\kappa = \pi_\kappa(a) \left\| \sum_j a_j \otimes \eta_j \right\|_\kappa
\]
\[
= \left\| \sum_j a a_j \otimes \eta_j \right\|_\kappa
\]
\[
= I_\kappa \left\| \sum_j \pi(aa_j) F \eta_j \right\|
\]
\[
= I_\kappa \pi(a) \left\| \sum_j \pi(a_j) F \eta_j \right\|.
\]

Taking into account that \( F_\kappa = I_\kappa F \), we obtain
\[
\kappa(a) = F_\kappa^* \pi_\kappa(a) F_\kappa = F^* \varrho \pi(a) F
\]
\[
= (\varrho^{1/2} F)^* \pi(a) (\varrho^{1/2} F) = K^* \pi(a) K,
\]
where \( K = \varrho^{1/2} F \).

\((\Leftarrow)\) Let \( \{ (a_{jm}) \}_{m, j = 1, \ldots, n} \) be a family of \( (\phi, \kappa) \) sequences. Then
\[
\left\| \sum_i \pi(a_{im}) F \eta_i \right\|_m \to 0
\]
and moreover
\[
0 = \lim_{m, r \to \infty} \sum_{i, k} \langle \eta_i | \kappa((a_{im} - a_{ir})^* (a_{km} - a_{kr})) \eta_k \rangle
\]
\[
= \lim_{m, r \to \infty} \sum_{i, k} \langle \eta_i | (\varrho^{1/2} F)^* \pi(a_{im} - a_{ir})^* \pi(a_{km} - a_{kr}) (\varrho^{1/2} F) \eta_k \rangle
\]
\[
= \lim_{m, r \to \infty} \left\langle \varrho^{1/2} \sum_i \pi(a_{im} - a_{ir}) F \eta_i | \varrho^{1/2} \sum_k \pi(a_{km} - a_{kr}) F \eta_k \right\rangle
\]
\[
= \lim_{m, r \to \infty} \left\| \varrho^{1/2} \sum_i \pi(a_{im}) F \eta_i \right\| - \varrho^{1/2} \sum \left\| \pi(a_{ir}) F \eta_i \right\|.
\]

Hence \( \varrho^{1/2} \left( \sum_i \pi(a_{im}) F \eta_i \right) \) is Cauchy, and since \( \varrho^{1/2} \) is closed,
\[
\varrho^{1/2} \left( \sum_i \pi(a_{im}) F \eta_i \right) \to 0.
\]
Then we have
\[
\lim_{m \to \infty} \sum_{i,k} \langle \eta_i | \kappa(a_{im}^* a_{km}) \eta_k \rangle = \lim_{m \to \infty} \sum_{i,k} \langle \eta_i | (q^{1/2} F)^* \pi(a_{im}) \pi(a_{km}) (q^{1/2} F) \eta_k \rangle
\]
\[
= \lim_{m \to \infty} \left\| q^{1/2} \sum_i \pi(a_{im}) F \eta_i \right\|^2 = 0.
\]
This means that \( \kappa \) is strongly completely absolutely continuous with respect to \( \phi \). This completes the proof of part (a).

Let us prove part (c) of our theorem.

(\( \Rightarrow \)) Suppose \( \kappa \) to be completely dominated by \( \phi \). As the condition (3) implies (2), therefore (a) holds. It remains to prove that \( q \) is bounded. The boundedness of \( q \) follows from the following calculations:
\[
\left\| q^{1/2} \sum_i \pi(a_i) F \eta_i \right\|^2 = \left\langle q^{1/2} \sum_i \pi(a_i) F \eta_i | q^{1/2} \sum_k \pi(a_k) F \eta_k \right\rangle
\]
\[
= \sum_{i,k} \langle \eta_i | \kappa(a_i^* a_k) \eta_k \rangle \leq \lambda \sum_{i,k} \langle \eta_i | \phi(a_i^* a_k) \eta_k \rangle
\]
\[
= \lambda \left\| \sum_i \pi(a_i) F \eta_i \right\|^2 \pi(a_k) F \eta_k \rangle
\]
\[
= \lambda \left\| \sum_i \pi(a_i) F \eta_i \right\|^2.
\]

(\( \Leftarrow \)) Suppose that \( q^{1/2} \) is bounded, then
\[
\sum_{i,k} \langle \eta_i | \kappa(a_i^* a_k) \eta_k \rangle = \sum_{i,k} \langle \eta_i | (q^{1/2} F)^* \pi(a_i^* a_k) (q^{1/2} F) \eta_k \rangle
\]
\[
= \left\langle q^{1/2} \sum_i \pi(a_i) F \eta_i | q^{1/2} \sum_k \pi(a_k) F \eta_k \right\rangle
\]
\[
= \left\| q^{1/2} \sum_i \pi(a_i) F \eta_i \right\|^2 \leq \left\| q^{1/2} \right\|^2 \left\| \sum_i \pi(a_i) F \eta_i \right\|^2
\]
\[
= \left\| q^{1/2} \right\|^2 \sum_{i,k} \langle \eta_i | \phi(a_i^* a_k) \eta_k \rangle.
\]

Hence \( \kappa \) is completely dominated by \( \phi \).

The uniqueness of \( q \) can be assured by choosing the smallest Hilbert space \( \mathcal{H}_\phi \) in which \( \phi(a) \) has the Steinpring form \( \phi(a) = F^* \pi(a) F \). Note that, if \( \phi = 1 \varphi, \mathcal{H}_\phi = \mathcal{H}_\varphi \), \( \pi_\phi (a) = 1 \otimes \varphi (a) \) and \( F = 1 \otimes f \), where \( \mathcal{H}_\varphi \ni f \) is the space of the cyclic representation \( \varphi(a) = f^* \pi_\varphi (a) f \) of a positive functional \( \varphi \) on \( \mathcal{A} \).

The formulation of complete absolute continuity for CP maps belongs to VPB, and the Main Theorem in the formulation of Parts (a) and (b) was originally given in [1].
References


